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Deficient and Ramified Small Functions for Admissible Solutions of Some Differential Equations II

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Abstract. Let $\alpha_j(z)$, $j=1, 2$, $a_i(z)$, $i=1, 2, \cdots, 6$ be meromorphic functions. Suppose the differential equation $(*)$ $w'^3+\alpha_2(z)w'^2+\alpha_1(z)w'=a_6(z)w^6+\cdots+a_1(z)w+a_0(z)$ possesses an admissible solution $w(z)$. If $\eta(z)$ is a solution of $(*)$ and small with respect to $w(z)$ and if $(*)$ is irreducible, then $\eta(z)$ is a deficient or a ramified function for $w(z)$.

1. Introduction.

We use here standard notations in Nevanlinna theory [2][6][8]. Let $f(z)$ be a meromorphic function. In this paper the term "meromorphic" will mean meromorphic in $|z|<\infty$. As usual, $m(r, f)$, $N(r, f)$, and $T(r, f)$ denote the proximity function, the counting function, and the characteristic function of $f(z)$, respectively. Let $\bar{N}(r, f)$ be the counting function for distinct poles of $f(z)$. Put $N_1(r, f)=N(r, f)-\bar{N}(r, f)$. For $\alpha\in\mathbb{C}$, the following quantities are defined

$$\delta(\alpha, f)=\liminf_{r\to\infty}\frac{m(r, 1/(f-\alpha))}{T(r, f)} \qquad (\textit{deficiency})$$

and

$$\theta(\alpha, f)=\liminf_{r\to\infty}\frac{N_1(r, 1/(f-\alpha))}{T(r, f)} \qquad (\textit{ramification index}).$$

A function $\varphi(r)$, $0\le r\le\infty$, is said to be $S(r, f)$ if there is a set $E\subset\mathbb{R}^+$ of finite linear measure such that $\varphi(r)=o(T(r, f))$ as $r\to\infty$, $r\notin E$. A meromorphic function $a(z)$ is said to be *small with respect to* $f(z)$ if $T(r, a)=S(r, f)$. We consider here the deficiency and the ramification for a small function $a(z)$ instead of complex number $\alpha\in\mathbb{C}$. We put, for a meromorphic function $a(z)$, $m(r, a; f)=m(r, 1/(f-a))$, $N(r, a; f)=N(r, 1/(f-a))$, and $\bar{N}(r, a; f)$, $N_1(r, a; f)$, $\delta(a, f)$, $\theta(a, f)$, etc., are defined in the same way as for a complex number $\alpha\in\mathbb{C}$, respectively. If $\delta(a, f)>0$ or $\theta(a, f)>0$, then $a(z)$ is said to be a

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deficient or *ramified* function for $f(z)$, respectively.

Let $\mathcal{M}$ be a finite collection of meromorphic functions. A transcendental meromorphic function $w(z)$ *is admissible with respect to* $\mathcal{M}$, if $T(r, a)=S(r, w)$ for any $a(z)\in\mathcal{M}$. Suppose a transcendental meromorphic function $w(z)$ is admissible with respect to $\mathcal{M}$. For $c\in\mathbb{C}\cup\{\infty\}$, z_0 *is admissible c-point with respect to* $\mathcal{M}$, if z_0 is c-point of $w(z)$ and neither zero nor pole of $a(z)$ which belongs to $\mathcal{M}$. Suppose $N(r, c; f)\neq S(r, f)$, $c\in\mathbb{C}\cup\{\infty\}$. We denote by $n^*_{C1}(r, c; f)$, the number of c-point z_0 of $f(z)$ in $|z|\leqq r$ so that z_0 satisfies some condition C1. $N^*_{C1}(r, c; f)$ is defined in the usual way. We use the word "*almost all*" c-point satisfy the condition C1, if

$$N(r, c; f)-N^*_{C1}(r, c; f)=S(r, f)\,.$$

REMARK 1. Let $\mathcal{M}$ be a finite collection of meromorphic functions. Suppose a transcendental meromorphic function $w(z)$ is admissible with respect to $\mathcal{M}$. Let $\eta(z)$ be rational of members of $\mathcal{M}$ and their derivatives. Then we have $T(r, \eta)\leq K\sum_{a_\nu\in\mathcal{M}}T(r, a_\nu)+S(r, w)$, for some $K>0$. Thus $\eta(z)$ is a small function with respect to $w(z)$. Assume that $N(r, w)\neq S(r, w)$, then there exists an admissible pole of $w(z)$ with respect to $\mathcal{M}$. If $\eta(z)$ vanishes at almost all poles of $w(z)$, then $\eta(z)\equiv 0$.

Let $\Omega(z, w, w', \cdots, w^{(n)})$ be a differential polynomial of w with meromorphic coefficients and let $\mathcal{M}$ be the collection of coefficients of Ω. We call $w(z)$ *an admissible solution* of the equation

$$\Omega(z, w, w', \cdots, w^{(n)})=0\,, \tag{1.1}$$

if $w(z)$ satisfies the above equation and $w(z)$ is admissible with respect to $\mathcal{M}$.

Let $\boldsymbol{M}$ be the field of meromorphic functions and let $\Omega(z, w, w')$ be a polynomial of w and w' with meromorphic (possibly transcendental) coefficients. We call the differential polynomial $\Omega(z, w, w')$ irreducible, if $\Omega(z, w, w')$ is irreducible over the field $\boldsymbol{M}$.

We know the following theorem due to Mokhoňko [7]:

THEOREM A. *Suppose the differential equation* (1.1) *possesses an admissible solution* $w(z)$. *If* $\eta(z)$ *is a deficient or ramified small function for* $w(z)$, *then* $\eta(z)$ *is a small solution of* (1.1), *i.e.*

$$\Omega(z, \eta, \eta', \cdots, \eta^{(n)})=0\,.$$

Our aim in this note is to get a converse of this result for the special case of (1.1), that is, for the equation of the form

$$P(z, w')=Q(z, w)\,, \tag{1.2}$$

where $P(z, w')$ and $Q(z, w)$ are polynomials of w' and w with meromorphic coefficients, respectively. In [3], we obtained the following theorem for the case $p=\deg_{w'}[P(z, w')]=2$.

THEOREM B. *Suppose the differential equation*

$$w'^2+\alpha_1(z)w'=a_4(z)w^4+\cdots+a_1(z)w+a_0(z) \tag{1.3}$$

possesses an admissible solution $w(z)$, *where the coefficients are meromorphic and* $|a_4|+|a_3|+|a_2|\not\equiv 0$. *If* $\eta(z)$ *is a small solution of* (1.3), *then* $\eta(z)$ *is a deficient or ramified function of* w, *unless* (1.3) *is reducible.*

In this note, we treat the case $p=3$ in (1.2), and prove the following theorem.

THEOREM 1. *Suppose the differential equation*

$$w'^3+\alpha_2(z)w'^2+\alpha_1(z)w'=a_6(z)w^6+\cdots+a_1(z)w+a_0(z) \tag{1.4}$$

possesses an admissible solution $w(z)$, *where the coefficients are meromorphic and* $|a_6|+|a_5|+|a_4|+|a_3|\not\equiv 0$. *If* $\eta(z)$ *is a small solution of* (1.4), *then* $\eta(z)$ *is a deficient or ramified function of* w, *unless* (1.4) *is reducible.*

2. Preliminary lemmas.

LEMMA 1 ([5]). *Suppose* (1.4) *possesses an admissible solution* $w(z)$. *If* $w(z)$ *satisfies the Riccati equation or the differential equation*

$$w'^2+B(z,w)w'+A(z,w)=0 \tag{2.1}$$

where $B(z,w)$ *and* $A(z,w)$ *are polynomials of* w *with small* (*w.r.t.* $w(z)$) *coefficients and* $\deg_w[B(z,w)]\le 2$, $\deg_w[A(z,w)]\le 4$, *then* (1.4) *is reducible.*

REMARK 2. Put $y=[a(z)w+b(z)]/[c(z)w+d(z)]$, $ad-bc\not\equiv 0$ in (2.1), where $a(z)$, $b(z)$, $c(z)$ and $d(z)$ are small (w.r.t. $w(z)$) functions. Then $y(z)$ satisfies the Riccati equation or the differential equation of the form

$$y'^2+\tilde{B}(z,y)y'+\tilde{A}(z,y)=0 \tag{2.1'}$$

where $\tilde{B}(z,y)$ and $\tilde{A}(z,y)$ are polynomials of y with small (w.r.t. $y(z)$) coefficients and $\deg_y[\tilde{B}(z,y)]\le 2$, $\deg_y[\tilde{A}(z,y)]\le 4$.

The equation (2.1) was treated by Steinmetz in [9] and by Eremenko in [1]. To state Lemma 2, we define some notations (see [4]).

Let $f(z)$ be a transcendental meromorphic function and let $\alpha_1(z),\cdots,\alpha_4(z)$, $\beta_1(z)$, $\cdots$, $\beta_4(z)$, $\gamma_1(z),\cdots,\gamma_4(z)$, $\delta_1(z),\cdots,\delta_4(z)$, $\lambda_1(z)$ and $\lambda_0(z)$ be small functions with respect to $f(z)$, where $\lambda_1(z)^2-4\lambda_0(z)\not\equiv 0$, $\alpha_4(z)^2-\lambda_1(z)\alpha_3(z)\alpha_4(z)+\lambda_0(z)\alpha_3(z)^2\not\equiv 0$, $\beta_4(z)^2-\lambda_1(z)\beta_3(z)\beta_4(z)+\lambda_0(z)\beta_3(z)^2\not\equiv 0$, $\gamma_4(z)^2-\lambda_1(z)\gamma_3(z)\gamma_4(z)+\lambda_0(z)\gamma_3(z)^2\not\equiv 0$, $\delta_4(z)^2-\lambda_1(z)\delta_3(z)\delta_4(z)+\lambda_0(z)\delta_3(z)^2\not\equiv 0$.

Let z_0 be a simple pole of $f(z)$. We call z_0 *strongly representable in the second kind sense by* $\alpha_1(z),\cdots,\alpha_4(z)$, $\beta_1(z),\cdots,\beta_4(z)$, $\gamma_1(z),\cdots,\gamma_4(z)$, $\delta_1(z),\cdots,\delta_4(z)$, $\lambda_1(z)$ *and* $\lambda_0(z)$, if $f(z)$ is written in the neighbourhood of z_0 as:

$$f(z)=\frac{R}{z-z_0}+\alpha+\beta(z-z_0)+\gamma(z-z_0)^2+\delta(z-z_0)^3+O(z-z_0)^4 \tag{2.2}$$

and

$$R^2+\lambda_1(z_0)R+\lambda_0(z_0)=0\,, \tag{2.3}$$

$$\begin{aligned} \alpha&=\frac{\alpha_1(z_0)R+\alpha_2(z_0)}{\alpha_3(z_0)R+\alpha_4(z_0)}\,, &\quad \beta&=\frac{\beta_1(z_0)R+\beta_2(z_0)}{\beta_3(z_0)R+\beta_4(z_0)}\,,\\ \gamma&=\frac{\gamma_1(z_0)R+\gamma_2(z_0)}{\gamma_3(z_0)R+\gamma_4(z_0)}\,, &\quad \delta&=\frac{\delta_1(z_0)R+\delta_2(z_0)}{\delta_3(z_0)R+\delta_4(z_0)}\,. \end{aligned} \tag{2.4}$$

For the sake of brevity, we call such simple pole, *SS2-kind pole.*

LEMMA 2. *Let $w(z)$ be a transcendental meromorphic function and let $\alpha_1(z), \cdots, \alpha_4(z), \beta_1(z), \cdots, \beta_4(z), \gamma_1(z), \cdots, \gamma_4(z), \delta_1(z), \cdots, \delta_4(z), \lambda_1(z)$ and $\lambda_0(z)$ be small functions with respect to $w(z)$. We denote by $n_{\langle SS2\rangle}(r, w)$ the number of the SS2-kind poles in $|z|\leqq r$. $N_{\langle SS2\rangle}(r, w)$ is defined in terms of $n_{\langle SS2\rangle}(r, w)$ in the usual way. If*

$$m(r, w)+(N(r, w)-N_{\langle SS2\rangle}(r, w))=S(r, w)\,, \tag{2.5}$$

then $w(z)$ satisfies a first order differential equation of the form (2.1).

The proof of Lemma 2 is given in [4].

LEMMA 3. *Suppose the differential equation*

$$u'(u'+\eta(z)u^2)^2=b_1(z)u^5+\cdots+b_5(z)u+b_6(z) \tag{2.6}$$

possesses an admissible solution $u(z)$. If $u(z)$ satisfies

$$N_1(r, u)+m(r, u)=S(r, u)\,, \tag{2.7}$$

then $u(z)$ satisfies the Riccati equation or an equation of the form (2.1).

PROOF. We write (2.6),

$$U(z, u, u')^2=V(z, u, u')\,, \tag{2.8}$$

where

$$U(z, u, u')=u'+\eta(z)u^2\,,\quad V(z, u, u')=[b_1(z)u^5+\cdots+b_5(z)u+b_6(z)]/u'\,.$$

Let $\mathscr{M}$ be the collection of coefficients of $U(z, u, u')$ and $V(z, u, u')$. Let z_0 be an admissible (w.r.t. $\mathscr{M}$) simple pole of u, and write in the neighbourhood of z_0,

$$u(z)=\frac{R}{z-z_0}+\alpha+O(z-z_0)\,. \tag{2.9}$$

Since the order of pole of $V(z)=V(z, u(z), u'(z))$ at z_0 is at most three, by (2.8), the order of pole of $U(z)=U(z, u(z), u'(z))$ at z_0 is at most one. Thus we have $-R+\eta(z_0)R^2=0$. Hence R is written by small function, that is, $R=1/\eta(z_0)$. For the sake of brevity, we put $R(z)=1/\eta(z)$ in this proof.

First we treat the case $N(r, U)=S(r, u)$. From (2.7), we have

$$m(r, U)\leq m(r, u'/u)+m(r, u)+m(r, u^2)+S(r, u)$$
$$\leq 3m(r, u)+S(r, u)\leq S(r, u)\,.$$

Hence $U(z)$ is a small function with respect to $u(z)$. Therefore $u(z)$ satisfies the Riccati equation in this case.

Secondly we treat the case $N(r, U)\neq S(r, u)$. We show that almost all admissible poles of $u(z)$ are simple poles of $U(z)$. By (2.7), we have to consider merely simple poles of $u(z)$.

We denote by $n^*(r, u)$ the number of admissible simple poles of $u(z)$ in $|z|\leqq r$ which are regular point of $U(z)$. $N^*(r, u)$ is defined in the usual way. Suppose $N^*(r, u)\neq S(r, u)$. There exists an admissible simple pole z_1 of $u(z)$, which is a regular point of $U(z)$. The order of pole of left-hand side of (2.6) at z_1 is at most two. If $|b_1|+|b_2|+|b_3|\not\equiv 0$, then by the definition of admissible pole, the order of pole of right-hand side of (2.6) at z_1 is at least three, which is a contradiction. Thus $b_1(z)=b_2(z)=b_3(z)\equiv 0$ in (2.6). Hence, by (2.6) $N(r, U)=S(r, u)$, which is a contradiction. Therefore, $N^*(r, u)=S(r, u)$ which implies that almost all admissible simple poles of $u(z)$ are simple poles of $U(z)$.

Let z_0 be an admissible simple pole of $u(z)$ and simple pole of $U(z)$. The order of pole of left-hand side of (2.6) at z_0 is four. If $b_1(z)\not\equiv 0$, then by the definition of admissible pole, the order of pole of right-hand side of (2.6) at z_0 is five, which is a contradiction. Thus $b_1(z)\equiv 0$, and from the above estimation, we have $b_2(z)\not\equiv 0$. By simple calculation in the neighbourhood of z_0,

$$V(z)=\frac{q(z_0)}{(z-z_0)^2}+\frac{p_1(z_0)+p_2(z_0)\alpha}{z-z_0}+O(1)\,,$$

$$\frac{q(z)}{R(z)}u'(z)=-\frac{q(z_0)}{(z-z_0)^2}+\frac{p_3(z_0)}{z-z_0}+O(1)\,,$$

where $q(z)=-b_2(z)R(z)^3$, $p_1(z)=-b_2'(z)R(z)^3-b_3(z)R(z)^2$, $p_2(z)=-4b_2(z)R(z)^2$ and $p_3(z)=-(q'(z)R(z)-q(z)R'(z))/R(z)$.

Hence near z_0

$$V(z)+\frac{q(z)}{R(z)}u'-\frac{p_1(z)+p_3(z)}{R(z)}u=\frac{p_2(z_0)\alpha}{z-z_0}+O(1)\,. \tag{2.10}$$

We have

$$U(z)+\frac{R'(z)}{R(z)}u=\frac{2\alpha}{z-z_0}+O(1)\,. \tag{2.11}$$

From (2.10) and (2.11), put

$$\varphi(z)=2\left[V(z, u, u')+\frac{q(z)}{R(z)}u'-\frac{p_1(z)+p_3(z)}{R(z)}u\right] \tag{2.12}$$
$$-p_2(z)\left[U(z, u, u')+\frac{R'(z)}{R(z)}u\right],$$

then $\varphi(z)$ is regular at z_0.

By (2.8), $V(z)$ is regular at zero of $u'(z)$. Thus, we have $N(r, \varphi)=S(r, u)$. From (2.7) and (2.8),

$$m(r, \varphi)\leq m(r, V)+m(r, U)+4m(r, u)+S(r, u)$$
$$\leq 3m(r, U)+4m(r, u)+S(r, u)\leq S(r, u).$$

Hence $\varphi(z)$ is a small function with respect to $u(z)$. From (2.8) and (2.12), $u(z)$ satisfies an equation of the form (2.1). Q.E.D.

3. Proof of Theorem 1.

Put $u=1/(w-\eta(z))$ in (1.4). Then by simple calculation (see [3])

$$\beta_1(z)u'u^4+\beta_2(z)u'^2u^2+\beta_3(z)u'^3 \tag{3.1}$$
$$=\Phi(z)u^6+b_1(z)u^5+\cdots+b_5(z)u+b_6(z),$$

where

$$\beta_k(z)=(-1)^k\sum_{j=k}^{3}\binom{j}{k}\alpha_j(z)\eta'(z)^{j-k}, \qquad \alpha_3(z)\equiv 1, \quad k=0, 1, 2, 3,$$

$$b_i(z)=\sum_{j=i}^{6}\binom{j}{i}a_j(z)\eta(z)^{j-i}, \qquad i=0, 1, \cdots, 6,$$

$$\Phi(z)=b_0(z)-\beta_0(z)=\sum_{j=0}^{6}a_j(z)\eta(z)^j-\sum_{j=0}^{3}\alpha_j(z)\eta'(z)^j.$$

We assume that $\eta(z)$ is a small solution of (1.4). Thus we have $\Phi(z)\equiv 0$ in (3.1). For the proof of Theorem 1, we show that $w(z)$ satisfies (2.1) under the condition that $\eta(z)$ is neither deficient nor ramified small function w.r.t. $w(z)$, that is

$$m(r, u)+N_1(r, u)=S(r, u). \tag{3.2}$$

Let z_0 be an admissible simple pole of $u(z)$. Write $u(z)$ near z_0 as:

$$u(z)=\frac{R}{z-z_0}+\alpha+\beta(z-z_0)+\gamma(z-z_0)^2+\delta(z-z_0)^3+O(z-z_0)^4. \tag{3.3}$$

From (3.1) and (3.3), since $\Phi(z)\equiv 0$,

$$\beta_1(z_0)R^2-\beta_2(z_0)R+\beta_3(z_0)=0\,, \tag{3.4}$$

$$[4\beta_1(z_0)R-2\beta_2(z_0)]\alpha=P_1(R;\,z_0)\,, \tag{3.5}$$

$$[3\beta_1(z_0)R^2-3\beta_3(z_0)]\beta=P_2(R,\,\alpha;\,z_0)\,, \tag{3.6}$$

$$[2\beta_1(z_0)R^2+2\beta_2(z_0)R-6\beta_3(z_0)]\gamma=P_3(R,\,\alpha,\,\beta;\,z_0)\,, \tag{3.7}$$

$$[-\beta_1(z_0)R^4-4\beta_2(z_0)R^3+9\beta_3(z_0)R^2]\delta=P_4(R,\,\alpha,\,\beta,\,\gamma;\,z_0)\,, \tag{3.8}$$

where $P_j(\,\cdot\,;\,z_0)$ $(j=1, 2, 3, 4)$ are polynomials of corresponding arguments with small coefficients.

Since $|a_6|+|a_5|+|a_4|+|a_3|\not\equiv 0$, the right-hand side of (3.1) does not vanish. Thus we have

$$|\beta_1|+|\beta_2|+|\beta_3|\not\equiv 0\,. \tag{3.9}$$

First we treat the case $\beta_1(z)\equiv 0$ or $\beta_3(z)\equiv 0$.

If $\beta_1(z)\equiv 0$, then we have $\beta_2(z)\not\equiv 0$ and $\beta_3(z)\not\equiv 0$. For, if $\beta_2(z)\equiv 0$ $(\beta_3(z)\equiv 0)$, then by (3.4) $\beta_3(z_0)=0$ $(\beta_2(z_0)=0)$. By Remark 1, we have $\beta_3(z)\equiv 0$ $(\beta_2(z)\equiv 0)$, which contradicts (3.9). Hence by (3.4) and (3.5), R and α are written by small functions, which implies that $u(z)$ satisfies the Riccati equation (see [9], pp. 47–48).

Similarly to the case $\beta_1(z)\equiv 0$, if $\beta_3(z)\equiv 0$, then $\beta_1(z)\not\equiv 0$ and $\beta_2(z)\not\equiv 0$, and we obtain that $u(z)$ satisfies the Riccati equation.

Secondly we treat the case $\beta_1(z)\not\equiv 0$ and $\beta_3(z)\not\equiv 0$.

If $(-\beta_2(z)/\beta_1(z))^2-4(\beta_3(z)/\beta_1(z))\equiv 0$, that is, $\beta_2(z)^2-4\beta_1(z)\beta_3(z)\equiv 0$, then the form of (3.1) is of the form (2.6). Thus by Lemma 3, $u(z)$ satisfies the Riccati equation or an equation of the form (2.1).

Hence, in the below, we assume that $\beta_2(z)^2-4\beta_1(z)\beta_3(z)\not\equiv 0$.

If any one of α, β, γ and δ is not written by the linear transformations of R with small (w.r.t. $u(z)$) coefficients, that is, if $4\beta_1(z_0)R-2\beta_2(z_0)=0$, $3\beta_1(z_0)R^2-3\beta_3(z_0)=0$, $2\beta_1(z_0)R^2+2\beta_2(z_0)R-6\beta_3(z_0)=0$ or $-\beta_1(z_0)R^4-4\beta_2(z_0)R^3+9\beta_3(z_0)R^2$ in (3.5)–(3.8), then by (3.4), $\beta_2(z_0)^2-4\beta_1(z_0)\beta_3(z_0)=0$ for each case. Hence by Remark 1, $\beta_2(z)^2-4\beta_1(z)\beta_3(z)\equiv 0$, which contradicts our assumption.

Here we have that for any admissible simple poles z_0, α, β, γ and δ are written by linear transformations of R with coefficients of small (w.r.t. $u(z)$) functions. Thus, almost all admissible poles are SS2-kind poles. Hence by (3.2) and Lemma 2, $u(z)$ satisfies the Riccati equation or a differential equation of the form (2.1). Thus by Remark 2, $w(z)$ satisfies a differential equation of the form (2.1). Therefore by Lemma 1, (1.4) is reducible, which implies that Theorem 1 is proved.

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