Tokyo J. Math.
Vol. 12, No. 1, 1989

Incompressibility of Measured Laminations in 3-Manifolds

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Introduction. In this paper we study codimension-1 measured laminations in 3-manifolds. A lamination is a foliation in a closed subset of a manifold. A codimension-1 lamination is said to be measured when it has a transverse invariant measure. The basic study of codimension-1 measured laminations was done by Morgan and Shalen in [2]. Our aim is a further study of measured laminations in a 3-manifold, especially about its incompressibility.

In 3-dimension there is close relationship between measured laminations and branched surfaces defined by Floyd and Oertel in [1]. In Morgan-Shalen [2], it was proved that each leaf of a measured lamination carried by an incompressible branched surface is incompressible. On the other hand, our main theorem in this paper states that for any transversely orientable measured lamination each of whose leaves is incompressible and whose support is not the whole of the manifold, there exists an incompressible branched surface carrying it. Note that as our definition of measured laminations is different from that of Oertel [5], it is not easy to see even that for a measured lamination, there exists a branched surface carrying it. Hence first we must construct a branched surface carrying the measured lamination using a handle decomposition of the manifold. This constitutes the first half part of this paper.

After completing this paper, it was informed that A. Hatcher proved that a lamination whose leaves are incompressible is carried by an incompressible branched surface. (This result is still unpublished.) But his definition of lamination is the same as that of Oertel and different from ours. D. Gabai and U. Oertel also proved the above result with a little different method in a part of their work.

Throughout this paper we work in C^∞-category. The symbol M always denotes a closed orientable irreducible 3-manifold. By a term

Received June 3, 1988
Revised December 15, 1988

lamination we always mean codimension-1 lamination. A general reference for laminations is Morgan-Shalen [2]. A lamination L is a foliation on a closed subset of M. For a lamination L, there exists a system of coordinate neighbourhoods $\{U_i\}$ where $U_i = V_i \times I$ for some open subset V_i of $\boldsymbol{R}^2$, such that $L \cap U_i = V_i \times X_i$ for some closed subset X_i of I, and such that the leaf structures $\{V_i \times \{x\}\}$ $(x \in X_i)$ and $\{V_j \times \{x\}\}$ $(x \in X_j)$ is compatible for two coordinate neighbourhoods U_i and U_j. The support of L is denoted by $|L|$.

A lamination L is said to be transversely orientable when there exists an open subset $U \supset L$ and a nonsingular vector field on U transverse to L.

A measured lamination is a lamination equipped with an invariant transverse measure.

A branched surface is a C^1-surface with singularities as depicted in Figure 0.A. A fibred neighbourhood $N(B)$ of a branched surface B is a neighbourhood of B fibred by intervals as Figure 0.B. The boundary of $N(B)$ consists of the part transverse to fibres which is called the horizontal boundary denoted by $\partial_h N(B)$ and the part consisting of fibres which is called the vertical boundary denoted by $\partial_v N(B)$. A branched surface B is said to be incompressible when the following three conditions are satisfied.

1. The horizontal boundary $\partial_h N(B)$ is incompressible in $M - \operatorname{int} N(B)$.
2. There are no disks of contact.

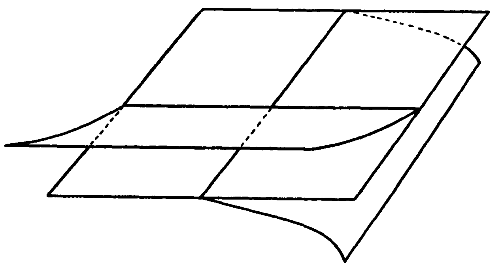


FIGURE 0.A

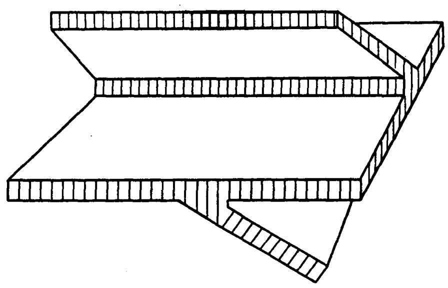
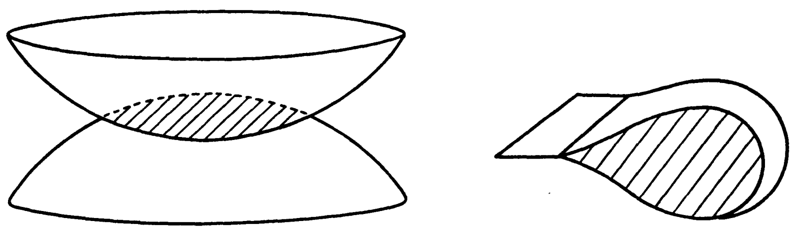


FIGURE 0.B



disk of contact monogon

FIGURE 0.C

3. There are no monogons.

See Floyd-Oertel [1] for details.

A measured lamination L is said to be carried by a branched surface B when L is isotoped so that L is contained in $N(B)$ transversely to fibres of $N(B)$. A measured lamination is said to be incompressible when it is carried by an incompressible branched surface.

In Morgan-Shalen [2], it was proved that every leaf of an incompressible measured lamination is incompressible. Our purpose in this paper is to prove that conversely a lamination whose leaves are incompressible is carried by an incompressible branched surface under an assumption that the measured lamination is transversely orientable.

THEOREM 1. *Let L be a transversely orientable measured lamination in M each of whose leaves is incompressible, and suppose $|L| \neq M$. Then there exists an incompressible branched surface B carrying L.*

For proving Theorem 1, we need several lemmas. As first of them, we prove the following lemma characterizing transversely orientable measured laminations in the 3-ball.

LEMMA 2. *Let L be a transversely orientable measured lamination in a 3-ball B^3 which is transverse to ∂B^3. Then L is decomposed into finite sets each of which consists of parallel compact leaves. Furthermore suppose that each leaf of L is incompressible. Then there are no leaves contained in the interior of B^3.*

PROOF. By Proposition 4.1 in Morgan-Shalen [2], each leaf of L is compact. By Theorem 3.2 in the same paper, L is decomposed into finite subsets, each of which consists of parallel compact leaves. An incompressible leaf of L cannot be contained in the interior of B^3 because if it is contained in the interior, it must be a closed incompressible surface, which does not exist in B^3. $\square$

The following definitions are due to Morgan-Shalen [3].

Let h be a handle decomposition of M. The handle decomposition h is said to be *admissible* when the following two conditions are satisfied.

1. For every distinct two 2-handles of h, there is at most one 1-handle meeting both of them.

2. Every 1-handle meets two or three 2-handles.

For an m-handle $\sigma = D^m \times D^{3-m}$, a disk $D^m \times \{y\}$ $(y \in D^{3-m})$ is called a horizontal disk and a disk $\{x\} \times D^{3-m}$ $(x \in D^m)$ is called a vertical disk. Let h be an admissible handle decomposition of M. A measured lamination

L is said to be *weakly normal* with respect to h when the following condition is satisfied.

○ For each handle σ, any component of $\sigma \cap L$ is a disk transverse to all vertical disks of σ. Especially L cannot intersect 3-handles.

Moreover L is said to be *normal* with respect to h when L is weakly normal with respect to h and satisfies the following condition.

○ For each handle τ of the handle decomposition of $\partial\sigma$ induced from h and each component λ of $L \cap \sigma$, the intersection $\lambda \cap \tau$ is an interval.

LEMMA 3. *Let L be a transversely orientable measured lamination each of whose leaves is incompressible, and suppose $|L| \neq M$. Then there exists an admissible handle decomposition of M with respect to which L is weakly normal.*

PROOF. Construct first arbitrarily an admissible handle decomposition h of M. In the following, we will construct a subdivision of h with respect to which L is weakly normal. As $|L| \neq M$, it is easy to see that we can isotope L so that L is disjoint from 3-handles of h. Furthermore as $|L| \neq M$, by isotoping L or h, we can assume that L is transverse to boundaries of handles of h. Then for any handle σ of h, $L \cap \sigma$ is decomposed into finitely many subsets each of which consists of parallel compact leaves by Lemma 2. They must be planar surfaces because leaves of L are incompressible by the assumption.

Let $\sigma = D^2 \times I$ be a 2-handle of h. Suppose that there exists a subset Δ of $L \cap \sigma$ consisting of parallel disks which are not transverse to the vertical intervals. (See Figure A.) Then for each component δ_i of Δ, there exists a disk δ_i' in $\partial D^2 \times I$ such that $\delta_i \cup \delta_i'$ bounds a 3-cell B_i in σ. Let δ_n be the outermost component of Δ such that B_n contains all the other B_i's. Then we can eliminate Δ from σ together with components of $L \cap \sigma$ contained in B_n by an isotopy. Hence we can assume that all disk components of $L \cap \sigma$ are transverse to the vertical intervals.

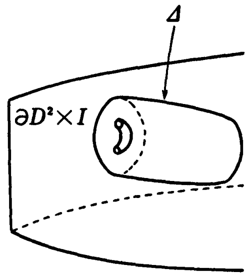


FIGURE A

Let S be the subset of $L\cap\sigma$ consisting of all leaves which are not disks. Our argument proceeds as follows. In (1), we assume that $L\cap\sigma=S$. In (2) we deal with the case without the assumption above. We subdivide (1) into three parts (1)-(a), (1)-(b), and (1)-(c). As we proceed (a), (b), (c), we deal with more general case. So there are the following relations between the parts: (1)-(a)$\subset$(1)-(b)$\subset$(1)-(c)$\subset$(2).

(1). Assume that $L\cap\sigma=S$. Then S is a disjoint union of $S_1,\cdots,S_n$ each of which is a parallel family of planar surfaces.

(1)-(a). We assume in (1)-(a) that for a leaf F_i of each S_i, components of ∂F_i bound disjoint disks in $\partial D^2\times I$. Then the surface F_i divides σ into two components one of which is homeomorphic to a 3-cell denoted by B_i by the Schoenflies theorem. The 3-cell B_i is regarded as a regular neighbourhood of a tree T_i embedded in σ which is possibly knotted such that the valencies of its vertices are at most 3. Let $\tilde{B}_0$ be the union of the outermost 3-cells $\{B_{nj}\}$ bounded by $\tilde{F}_0=\cup F_{nj}$ and let $\pi\colon \tilde{F}_0\to\tilde{T}_0$ $(=\cup T_{nj})$ be a bundle projection which has a singularity at a vertex and whose fibre is S^1 except on vertices. We can isotope $\tilde{T}_0$ so that for the canonical projection $D^2\times I\to D^2$, $p|\tilde{T}_0$ is a generic immersion all of whose double points are transverse. Each double point x of $p(\tilde{T}_0)$ corresponds to two circles C_x^1 and C_x^2 whose union is equal to $\pi^{-1}p^{-1}(x)$ in $\tilde{F}_0$, and each vertex y of $p(\tilde{T}_0)$ corresponds to a closed curve $K_y=\pi^{-1}p^{-1}(y)$ which has one double point. By cutting $\tilde{F}_0$ at C_x^i $(i=1,2)$ and K_y for every double point x and every vertex y of $p(\tilde{T}_0)$, we obtain a family of unknotted annuli in σ. For each annulus of this family, we construct a complex consisting of three 1-handles and 2-handles which forms a regular neighbourhood of a triangular cylinder as depicted in Figure B.

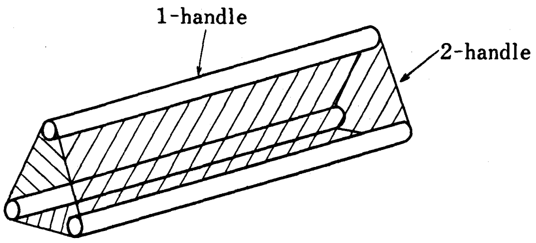


FIGURE B

At each C_x^i $(i=1,2)$, we connect two complexes by attaching a complex Γ_x^i consisting of three 0-handles and 1-handles which forms a regular neighbourhood of three sides of a triangle. After that we connect two 0-handles one of which is in Γ_x^1 and the other of which is in Γ_x^2 by a 1-handle vertical with respect to the projection p. (See Figure C.) At each closed curve K_y, we connect two complexes by a complex which forms a regular neighbourhood of a saddle as depicted in Figure D. By this construction, we obtain a 2-complex Σ_0' into which $\tilde{F}_0$ can be embedded normally. Let Δ be one of the disks obtained by cutting D^2 at $p(\tilde{T}_0)$. Then $\mathrm{Fr}_{D^2}\Delta$ lifts to a circle or an arc in 1-skeleton of Σ_0' which bounds a disk (rel. $\partial D^2\times I$) in $D^2\times I-\mathrm{int}(\Sigma_0')$ which projects to $\Delta-\mathrm{int}\,N(\mathrm{Fr}_{D^2}\Delta)$ by p. We attach the disks for all the disks obtained by cutting D^2 at $p(\tilde{T}_0)$ and obtain Σ_0. We take these disks disjointly. Then the outer complement of Σ_0 in σ consists of two 3-cells each of which is isotopic to $D^2\times J$ where J is a closed interval in I.

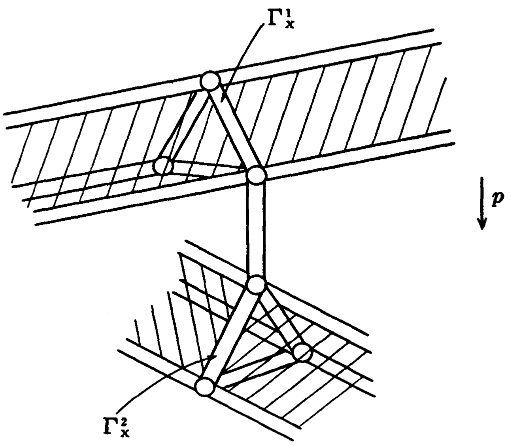
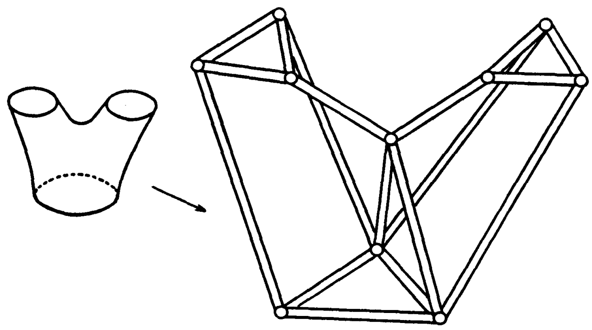


FIGURE C



(the picture of 1-skeleton)

FIGURE D

We do the similar construction for all of the B_i's which are outermost in $\tilde{B}_0$. The differences from the case of $\tilde{B}_0$ are that we replace σ by $\tilde{B}_0$ and that we must fill the spaces between $\partial\tilde{B}_0$ and the constructed complex by 3-handles in the last step. We repeat this construction for the B_i's from outward and obtain a 3-complex Σ into which S can be embedded normally. We replace σ by Σ after subdividing 1-handles surrounding σ so that $\Sigma\cup h-\sigma$ is a handle decomposition of M.

(1)-(b). We assume that components of ∂F_i bound disks in $\partial D^2\times I$ for each S_i, but they may not be disjoint. Let $\{\delta_i^j\}$ be the disks in $\partial D^2\times I$ bounded by the components of ∂F_i. We attach the δ_i^j's to ∂F_i and push off into σ so that if δ_1 is contained in δ_2 ($\delta_1, \delta_2\in\{\delta_i^j\}$), δ_1 is pushed higher than δ_2. Then we obtain a family of disjoint spheres which bound 3-cells in σ. Hence the closure of every component of $\sigma-\cup F_i$ is obtained from a 3-cell with finite holes by finite processes of removing and adding possibly knotted 1-handles one of whose attaching disks is contained in $\partial D^2\times I$. (See Figure E.)

Let $\{h\}$ be the set of these 1-handles. As $F_i - \cup_h h \cap F_i$ is an unknotted planar surface in σ, it is easy to construct a 2-complex K_i into which $F_i - \cup_h h \cap F_i$ is embedded normally and such that each hole of K_i is a regular neighbourhood of a triangle consisting of three 1-handles and 0-handles. For each of these 1-handles h, $h \cap F_i$ is an annulus contained in the closure of two components of σ, b^1 and b^2. If h is a removed 1-handle for b^1 (resp. b^2), then it is an added 1-handle for b^2 (resp. b^1). For each component B_j of $\sigma - \cup_i F_i$, let $h_j^1, \cdots, h_j^k$ be the removed 1-handles for B_j. Then the handles $h_j^1, \cdots, h_j^k$ are regarded as regular neighbourhoods of possibly knotted arcs in $B_j \cup h_j^1 \cup \cdots \cup h_j^k$ which is homeomorphic to a 3-cell with holes. Hence by the same method as in the case (1)-(a) using an isotopy and the canonical projection, we can construct a 2-complex Σ_j into which $h_j^1, \cdots, h_j^k$ are embedded normally such that $B_j - \Sigma_j$ is a union of 3-cells. By attaching the K_i's to the Σ_j's and filling the complement by 3-handles, we obtain a 3-complex Σ. We replace σ by Σ after subdividing 1-handles surrounding σ.

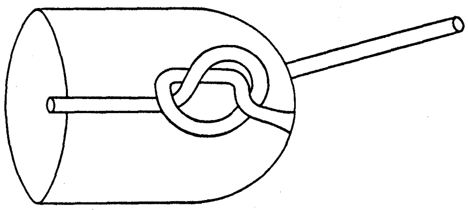


FIGURE E

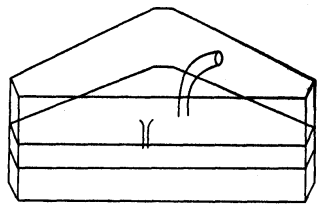


FIGURE F

(1)-(c). In this part we allow that some components of ∂F_i are essential in $\partial D^2 \times I$. By the consideration similar to the case (1)-(b), we can see that each component of $\sigma - \cup_i F_i$ is obtained by adding and removing 1-handles $\{h\}$ from 3-cells with holes bounded by 2-spheres and normal disks in σ. (See Figure F.) It is easy to construct a 2-complex with triangular holes into which $F_i - \cup_h h$ is embedded normally. Hence by the same construction as (1)-(b), we obtain a 3-complex Σ into which the F_i's are embedded normally.

(2). In (1) we assumed that $S = L \cap \sigma$. Now we consider the case when S is a proper subset of $L \cap \sigma$. Let $\nu_1, \cdots, \nu_k$ be the families of the decomposition of leaves of $L \cap \sigma$ which consists of parallel normal disks. Let $N_1, \cdots, N_k$ be disjoint 2-handles with the form $D^2 \times J_i$ (where $J_i \subset I$) containing ν_i respectively, and such that $N_i \cap S = \emptyset$. Then consider each of components $\sigma_1, \cdots, \sigma_{k+1}$ of $\sigma - \cup_{i=1}^k N_i$ instead of σ in the above construction and we obtain $\Sigma_1, \cdots, \Sigma_{k+1}$. Let Σ be the union of the Σ_i's, the N_i's regarded as 2-handles, and 3-handles filling $\sigma - \mathrm{int}(\cup_i \Sigma_i \cup \cup_j N_j)$.

We assumed that σ is a 2-handle above. For 1-handles and 0-handles, we do similar but simpler construction and we can make S contained in it normally.

The handle decomposition constructed above may not be admissible. But it is easy to make it admissible by the small changes as depicted in Figure G. $\square$

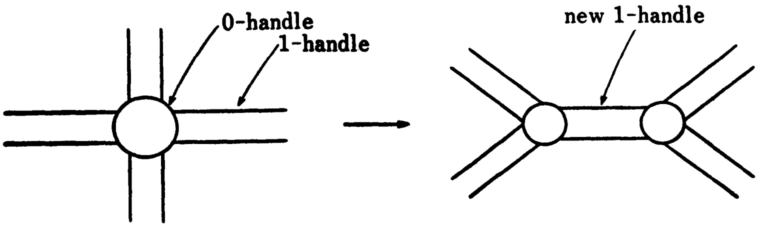


FIGURE G

LEMMA 4. *Let h be an admissible handle decomposition. Let L be a transversely orientable measured lamination which is weakly normal with respect to h and each of whose leaves is incompressible. Then we can isotope L so that L is normal with respect to h.*

PROOF. What we have to do is to remove folds in 1-handles and 0-handles as illustrated in Figure H. We call spaces between upper leaves and lower leaves of folds gaps. Let σ be a 2-handle such that a 1-handle or a 0-handle surrounding σ has folds coming from and returning to σ. Let S_σ be the union of 1-handles and 0-handles surrounding σ. We fix a transverse orientation of l so that the orientation of upper leaves is upward.

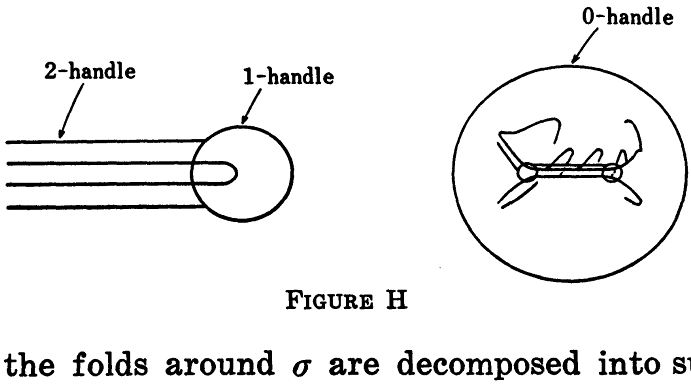


FIGURE H

The set of the folds around σ are decomposed into subsets $\Lambda_1, \cdots, \Lambda_k$ each of which consists of parallel folds. If Λ_i is inside of the gap of Λ_j, the number of 1-handles of S_σ intersecting Λ_i is greater than that of Λ_j. (See Figure I.) Let I_i be an isotopy moving Λ_i to the other side $S_\sigma - (S_\sigma \cap \Lambda_i)$. (Figure J.) We define the fold removing move across σ to be the composition of the I_i's beginning from the innermost Λ_i to the outermost one. This move may produce new folds in 1-handles of S_σ coming from and returning to 2-handles $\sigma_1, \cdots, \sigma_j$ adjacent to σ. Then we next carry out the fold removing moves across $\sigma_1, \cdots, \sigma_j$ one after another. We continue this process. If it terminates in finite steps, we can remove the fold which we chose first, and the same procedure removes all folds so that L would be normal with respect to h. Hence what we must show is that this process terminates in finite steps. The set of fold removing moves $\phi_1, \cdots, \phi_k$ across $\sigma_1, \cdots, \sigma_k$ where σ_{i+1} is adjacent to σ_i and such that ϕ_{i+1} is a move removing a fold made by ϕ_i is called a sequence of fold removing moves. We only need to prove that there are no infinite sequence of fold removing moves.

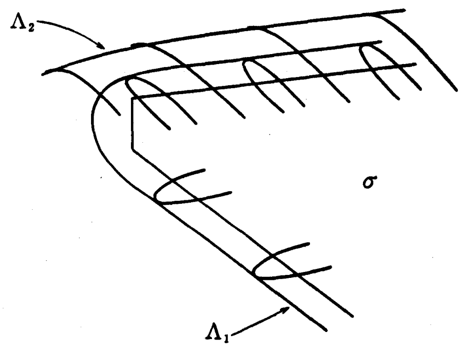


FIGURE I

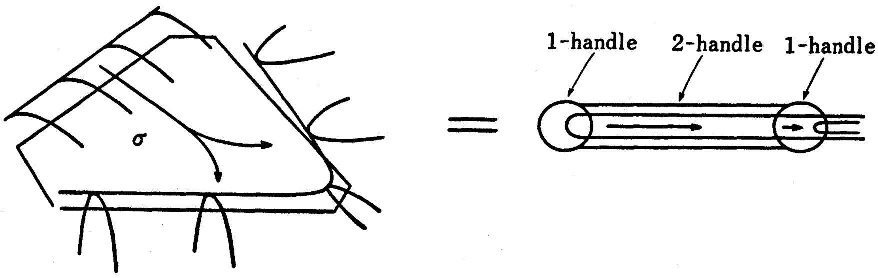


FIGURE J

CLAIM 1. *A sequence of fold removing moves cannot be continued infinitely.*

PROOF. Suppose that there is an infinitely long sequence of fold removing moves. As there are only finitely many 2-handles, there exists a 2-handle τ where the sequence returns infinitely many times. There are two possibilities. One is that there are disjoint infinite gaps in τ, and the other is that the latter gap includes the former gap and that there is a monotone increasing infinite sequence of gaps. In the first case leaves with opposite orientation approach infinitely. (See Figure K.) It contradicts the assumption that L is transversely orientable. In the second case, by finiteness of the measure in τ, the measure of the upper and the lower leaves must tend to zero. This happens only when the transverse measure of the fold decreases while it returns to τ again by continuing fold removing moves. The transverse measure of the fold decreases only when it passes the situation illustrated in Figure L. (Figure L is the situation when the leaves above a gap which are

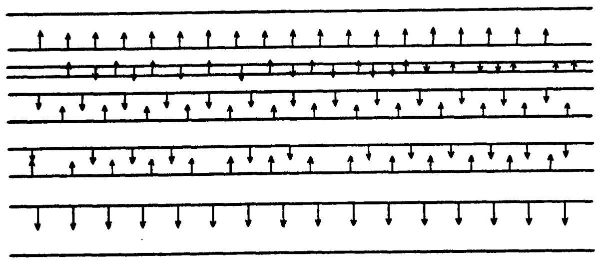


FIGURE K

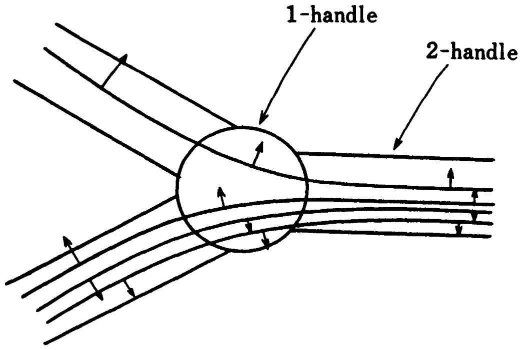


FIGURE L

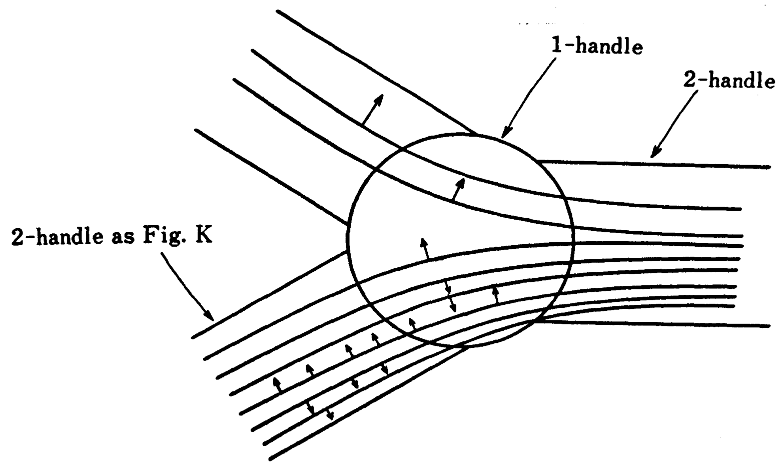


FIGURE M

contained in a fold are divided into two parts flowing into different 2-handles.) As the transverse measure tends to zero, the situation illustrated in Figure L must happen infinitely many times. As there are only finite 1-handles, there exists a 1-handle which contains infinite sets of leaves as Figure L. As there are only finite choices, we may assume that they situate as Figure M. This yields a 2-handle as Figure K adjacent to the 1-handle. Thus it contradicts the transverse orientability of L. $\square$

Let B_L be a branched surface obtained by identifying leaves in handles with the same disk types. Then obviously the branched surface B_L carries L. The next step is to make B_L an incompressible branched surface. The argument is similar to that of Floyd-Oertel [1].

LEMMA 5. *If B_L carries a 2-sphere then there exists a disk $D \subset L$ and a subsurface $P \subset D$ yielding a surface of contact such that $\partial D \subset \partial P$.*

PROOF. Let N_L be a fibred neighbourhood of B_L. We think of L as is contained in $\operatorname{int} N_L$. Let S be a 2-sphere carried by B_L. We can make S transverse to L and fibres of N_L. Then $L \cap S$ consists of finite families of parallel simple closed curves. We subdivide the families so that in a family all leaves of L intersects S from the same side. (The side is determined by the fibering of N_L.) Moreover as L is transversely orientable, we can subdivide the families so that the induced transverse orientations of all leaves of a family coincide.

Let Γ be an innermost family in S. Let γ_1 be the outermost leaf of Γ and γ_2 the innermost leaf of Γ. Let Δ_i be a disk on L bounded by γ_i for $i=1, 2$. Let D_1 be a disk on S bounded by γ_1 inside. If Δ_1 intersects S from inward (Figure N (a)), we let γ be γ_1, Δ be Δ_1, and D be a disk on S which γ_1 bounds outside. Next suppose that Δ_1 intersects S from outward (Figure N (b)). Then $\Delta_1 \cup D_1$ is an embedded sphere as Γ is innermost and the transverse orientations of leaves of Γ coincide. Hence it bounds a 3-cell B. Suppose that it bounds B outside, i.e. B contains $S-D_1$. Perturb L so that $\partial B \cap L = \Gamma$, and the intersection is transverse. Then by Lemma 2, $\partial \Delta_1$ is a boundary of a planar surface which is a leaf of $B \cap L$. But as $\partial B \cap L = \Gamma$ and there transverse orientations of leaves of Γ coincide, it must be a disk. This contradicts the fact that there are no leaves of L homeomorphic to spheres. Therefore $\Delta_1 \cup D_1$ bounds B inside. By the same argument as above, γ_2 bounds a disk in B. Hence Δ_2 also intersects S from outward. We let γ be γ_2, Δ be Δ_2, and D be a disk on S which γ_2 bounds inside in this case. If

$\operatorname{int} \Delta \cap S \neq \emptyset$, let c be an innermost simple closed curve of Δ bounding D' in S and Δ' in L. We choose D' so that $D' \cap \Delta'$ is tangent to fibres. We replace S by $(S-D') \cup \Delta'$. (More precisely we need to perturb $(S-D') \cup \Delta'$ so that it is apart from Δ' and transverse to L. Notice that this move makes no double points. From now on, cuts and pastes always include this sort of perturbation implicitly.) If $D' \cap \Gamma \neq \emptyset$, then we replace Γ by $\Gamma \cap (S-D')$. Repeating this, we can assume that $\operatorname{int} \Delta \cap S = \emptyset$. Replacing S by $D \cup \Delta$, we can eliminate Γ from $S \cap L$.

(a) (b)

FIGURE N

Therefore we can assume that $L \cap S = \emptyset$ by repeating the process above. As M is irreducible, S bounds a 3-cell B and by Lemma 2, $B \cap L = \emptyset$. Then by the same argument as the proof of Claim 1 of Floyd-Oertel [1] or Oertel [4], the proof of this lemma completes. □

LEMMA 6. *Suppose that B_L has a disk of contact and that B_L carries no 2-spheres. Then B_L has a disk of contact contained in L.*

PROOF. Let D be a disk of contact for B_L. We assume that ∂D is contained in $\partial(\partial_v B_L)$. Suppose that $L \cap \operatorname{int} D \neq \emptyset$. Then $L \cap \operatorname{int} D$ is decomposed into finite families each of which consists of parallel simple closed curves. Take an outermost simple closed curve γ of $L \cap \operatorname{int} D$. The simple closed curve γ bounds a disk Δ in L and δ in D. The disk Δ may intersect with D in the interior. Let c be an innermost component in Δ of $\operatorname{int} \Delta \cap D$ bounding Δ' in Δ and D' in D. As $\Delta' \cup D'$ is a 2-sphere, we can see that $\Delta' \cup D'$ must be tangent to fibres at $\Delta' \cap D'$ by the assumption that B_L carries no 2-spheres. Hence we can replace D by $(D-D') \cup \Delta'$ which is also a disk of contact. Repeating this we may assume that $\operatorname{int} D \cap \operatorname{int} \Delta = \emptyset$. Finally we replace D by $(D-\delta) \cup \Delta$. Repeating this for all outermost simple closed curves we can assume that $L \cap \operatorname{int} D = \emptyset$. The boundary ∂D bounds a disk G in L. As $G \cup D$ is a 2-sphere and is not carried by B_L by the assumption, G is also a disk of contact. □

PROOF OF THEOREM 1. Suppose that N_L contains a monogon. Then, as is shown in Floyd-Oertel [1], there is a solid torus V bounded by a component of $\partial_v N_L$ and $\partial_h N_L$. Isotope all leaves flowing along $V \cap \partial_h N_L$ across V as is depicted in Figure O. Then L becomes not normal again, but we can eliminate one monogon. We isotope L as in the proof of Lemma 4 and make it normal. If it stops again in a solid torus bounding a monogon, then continue to isotope L. By the same argument as in the proof on Lemma 4, this process terminates in finite steps. We carry out this isotopic move of L for all monogons. After that we eliminate sectors of N_L which do not contain leaves of L, and we can make N_L have no monogons.

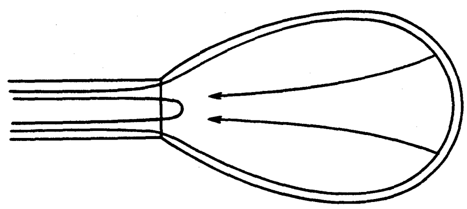


FIGURE O

Suppose that N_L contains a disk of contact D. Then by Lemma 5 and Lemma 6, there exists a surface of contact G contained in L. We eliminate $G \times I$ from N_L. We carry out this operation for all disks of contact of N_L, and we obtain $\tilde{N}_L$. By construction, $\tilde{N}_L$ contains no disks of contact. Moreover as $\partial_h \tilde{N}_L$ can be regarded as a part of L, it is incompressible. The operation above does not make new monogons. Therefore by letting N be $\tilde{N}_L$, we obtain an incompressible branched surface carrying L. $\square$

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