TOKYO J. MATH.
VOL. 15, NO. 1, 1992

Coboundaries under Integrable Exponentiation

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(Communicated by T.Nagano)

Abstract. It is known that if X is a Lebesgue probability space, $T: X \to X$ an ergodic measure preserving automorphism, and n a fixed nonzero integer, then a coboundary for the automorphism T^n is also a coboundary for T. In this paper, the result is extended to include the case where the exponent $n = \mathrm{m}(x)$ is an arbitrary integrable integer valued function on X.

1. Introduction.

For an invertible measure preserving transformation $T: X \to X$, a real-valued measurable function f of the form $f(x) = g(x) - g(Tx)$ is called a coboundary for T. The measurable function g is called a transfer function. It is well known that if $f(x) = g(x) - g(T^n x)$, then $f(x) = K(x) - K(Tx)$ where $K(x) = \sum_{i=0}^{n-1} g(T^i x)$ if $n > 0$ and $K(x) = -\sum_{i=1}^{n} g(T^{-i} x)$ if $n < 0$. That is, a coboundary for T^n is also a coboundary for T. A natural question one asks is what happens when the exponent is not a constant. More precisely, if S is an automorphism with $Sx = T^{m(x)} x$ for some measurable integer valued function m defined on X and $f(x) = g(x) - g(Sx)$, is f a coboundary for T? In this paper we show that the result is true for integrable exponents.

Let T be an invertible ergodic measure preserving transformation on a Lebesgue probability space $(X, \mathscr{B}, \mu)$. Let $[T]$ denote the full group of T. That is, $[T]$ consists of all invertible bimeasurable transformations S on X for which there exists an integer valued measurable function m on X such that for every $x \in X$, $Sx = T^{m(x)} x$.

DEFINITION 1.1. If $f: X \to \boldsymbol{R}$ is a measurable function, the T cocycle f^* of f is the measurable function defined on $Z \times X$ as follows:

$$f^*(n, x) = \begin{cases} \sum_{i=0}^{n-1} f(T^i x) & \text{if } n > 0 \\ -f(-n, T^n x) & \text{if } n < 0 \end{cases}$$

and we set $f^*(0, x) = 0$. Then f^* satisfies the so called cocycle identity: $f^*(n+m, x) = f^*(n, x) + f^*(m, T^n x)$.

Received February 8, 1991

DEFINITION 1.2. A T cocycle f^* is said to be a T coboundary if there exists a measurable function $g: X \to \boldsymbol{R}$ such that

$$f^*(n, x) = g(x) - g(T^n x) \qquad \text{for all } n \in \boldsymbol{Z} \text{ and a.e. } x \in X .$$

The function g is called a transfer function of f^*. In this case we also refer to the generating function $f = f^*(1, \cdot)$ as T coboundary.

The following is a theorem due to K. Schmidt which appears as lemma 11.7 of [S] and gives necessary and sufficient conditions for a cocycle to be a coboundary.

THEOREM 1.3. *A cocycle f^* is a coboundary if and only if for every $\varepsilon > 0$, there exists a positive real number A such that for each $n \in \boldsymbol{Z}$, $|f^*(n, x)| < A$ for all x in a set E_n of measure at least $1 - \varepsilon$.*

2. The orbit equivalence case.

In this section we show that if $[S] = [T]$ and $Tx = S^{m(x)}x$ with $m(x) \in L^1(X)$, then a coboundary for S is a coboundary for T. In this case one has, by Belinskaya' theorem [B], that $\int_X m(x)d\mu(x) = \mp 1$ and either $m(x) - 1 = l(x) - l(Tx)$ if the integral is positive, or $m(x) + 1 = l(x) - l(T^{-1}x)$ if it is negative; see [K].

THEOREM 2.1. *Let $[S] = [T]$ with $Tx = S^{m(x)}x$ and $m(x) \in L^1(X)$. If f is an S coboundary, then f is a T coboundary.*

PROOF. Let $f(x) = g(x) - g(Sx)$ be an S coboundary. We shall assume that $\int_X m(x)d\mu(x) = 1$ (the other case is proved similarly). Then by the above discussion $m(x) - 1 = l(x) - l(Tx)$. If $x \in X$, then for any nonzero integer n,

$$T^n x = S^{m^*(n,x)}x = S^{n + l(x) - l(T^n x)}x$$

where m^* denotes the T cocycle of m as defined in 1.1.

Let $0 < \varepsilon < 1/4$ be given. Choose $A > 0$ a sufficiently large integer so that the set

$$C = \left\{ x : |l(x)| < A \text{ and } \left| \frac{1}{n}(l(x) - l(T^n x)) \right| < \varepsilon \text{ for all } |n| \geq A \right\}$$

satisfies $\mu(C) > 1 - \varepsilon/16$. Applying the ergodic theorem to χ_C we can find an integer $B \geq A$ such that the set

$$D = \left\{ x \in C : \left| \frac{1}{n} \sum_{i=0}^{n-1} \chi_C(T^i x) - \mu(C) \right| < \varepsilon/2 \text{ for all } n \geq B \right\}$$

has measure greater than $1 - \varepsilon/8$.

We want to show that for each integer n the T cocycle f^* of f satisfies the hypothesis of theorem 1.2. This will only be done for positive integers since the cocycle identity or a similar proof gives the other case. Let $n > 10^{10}B$, since T is measure preserving

$\mu(D\cap T^{-n}C)>1-\varepsilon/4$. If $x\in D\cap T^{-n}C$, then at least $(1-\varepsilon)n$ of the points $\{x, Tx, \cdots, T^{n-1}x\}$ lie in C. We first show that $\{T^{2A}x, \cdots, T^{n-4A}x\}\subseteq\{x, Sx, \cdots, S^{n-1}x\}$. To this end, if $T^ix\in C$ and $2A\leq i\leq n-4A$, then $|l(x)|, |l(T^ix)|<A$ which implies $i-2A\leq l(x)-l(T^ix)+i\leq 2A+i$ so that $T^ix\in\{x, Sx, \cdots, S^{n-1}x\}$. Now let $r<\varepsilon n$ and $i_1, i_2, \cdots, i_r$ the indices for which $T^{i_k}x\notin C$ and $2A\leq i_k\leq n-4A$. If $2A\leq i_k\leq n(1-\varepsilon)$, then since $x\in C$ we have $(1/i_k)|l(x)-l(T^{i_k}x)|\leq\varepsilon$, so that

$$0<i_k(1-\varepsilon)<l(x)-l(T^{i_k}x)+i_k<i_k(1+\varepsilon)\leq n(1-\varepsilon^2)<n\,,$$

hence $T^{i_k}x\in\{x, Sx, \cdots, S^{n-1}x\}$. On the other hand, if $n(1-\varepsilon)<i_k\leq n-4A$, then since $T^nx\in C$ and $i_k-n<-A$, we have $|l(T^nx)-l(T^{i_k}x)|<(n-i_k)\varepsilon$. Then

$$|l(x)-l(T^{i_k}x)|\leq|l(x)-l(T^nx)|+|l(T^nx)-l(T^{i_k}x)|<2A+(n-i_k)\varepsilon\,.$$

Now,

$$\begin{aligned} l(x)-l(T^{i_k}x)+i_k&<2A+(n-i_k)\varepsilon+i_k\\ &=2A+n\varepsilon+i_k(1-\varepsilon)\\ &\leq 2A+n\varepsilon+(n-4A)(1-\varepsilon)\\ &=2A(2\varepsilon-1)+n<n\,, \end{aligned}$$

and

$$\begin{aligned} l(x)-l(T^{i_k}x)+i_k&>-2A-(n-i_k)\varepsilon+i_k\\ &>-2A-n\varepsilon+n-n\varepsilon^2>0 \end{aligned}$$

if n is chosen sufficiently large. Hence, $T^{i_k}x\in\{x, Sx, \cdots, S^{n-1}x\}$ and therefore $x\in D\cap T^{-n}C$,

$$\{T^{2A}x, \cdots, T^{n-4A}x\}\subseteq\{x, Sx, \cdots, S^{n-1}x\}\,. \tag{$*$}$$

We now show that for $x\in D\cap T^{-n}C$, $\{S^{4A}x, \cdots, S^{n-4A}x\}\subseteq\{T^{-A}x, \cdots, T^{n+A}x\}$. So let $4A\leq j<n-4A$, since $[S]=[T]$ it follows that there exists an integer i such that

$$S^jx=T^ix=S^{l(x)-l(T^ix)+i}x\,.$$

This implies that

$$4A\leq j=l(x)-l(T^ix)+i\leq n-4A\,. \tag{1}$$

We show that (1) implies that $i\in\{-A, \cdots, n+A\}$. For this we consider two cases:
Case 1. If $T^ix\in C$, then $|l(T^ix)|<A$ and by (1)

$$-A<2A<4A-(l(x)-l(T^ix))\leq i\leq n-4A-(l(x)-l(T^ix))<n-2A<n+A\,.$$

Case 2. Let $T^ix\notin C$. If $i>n+A$, then $i-n>A$ and $T^nx\in C$ implies that

$$|l(T^ix)-l(T^nx)|<\varepsilon(i-n)$$

so that

$$|l(x)-l(T^i x)| \leq |l(x)-l(T^n x)| + |l(T^n x)-l(T^i x)| < 2A+(i-n)\varepsilon .$$

Thus by (1) we have

$$i \leq n-4A-(l(x)-l(T^i x)) < n-2A+\varepsilon i-\varepsilon n$$

and hence $(1-\varepsilon)i<n(1-\varepsilon)-2A$ which implies that $i<n-2A/(1-\varepsilon)<n+A$, a contradiction. On the other hand, if $i<-A$, then $|l(x)-l(T^i x)|<\varepsilon|i|$ and so by (1)

$$i \geq 4A-(l(x)-l(T^i x)) > 4A+\varepsilon i .$$

This gives that $i(1-\varepsilon)>4A$ or $i>4A/(1-\varepsilon)>0$, which is a contradiction. Thus, for all $x \in D \cap T^{-n}C$

$$\{S^{4A}x, \cdots, S^{n-4A}x\} \subseteq \{T^{-A}x, \cdots, T^{n+A}x\} . \tag{**}$$

For $x \in D \cap T^{-n}C$, let

$$\alpha(x)=\{-A \leq i \leq n+A : T^i x \notin \{S^{4A}x, \cdots, S^{n-4A}x\}\}$$

and

$$\gamma(x)=\alpha(x) \cap \{2A, \cdots, n-4A\} .$$

Then by (**) we have

$$\{T^i x : -A \leq i \leq n+A\} = \{S^i x : 4A \leq i \leq n-4A\} \cup \{T^i x : i \in \alpha(x)\} .$$

Also, if $i \in \gamma(x)$, then by (*) $T^i x \in \{x, Sx, \cdots, S^{4A-1}x\} \cup \{S^{n-4A}x, \cdots, S^{n-1}x\}$ so that

$$f(T^i x) = g(S^{j(i)}x) - g(S^{j(i)+1}x)$$

for some $j(i) \in \{0, 1, \cdots, 4A-1\} \cup \{n-4A, \cdots, n-1\}$. If $i \in \alpha(x) \setminus \gamma(x)$, then

$$f(T^i x) = g(T^i x) - g(ST^i x)$$

for some $i \in \{-A, \cdots, 2A-1\} \cup \{n-4A+1, \cdots, n+A\}$. Then

$$\begin{aligned}\sum_{i=-A}^{n+A} f(T^i x) &= \sum_{i=4A}^{n-4A} f(S^i x) + \sum_{i \in \gamma(x)} f(T^i x) + \sum_{i \in \alpha(x) \setminus \gamma(x)} f(T^i x) \\ &= g(S^{4A}x) - g(S^{n-4A}x) + \sum_{i \in \gamma(x)} g(S^{j(i)}x) - g(S^{j(i)+1}x) \\ &\quad + \sum_{i \in \alpha(x) \setminus \gamma(x)} g(T^i x) - g(ST^i x) .\end{aligned} \tag{***}$$

Choose $\beta>0$ sufficiently large so that the set $F=\{x \in X : |g(x)|<\beta\}$ has measure greater than $1-\varepsilon/100A^2$. Since S is measure preserving $\mu(S^{-1}F)>1-\varepsilon/100A^2$. Let $F'=F \cap S^{-1}F$ and

$$F'' = \bigcap_{i=-A}^{2A} T^{-i}F' \cap \bigcap_{i=n-4A}^{n-1} T^{-i}F' .$$

Then $\mu(F'') > 1 - 7\varepsilon/50A$. Let

$$E_n = F'' \cap \bigcap_{i=0}^{4A} S^{-i}F \cap \bigcap_{i=n-4A}^{n-1} S^{-i}F \cap T^{-n}C \cap D .$$

Then $\mu(E_n) > 1 - \varepsilon$ and for $x \in E_n$ by (*), (**), and (***) we have

$$\begin{aligned}
|f^*(n, x)| &= \left| \sum_{i=0}^{n-1} f(T^i x) \right| = \left| \sum_{i=-A}^{n+A} f(T^i x) - \sum_{i=-A}^{-1} f(T^i x) - \sum_{i=n}^{n+A} f(T^i x) \right| \\
&\leq \left| \sum_{i=-A}^{n+A} f(T^i x) \right| + \left| \sum_{i=-A}^{-1} f(T^i x) \right| + \left| \sum_{i=n}^{n+A} f(T^i x) \right| \\
&\leq |g(S^{4A}x)| + |g(S^{n-4A}x)| + \sum_{i=0}^{4A-1} |g(S^i x)| + |g(S^{i+1}x)| \\
&\quad + \sum_{i=n-4A}^{n-1} |g(S^i x)| + |g(S^{i+1}x)| + 2 \sum_{i=-A}^{2A-1} |g(T^i x)| + |g(ST^i x)| \\
&\quad + 2 \sum_{i=n-4A+1}^{n+A} |g(T^i x)| + |g(ST^i x)| \\
&< 2\beta + 8A\beta + 8A\beta + 12A\beta + 20A\beta = 50A\beta .
\end{aligned}$$

Therefore, by theorem 1.2 f is a T coboundary. □

REMARK 2.2. The assumption in theorem 2.1 that the exponent $m(x)$ is integrable can be replaced by the weaker assumption that the cardinality of one of the sets $\{S^i x\}_{i \in N} \Delta \{T^i\}_{i \in N}$ or $\{S^i x\}_{i \in N} \Delta \{T^{-i}\}_{i \in N}$ is finite a.e.; for this implies that either $m(x) - 1 = l(x) - l(Tx)$ or $m(x) + 1 = l(x) - l(T^{-1}x)$ (see [K]).

3. The general integrable case.

Let $S \in [T]$ with $Sx = T^{m(x)}x$ where $m(x) \in L^1(X)$. We first outline a technique employed by Katznelson in [K] to reduce the verification to the case $m(x) > 0$. In [K] it was shown that $\int_X m(x) d\mu(x) = p$ is an integer (which we will assume with no loss of generality that it is positive, otherwise the result will be proved for S and T^{-1} and hence for S and T); this says that p is the number of S orbits contained in a T orbit. Also, if for each $x \in X$ one considers $q(x)$ to be the smallest positive integer such that $S_1 x = T^{q(x)}x \in \{S^i x\}_{i \in Z}$, then one can easily show that $[S_1] = [S]$ and that the cardinality of the set $\{S_1{}^j\}_{j \in N} \Delta \{S^j x\}_{j \in N}$ is finite a.e. Thus by theorem 2.1 and remark 2.2, any coboundary for S is a coboundary for S_1.

THEOREM 3.1. *Let $S\in[T]$ with $Sx=T^{m(x)}x$. If $m(x)\in L^1(X)$, then any S coboundary is a T coboundary.*

PROOF. By the above discussion we shall assume with no loss of generality that $m(x)>0$ a.e. For $x\in X$ set $r_0(x)=s_0(x)=0$. For $1\le i\le p-1$, let $r_i(x)$ be the smallest positive integer such that

$$T^{r_i(x)}x\in\{T^jx\}_{j\in N}\setminus\bigcup_{k\le i-1}\{S^jT^{r_k(x)}x\}_{j\in N},$$

and let $s_i(x)$ be the largest negative integer such that

$$T^{s_i(x)}x\in\{T^{-j}x\}_{j\in N}\setminus\bigcup_{k\le i-1}\{S^{-j}T^{s_k(x)}x\}_{j\in N}.$$

Then $s_{p-1}(x)<\cdots<s_1(x)<0<r_1(x)<\cdots<r_{p-1}(x)$.

Let $\varepsilon>0$ be given, choose a sufficiently large integer $N>0$ so that each one of the sets $A=\{x\in X: r_{p-1}(x)\le N\}$ and $B=\{x\in X: s_{p-1}(x)\ge -N\}$ has measure greater than $1-\varepsilon/4$. Let $|n|>10^{10}N$ be a sufficiently large integer and let $f(x)=g(x)-g(Sx)$ be a coboundary for S. Consider the T cocycle $f^*(n,x)$ of f, we shall assume without loss of generality that $n>0$ since the other case is proved similarly. Since $m(x)>0$, for each $x\in A\cap T^{-(n-1)}B$ we have

$$\{T^ix: 0\le i\le n-1\}\Delta\{ST^ix: 0\le i\le n-1\}=\{T^{r_i(x)}x,\ ST^{n-1+s_i(T^{n-1}x)}x: 0\le i\le p-1\}$$

so that

$$f^*(n,x)=\sum_{i=0}^{n-1}g(T^ix)-g(ST^ix)=\sum_{i=0}^{p-1}g(T^{r_i(x)}x)-g(ST^{n-1+s_i(T^{n-1}x)}x)$$

and hence

$$|f^*(n,x)|\le\sum_{i=0}^{r_{p-1}(x)}|g(T^ix)|+\sum_{i=n+s_{p-1}(T^{n-1}x)-1}^{n-1}|g(ST^ix)|.$$

Now choose $M>0$ a sufficiently large real number so that the set $C=\{x: |g(x)|<M/2(N+1)\}$ has measure greater than $1-\varepsilon/4(N+1)$. Let $E_n=A\cap T^{-(n-1)}B\cap\bigcap_{i=0}^{N}T^{-i}C\cap\bigcap_{i=n-N-1}^{n-1}T^{-i}S^{-1}C$, then $\mu(E_n)>1-\varepsilon$ and for every $x\in E_n$ we have $|f^*(n,x)|<M$. Thus f^* is a T coboundary. □

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