TOKYO J. MATH.
VOL. 17, NO. 2, 1994

Determinant Surfaces of Rank 2 Bundles on P^3

Shigeharu TAKAYAMA

Tokyo Metropolitan University
(Communicated by M. Sakai)

§1. Introduction.

The aim of this paper is to study the relationship between stable vector bundles $\mathscr{E}$ of rank two on P^3 and their determinant surfaces S determined by two sections of $\mathscr{E}$. We discuss specifically the case $c_1(\mathscr{E})=4$ in detail.

Vector bundles on a variety are closely related to its special subvarieties. On P^3, a general surface has Picard number one by the Noether-Lefschetz theorem (cf. [Lo]): If S is a general surface of degree $d\geq 4$ in P^3, then $\mathrm{Pic}\, S\cong Z$ with the generator $\mathcal{O}_S(1)$. On the other hand, a smooth determinant surface S is not general because its Picard number is at least two by Theorem 3.1:

THEOREM 1.1. *A smooth surface S in P^3 occurs as a determinant surface of a rank two vector bundle $\mathscr{E}$ on P^3 if and only if S has a surjective morphism onto P^1.*

In this paper we give an estimate of $\rho(S)$ from below in terms of the behaviour of $\mathscr{E}$ under the restriction to lines and planes. Defining the jumping planes in (5.5), we can state a sufficient condition for S to have Picard number ≥ 3 in (5.6). Moreover we have the following estimate:

THEOREM 1.2. *Let $\mathscr{E}$ be a stable vector bundle of rank two on P^3 with $c_1(\mathscr{E})=4$ and $c_2(\mathscr{E})\geq 9$. Suppose that $\mathscr{E}$ has a smooth determinant surface S and that $c_2(\mathscr{E})/(h^1(\mathscr{E}(-4))+1)=$(degree of a fibre of the Stein factorization of the morphism $S\to P^1$ as in Theorem 1.1)≥ 4. Then*

$$\rho(S)\geq 2+\tfrac{1}{2}\# J(\mathscr{E}),$$

where $\# J(\mathscr{E})$ is the number of jumping planes for $\mathscr{E}$.

As a corollary of these theorems and (2.13), we have:

COROLLARY 1.3. *For any given $c_2\geq 5$, there exists a stable vector bundle $\mathscr{E}$ of rank*

Received February 9, 1993
Revised July 6, 1993

two with $c_1(\mathscr{E})=0$ and $c_2(\mathscr{E})=c_2$ on $\boldsymbol{P}^3$ such that the restriction $\mathscr{E}|_H$ is stable on any plane H in $\boldsymbol{P}^3$.

If $c_1(\mathscr{E})=0$ and $c_2(\mathscr{E})=2$, there exists a plane H in $\boldsymbol{P}^3$ such that the restriction $\mathscr{E}|_H$ is not stable on H [Ha 2, Proposition 9.10], and for any *null-correlation bundle* $\mathscr{N}$ (a stable rank 2 bundle with $c_1(\mathscr{N})=0$ and $c_2(\mathscr{N})=1$), $\mathscr{N}|_H$ is not stable for any plane H in $\boldsymbol{P}^3$ [Ba].

Throughout the paper, we work over a complex number field $\boldsymbol{C}$, and we use the standard notation of algebraic geometry [Ha 1].

I would like to thank Professors N. Sasakura, T. Urabe and T. Sasai for many valuable comments and suggestions during the preparation of this paper. Without their encouragement this work may not have been completed. In particular, (2.13) was communicated to me by Professor Urabe.

§2. Preliminaries.

In this section we review known results about rank 2 vector bundles on $\boldsymbol{P}^3$. Let $\mathscr{E}$ be a rank 2 vector bundle on $\boldsymbol{P}=\boldsymbol{P}^3$ and let $s\in H^0(\boldsymbol{P}^3, \mathscr{E})$ be a section whose scheme of zeros has codimension 2, then we obtain a curve $Y=(s)_0$ (By a curve we mean a 1-dimensional closed subscheme of $\boldsymbol{P}^3$.). In this case we say that the bundle $\mathscr{E}$ corresponds to the cuve Y. For any curve $Y\subset\boldsymbol{P}^3$, let $\omega_Y=\mathscr{E}xt^2_{\boldsymbol{P}}(\mathcal{O}_Y, \omega_{\boldsymbol{P}})$ denote its dualizing sheaf. The following proposition is well known and one of the foundation of our theory.

PROPOSITION 2.1 (Serre [Ha 2, Theorem 1.1]). *A curve Y in $\boldsymbol{P}^3$ occurs as the scheme of zeros of a section of a rank 2 vector bundle $\mathscr{E}$ on $\boldsymbol{P}^3$ if and only if Y is a local complete intersection and ω_Y is isomorphic to the restriction to Y of some invertible sheaf on $\boldsymbol{P}^3$.*

COROLLARY 2.2 ([Ha 2, Corollary 1.2]). *If a bundle $\mathscr{E}$ corresponds to a curve Y, then Y is a complete intersection if an only if $\mathscr{E}$ is a direct sum of line bundle.*

PROPOSITION 2.3 ([Ha 2, Proposition 2.1]). *Let $\mathscr{E}$ correspond to a curve Y, and let Y have degree d and arithmetic genus p_a. Then $d=c_2$ and $2p_a-2=c_2(c_1-4)$.*

DEFINITION 2.4. A vector bundle $\mathscr{E}$ of rank 2 on $\boldsymbol{P}^n$ $(n\geq 2)$ is *stable* (respectively, *semistable*) if for every invertible subsheaf $\mathscr{L}$ of $\mathscr{E}$,

$$c_1(\mathscr{L})<\tfrac{1}{2}c_1(\mathscr{E}) \qquad \text{(respectively, } \leq\text{)}\,.$$

REMARK 2.5. (1) Let $c_1(\mathscr{E})$ (respectively, $c_2(\mathscr{E})$) denote the first (respectively, the second) Chern class of $\mathscr{E}$, a rank 2 vector bundle on $\boldsymbol{P}^3$. Since the Chow ring of $\boldsymbol{P}^3$ is isomorphic to $\boldsymbol{Z}[h]/h^4$, we will regard c_1 and c_2 as integers. From the general theory it follows that $\bigwedge^2\mathscr{E}=\mathcal{O}(c_1(\mathscr{E}))$, $c_1(\mathscr{E}(m))=c_1(\mathscr{E})+2m$ and $c_2(\mathscr{E}(m))=c_2(\mathscr{E})+mc_1(\mathscr{E})+m^2$ for any $m\in\boldsymbol{Z}$. Since $\mathscr{E}$ has rank 2, the natural map $\mathscr{E}\otimes\mathscr{E}\to\bigwedge^2\mathscr{E}$ is a perfect pairing, whence $\mathscr{E}^{\vee} \cong \mathscr{E}(-c_1)$.

(2) A vector bundle $\mathscr{E}$ of rank 2 on $\boldsymbol{P}^n$ $(n \geq 2)$ is stable if and only if $\mathscr{E}(m)$ is stable for any $m \in \boldsymbol{Z}$. Since twisting a rank 2 bundle by m changes its first Chern class by $2m$, we can twist any bundle so that its first Chern class becomes 0 or -1. In this case we will say that $\mathscr{E}$ is *normalized.* If $\mathscr{E}$ is normalised, then $\mathscr{E}$ is stable if and only if $H^0(\mathscr{E})=0$. In case $c_1=0$, $\mathscr{E}$ is semistable if and only if $H^0(\mathscr{E}(-1))=0$.

PROPOSITION 2.6 ([Ha 2, Corollary 8.4]). *The possible values of c_1, c_2, α for a normalised stable rank 2 bundle on $\boldsymbol{P}^3$ are*

$$c_1=0, \alpha=0, c_2 \geq 1\,. \quad c_1=0, \alpha=1, c_2 \geq 3\,. \quad c_1=-1, c_2 \text{ even} \geq 2\,.$$

The following proposition gives a criterion for a bundle to be stable:

PROPOSITION 2.7 ([Ha 2, Proposition 3.1]). *Let $\mathscr{E}$ be a rank 2 bundle on $\boldsymbol{P}^3$ corresponding to a curve Y in $\boldsymbol{P}^3$. Then $\mathscr{E}$ is stable (respectively, semistable) if and only if*

(1) *$c_1(\mathscr{E})>0$ (respectively, $c_1(\mathscr{E}) \geq 0$) and*

(2) *Y is not contained in any surface of degree $\leq \frac{1}{2}c_1(\mathscr{E})$ (respectively, $<\frac{1}{2}c_1(\mathscr{E})$).*

DEFINITION 2.8. Let $\mathscr{E}$ be a stable rank 2 vector bundle on $\boldsymbol{P}^n$ $(n \geq 2)$ with $c_1(\mathscr{E})=0$. Since any vector bundle on $\boldsymbol{P}^1$ is a direct sum of line bundles and $c_1(\mathscr{E})=0$, we know that for any line L in $\boldsymbol{P}^n$, $\mathscr{E}|_L \cong \mathcal{O}_L(-a) \oplus \mathcal{O}_L(a)$ for some $a \geq 0$. By the theorem of Grauert-Mülich [Ba, Theorem 1], $\mathscr{E}|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L$ for almost all lines. The lines for which this does not hold are called *jumping lines of order a* (>0) for $\mathscr{E}$.

For the restriction $\mathscr{E}|_H$ to a plane H in $\boldsymbol{P}^3$, the following is known.

PROPOSITION 2.9 ([Ba, Theorem 3]). *Let $\mathscr{E}$ be a stable rank 2 vector bundle on $\boldsymbol{P}^3$, then for almost all planes H in $\boldsymbol{P}^3$, the restriction $\mathscr{E}|_H$ is stable, unless $=\mathscr{N}(a)$ for some $a \in \boldsymbol{N}$. ($\mathscr{N}$ denotes a null-correlation bundle.)*

DEFINITION 2.10. Let $\mathscr{E}$ be as in (2.9). The planes for which the restriction $\mathscr{E}|_H$ is not stable are called *unstable planes* for $\mathscr{E}$.

PROPOSITION 2.11 ([Ha 2, Theorem 3.3]). *Let $\mathscr{E}$ be a semistable rank 2 bundle on $\boldsymbol{P}^3$, then for almost all planes H in $\boldsymbol{P}^3$, the restriction $\mathscr{E}|_H$ is semistable. $\mathscr{N}|_H$ is semistable for any H in $\boldsymbol{P}^3$.*

We will need the following two technical propositions. Professor T. Urabe pointed out the following fact (2.13) by using the theory of period of K-3 surfaces.

PROPOSITION 2.12. *Let X be a smooth hypersurface of $d=\deg X \geq 2$ in $\boldsymbol{P}^n$ $(n \geq 3)$ and H be a hyperplane in $\boldsymbol{P}^n$. Then the hyperplane section $X \cap H$ has only isolated singularities.*

PROOF. Let f, h and g be defining polynomials of X, H and $X \cap H$ respectively. Then $f=hf_1+g$ for some f_1 of degree $d-1$. Let Σ be the singular locus of $X \cap H$. If

$\dim \Sigma \geq 1$, then $\Sigma \cap \{f_1 = 0\} \neq \emptyset$. By calculating the differential of f, we can see that X has singularities along $\Sigma \cap \{f_1 = 0\}$. Q.E.D.

PROPOSITION 2.13. *For any given $d \geq 3$, there exists an elliptic K-3 surface S of degree 4 in $\boldsymbol{P}^3$ such that $\rho(S) = 2$ and* deg(*fibre of the elliptic fibration*) $= d$.

§3. The correspondence between vector bundles and surfaces.

In this section we study a correspondence between vector bundles and surfaces induced by the determinant. Let $\mathscr{E}$ be a rank 2 vector bundle on $\boldsymbol{P} = \boldsymbol{P}^3$. Let s_1 and s_2 be nonzero global sections of $\mathscr{E}$. Assume that s_1 and s_2 are linearly independent, and that both of the scheme of zeros of s_1 and s_2 are curves. Then the scheme of zeros of $s_1 \wedge s_2 \in H^0(\boldsymbol{P}^3, \overset{2}{\wedge} \mathscr{E})$, of degree $c_1(\mathscr{E})$ in $\boldsymbol{P}^3$, is called a *determinant surface* of $\mathscr{E}$ and is denoted by $(s_1 \wedge s_2)_0$. We also say that $\mathscr{E}$ has a determinant surface S.

Our first result is to characterize the smooth surface S which occur in this way, and to show how to recover the bundle $\mathscr{E}$ from S. The following theorem is well known (cf. [Ma]), but we have to give the proof because the important thing is the relationships of bundles, sections, surfaces, morphisms and fibres.

THEOREM 3.1. *A smooth surface S in $\boldsymbol{P}^3$ occurs as a determinant surface of a rank two vector bundle $\mathscr{E}$ on $\boldsymbol{P}^3$ if and only if S has a surjective morphism to $\boldsymbol{P}^1$.*

PROOF. (1) Only if part: We can write $S = (s_1 \wedge s_2)_0$ for some $s_1, s_2 \in H^0(\boldsymbol{P}^3, \mathscr{E})$. Since S is smooth, $\{\text{support of } (s_1)_0\} \cap \{\text{support of } (s_2)_0\} = \emptyset$ and $\dim(\eta_1 s_1 + \eta_2 s_2)_0 = 1$ for any $\eta = (\eta_1 : \eta_2) \in \boldsymbol{P}^1$. By sending $(s_\eta := \eta_1 s_1 + \eta_2 s_2)_0$ to η, we obtain a projection from S onto $\boldsymbol{P}^1$.

(2) If part: By considering the Stein factorization of the projection $\varphi := S \to \boldsymbol{P}^1$, we get a surjective morphism (say, π) from S onto $\boldsymbol{P}^1$ with connected fibres. Note that the target of π is $\boldsymbol{P}^1$ since $q(S) = h^0(S, \Omega_S^1) = 0$. Let F_i $(1 \leq i \leq \lambda)$ be mutually distinct smooth fibres of π, and put $e = \deg F_1 = \deg F_i$, $g =$ genus of F_i. Let Y be the disjoint union of $F_1, F_2, \cdots, F_\lambda$. In this notation, we have:

Claim. There exists a rank 2 vector bundle $\mathscr{E}$ on $\boldsymbol{P}^3$ and a nonzero global section $s \in H^0(\boldsymbol{P}^3, \mathscr{E})$ such that $Y = (s)_0$.

Proof of the claim. By (2.1), it is sufficient to show that Y is a local complete intersection and ω_Y is isomorphic to the restriction to Y of some line bundle on $\boldsymbol{P}^3$. Y is smooth, and hence a local complete intersection. By adjunction formula $\omega_S \otimes \mathcal{O}_Y \cong \omega_Y \otimes (\mathcal{O}_S(-Y) \otimes \mathcal{O}_Y) \cong \omega_Y$ and $\omega_S \cong \omega_{\boldsymbol{P}} \otimes \mathcal{O}_{\boldsymbol{P}}(d) \otimes \mathcal{O}_S$, where $d = \deg S$. So ω_Y is isomorphic to the restriction to Y of $\mathcal{O}_{\boldsymbol{P}}(d-4)$. These imply the claim by (2.1).

Continuation of the proof. By (2.3), $c_1(\mathscr{E}) = d = \deg S$ and $c_2(\mathscr{E}) = e\lambda = \deg Y$. Applying the functor $\mathscr{H}om(\cdot, \mathcal{O}_{\boldsymbol{P}})$ to a map $\mathcal{O}_{\boldsymbol{P}} \xrightarrow{s} \mathscr{E}$, we get a map $\mathscr{E}^\vee \xrightarrow{s^\vee} \mathcal{O}_{\boldsymbol{P}}$ whose image is a sheaf of ideal $\mathscr{I}_Y$ in $\mathcal{O}_{\boldsymbol{P}}$. Since Y has codimension 2, locally the two generators of $\mathscr{I}_Y$ form a regular sequence in $\mathcal{O}_{\boldsymbol{P}}$, so the local Koszul complexes glue together to give a resolution of $\mathscr{I}_Y$:

$$0 \longrightarrow \overset{2}{\wedge}(\mathscr{E}^\vee) \longrightarrow \mathscr{E}^\vee \xrightarrow{s^\vee} \mathscr{I}_Y \longrightarrow 0\,.$$

By identifying $\mathscr{E}^\vee$ to $\mathscr{E}(-c_1(\mathscr{E}))$, we obtain:

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}}(-c_1(\mathscr{E})) \longrightarrow \mathscr{E}(-c_1(\mathscr{E})) \longrightarrow \mathscr{I}_Y \longrightarrow 0\,.$$

Moving to the long exact sequence, we get the following exact sequence

$$0 \longrightarrow H^0(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}}) \xrightarrow{\mu} H^0(\boldsymbol{P}^3, \mathscr{E}) \xrightarrow{\nu} H^0(\boldsymbol{P}^3, \mathscr{I}_Y(d)) \longrightarrow 0\,,$$

where $\mu(\cdot) = \cdot s$, $\nu(\cdot) = \cdot \wedge s$. Since Y is contained in S, there exists an $f \in H^0(\boldsymbol{P}^3, \mathscr{I}_Y(d))$ such that $S = \{f = 0\}$. Let $s_1 \in H^0(\boldsymbol{P}^3, \mathscr{E})$ be an element with $f = \nu(s_1) = s_1 \wedge s$. One knows that $S = (s_1 \wedge s)_0$ is a determinant surface. Q.E.D.

REMARK 3.2. (1) In the proof above, if we take $\lambda \in \boldsymbol{N}$ satisfying the inequality $e\lambda > d^2/2$, then $\mathscr{E}$ is stable by (2.7).

(2) If $e\lambda > d^2$, then $h^0(\boldsymbol{P}^3, \mathscr{I}_Y(d)) = 1$, so $h^0(\mathscr{E}) = 2$.

(3) $\rho(S) = $(Picard number of $S) \geq 2$.

COROLLARY 3.3. *If a bundle $\mathscr{E}$ has a smooth determinant surface S, then $\mathscr{E}$ is a direct sum of line bundles if and only if Y is a complete intersection, where Y is a general fibre of the natural projection $\varphi := S \to \boldsymbol{P}^1$ as in the proof of* (3.1).

PROOF. It follows from (2.2) and (3.1). Q.E.D.

§4. Determinant surfaces of stable bundles with $c_1(\mathscr{E}) = 4$.

Throughout this section $\mathscr{E}$ will denote a stable vector bundle of rank two on $\boldsymbol{P} = \boldsymbol{P}^3$ with $c_1(\mathscr{E}) = 4$, and assume that $\mathscr{E}$ has a smooth determinant surface $S = (s_1 \wedge s_2)_0$ for some $s_1, s_2 \in H^0(\mathscr{E})$. We denote the natural morphism $S \supset (\eta_1 s_1 + \eta_2 s_2)_0 \to \eta = (\eta_1 : \eta_2) \in \boldsymbol{P}^1$ by φ. Let $\pi : S \to \boldsymbol{P}^1$ be the Stein factorization of φ. By Y we denote a general fibre of φ.

PROPOSITION 4.1. *Let S be as above, then S is an elliptic K-3 surface. Conversely, every elliptic K-3 surface of degree 4 in $\boldsymbol{P}^3$ with* deg(*fibre of* φ) ≥ 5 *occurs in this way.*

PROOF. Since $\deg S = c_1(\mathscr{E}) = 4$, S is a K-3 surface. We may assume that $Y = (s_1)_0$ is a disjoint union of smooth curves. $Y = Y_1 \coprod Y_2 \coprod \cdots \coprod Y_\lambda$ for some $\lambda \in \boldsymbol{N}$. Since Y_1 is linearly equivalent to Y_i $(1 \leq i \leq \lambda)$, $\deg Y_1 = \cdots = \deg Y_\lambda$ and $p_a(Y_1) = \cdots = p_a(Y_\lambda)$. Hence $p_a(Y) = 1 - \chi(\mathcal{O}_Y) = 1 - \lambda(1 - p_a(Y_1))$. By (2.3), we can see $\deg Y_1 = c_2/\lambda$ and $p_a(Y_1) = 1$. Hence the morphism $\pi : S \to \boldsymbol{P}^1$ gives the structure of elliptic surface. The second statement follows from (3.1). Q.E.D.

REMARK 4.2. (1) For any given $d \geq 3$, there is a smooth elliptic curve of degree

d on a smooth quartic surface in $\boldsymbol{P}^3$ [Mo]. So, given $d \geq 3$, there exists an elliptic K-3 surface $\pi : S \to \boldsymbol{P}^1$ (with connected fibres) such that deg(fibre of π) $= d$.

(2) By (2.6), we have $c_2 \geq 5$ by stability. By (2.7) and (1) just above, there is a stable rank 2 bundle $\mathscr{E}$, $c_1(\mathscr{E}) = 4$ which has a smooth determinant surface.

PROPOSITION 4.3. *$\lambda :=$ (number of irreducible components of Y) $= h^1(\mathscr{E}(-4)) + 1$.*

PROOF. As in the proof of (3.1), we get the following resolution of $\mathscr{I}_Y$

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}}(-4) \longrightarrow \mathscr{E}(-4) \longrightarrow \mathscr{I}_Y \longrightarrow 0\,,$$

from which follows $h^1(\mathscr{E}(-4)) = h^1(\mathscr{I}_Y)$. Viewing the exact sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathcal{O}_{\boldsymbol{P}} \longrightarrow \mathcal{O}_Y \longrightarrow 0\,,$$

we have $h^0(\mathcal{O}_Y) = h^0(\mathcal{O}_{\boldsymbol{P}}) + h^1(\mathscr{I}_Y)$. Since $h^0(\mathcal{O}_Y) = \lambda$, we have $\lambda = h^1(\mathscr{E}(-4)) + 1$. Q.E.D.

A determinant surface S with $c_2/\lambda = 3$ has a special property as below.

PROPOSITION 4.4. *In our situation, $c_2/\lambda = 3$ if and only if there exists a line L_0 on S such that $(L_0, Y) = c_2$. Moreover, such L_0 is unique if it exists and π corresponds to $\Phi_{|H-L_0|}$, where H is the hyperplane section of S and $\Phi_{|H-L_0|}$ is a morphism to $\boldsymbol{P}^1$ associated to the linear system $|H - L_0|$.*

PROOF. (1) Assume that there exists a line L_0 on S such that $(L_0, Y) = c_2$. Let H_0 be a generic hyperplane section of S containing L_0. Then $H_0 = L_0 + R_0$ as a divisor on S, where R_0 is a smooth plane cubic curve on S. Since $c_2 = (H_0, Y) = (L_0, Y) + (R_0, Y)$, we get $(R_0, Y) = 0$. Hence R_0 is contained in some fibre of π. By genericity of $R_0 = H_0 - L_0$, $H_0 - L_0$ is linearly equivalent to the fibre of π. Hence $c_2/\lambda = 3$ and π corresponds to $\Phi_{|H-L_0|}$. If there is a line L_1 on S such that $(L_1, Y) = c_2$, for any plane H_1 containing L_1, we have $H_0 - L_0 \sim Y_1$ and $H_1 - L_1 \sim Y_1$ as above, where Y_1 is a fibre of π and $\sim$ means linearly equivalence. By $4 = (H_0, H_1) = (L_0 + Y_1, L_1 + Y_1) = 6 + (L_0, L_1)$, we have $L_0 = L_1$.

(2) Next assume that $c_2/\lambda = 3$. Let Y_1 be a fibre of π. Y_1 is a plane cubic curve, so there is a unique plane H containing Y_1. We also denote the hyperplane section of S by H. Then $(L_0, Y) = (L_0, \lambda Y_1) = 3\lambda = c_2$ and π corresponds to $\Phi_{|H-L_0|} : S \to \boldsymbol{P}^1$, since $H - L_0 = Y_1$. Q.E.D.

PROPOSITION 4.5. *If S contains a smooth rational curve $C \cong \boldsymbol{P}^1$, then $\rho(S) \geq 3$, unless $c_2/\lambda = 3$ and $C = L_0$ such that $(L_0, Y) = c_2$ as above.*

PROOF. Let M be the intersection matrix with respect to three divisors H (hyperplane section of S), Y and C. $\det M = -2(2(Y, C)^2 - c_2(Y, C)\deg C - c_2^2)$, so $\det M = 0$ if and only if $\deg C = 1$ and $(Y, C) = c_2$. Hence this proposition follows from (4.4). Q.E.D.

PROPOSITION 4.6. *Let H be a plane in $\boldsymbol{P}^3$. The restriction $\mathscr{E}|_H$ to H is semistable,* *unless $c_2/\lambda=3$ and H contains a line L_0 lying on S such that $(Y, L_0)=c_2$.*

PROOF. Since $c_1(\mathscr{E}(-2)|_H)=0$, it is sufficient to show that $H^0(\mathscr{E}(-3)|_H)=0$. If H contains an irreducible component Y_i of Y, then $\deg Y_i=3$ and $c_2/\lambda=3$. In this case, by (4.4), there exists a line L_0 such that $H\cap S=Y_i+L_0$ and that $c_2=(L_0, Y)$. So we may assume that $\dim Y\cap H=0$. The exact sequence

$$0 \longrightarrow \mathcal{O}_P(-3) \longrightarrow \mathscr{E}(-3) \longrightarrow \mathscr{I}_Y(1) \longrightarrow 0\,,$$

induces the exact sequence

$$0 \longrightarrow \mathcal{O}_H(-3) \longrightarrow \mathscr{E}(-3)|_H \longrightarrow \mathscr{I}_Z(1) \longrightarrow 0\,,$$

where $Z=Y\cap H$ is a zero-dimensional scheme of degree c_2 in H, and $h^0(\mathscr{E}(-3)|_H)=h^0(\mathscr{I}_Z(1))$. Z lies on the reduced plane quartic curve $C=S\cap H$. Since $c_2\geq 5$, the scheme Z is contained in some line on H if and only if H contains a line L on S such that $(L, T)=c_2$. Hence this follows from (4.4). Q.E.D.

PROPOSITION 4.7. *Let H be a plane in $\boldsymbol{P}^3$. Assume that $S\cap H$ is irreducible and that $c_2(\mathscr{E})>8$. Then the restriction $\mathscr{E}|_H$ is stable on H.*

PROOF. Since $c_1(\mathscr{E}(-2)|_H)=0$, it is sufficient to show that $H^0(\mathscr{E}(-2)|_H)=0$. This follows from the same argument as in (4.6). Q.E.D.

COROLLARY 4.8. *Let H be a plane in $\boldsymbol{P}^3$. Assume that $c_2(\mathscr{E})>8$ and that $\rho(S)=2$. Then the restriction $\mathscr{E}|_H$ is stable on H, unless $c_2/\lambda=3$ and H contains a line L_0 on S such that $(L_0, Y)=c_2$.*

PROOF. It follows from (4.5) and (4.7), since any reducible plane quartic curve contains a smooth rational curve as a component. Q.E.D.

As a corollary of (2.13), (3.1) and (4.8), we get:

COROLLARY 4.9. *For any given $c_2\geq 5$, there exists a stable vector bundle $\mathscr{E}$ of rank two with $c_1(\mathscr{E})=0$ and $c_2(\mathscr{E})=c_2$ on $\boldsymbol{P}^3$ such that the restriction $\mathscr{E}|_H$ is stable for any plane H in $\boldsymbol{P}^3$.*

§5. Jumping planes.

From (4.8), if $c_2(\mathscr{E})\geq 9$, the unstability of $\mathscr{E}|_H$ gives some information on the Picard number $\rho(S)$. Our main object is to estimate the value $\rho(S)$ by the grade of unstability of $\mathscr{E}|_H$. First of all we shall study jumping lines and pairs of jumping lines and unstable planes.

NOTATIONS.
$\mathscr{E}$: a stable vector bundle of rank two with $c_1(\mathscr{E})=4$ and $c_2(\mathscr{E})\geq 9$ on $\boldsymbol{P}^3$.
$S=(s_1\wedge s_2)_0$: a smooth determinant surface of $\mathscr{E}$, $\deg S=4$.

$\varphi : S \supset (\eta_1 s_1 + \eta_2 s_2)_0 \to (\eta_1 : \eta_2) \in \boldsymbol{P}^1$ denotes the natural projection.
Y : a generic fibre of φ, deg $Y = c_2$.
$\lambda = h^1(\mathscr{E}(-4)) + 1 =$ the number of the irreducible components of Y.
$\pi : S \to \boldsymbol{P}^1$ the Stein factorization of φ.
Y_1 : a generic fibre of π, deg $Y_1 = c_2/\lambda$.

5.1. Let L be a line in $\boldsymbol{P}^3$. $\mathscr{E}(-2)|_L = \mathcal{O}_L(-a) \oplus \mathcal{O}_L(a)$ for some $a \geq 0$. Pick a section $s_\eta = \eta_1 s_1 + \eta_2 s_2 \in H^0(\mathscr{E})$. Put $Y_\eta = (s_\eta)_0$ and let

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}} \xrightarrow{s_\eta} \mathscr{E} \longrightarrow \mathscr{I}_{Y_\eta}(4) \longrightarrow 0$$

be the corresponding exact sequence as in the proof of (3.1). If Y_η meets L at finite number of points $P_1, P_2, \cdots, P_r$ $(0 \leq r \leq c_2)$, then the tensor products with $\mathcal{O}_L$ give an exact sequence

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathscr{E}|_L \longrightarrow \mathcal{O}|_L(4 - \sum P_i) \oplus \sum k_{P_i} \longrightarrow 0\,,$$

where k_{P_i} is the skyscraper sheaf $\boldsymbol{C}$ at P_i. If $r = 0$, then $H^0(\mathscr{E}(-5)|_L) = 0$ so we must have $\mathscr{E}(-2)|_L \cong \mathcal{O} \oplus \mathcal{O}$, $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ or $\mathcal{O}(-2) \oplus \mathcal{O}(2)$. If $r = 1$, $\mathscr{E}(-2)|_L \cong \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. If $r \geq 2$, $\mathscr{E}(-2)|_L \cong \mathcal{O}(-r+2) \oplus \mathcal{O}(r-2)$.

DEFINITION 5.2. Let L be a line in $\boldsymbol{P}^3$. If L is the jumping line of order a (>0) for $\mathscr{E}(-2)$ in the sense of (2.8), we call that L is the *jumping line of order* a (>0) for $\mathscr{E}$.

PROPOSITION 5.3. (1) *Let L be a jumpimg line of order $a \geq 3$ for $\mathscr{E}$. Then $L \subset S$ and the intersection number $(Y, L) = a + 2$.*

(2) *Let H be an unstable plane for $\mathscr{E}$. Then H contains at most two jumping lines with order greater than* 2.

PROOF. (1) By (5.1), $Y_\eta \cap L = (a+2)$-points scheme, but if $L \not\subset S$ then $\deg(L \cap S) = 4$. So $L \subset S$ and $(Y, L) = (Y_\eta, L) = a + 2$.

(2) By (4.7), $S \cap H$ is reducible (but reduced, since s is smooth). From the proof of (4.6) and (4.7), $\mathscr{E}|_H$ is not stable (possibly not semistable) if and only if $h^0(\mathscr{E}(-2)|_H) = h^0(H, \mathscr{I}_Z(2)) \neq 0$, where $Z = H \cap Y$ is a c_2-points scheme in H. Take a nonzero section $\tau \in H^0(H, \mathscr{I}_Z(2))$, then $(\tau)_0$ is a conic on H containing Z (possibly $(\tau)_0 \not\subset S$). Assume that $(\tau)_0$ is smooth. $Z \subset (\tau)_0 \subset S \cap H$ and H has no jumping line with order greater than 2 by considering the intersection $(\tau)_0 \cap$(a line) of degree two. Assume that $(\tau)_0$ is a singular conic. Let L be a jumping line of order $a \geq 3$ on H. Then $(\tau)_0 \cap L$ contains 5-points scheme, so $L \subset (\tau)_0$. Q.E.D.

5.4. Let H be an unstable plane (2.10) in $\boldsymbol{P}^3$. Then continuing the argument of the proof of (5.3), we get the following (5.4.1)–(5.4.3). Note that $S \cap H$ is reduced by (2.12).

(5.4.1) Assume that H has exactly one jumping line L with order greater than 2. Then $S \cap H = L + R$ as a divisor on S, where R is the residual curve. In this case $(L, Y) \geq c_2 - 4$ and there exists a line L' on H (possibly $L' \not\subset S$) such that $Z \subset L \cup L'$.

(5.4.2) Assume that H has two jumping lines L_1 and L_2 with order greater than 2. Then $S \cap H = L_1 + L_2 + R$, where R is the residual curve and contained in a singular fibre of the elliptic fibration $\pi: S \to \boldsymbol{P}^1$. In this case $Z \subset L_1 \cup L_2$ and $(Y, L_1) + (Y, L_2) = c_2$.

(5.4.3) Assume that H has no jumping line with order greater than 2. Then $S \cap H = C + R$, where C is a smooth conic such that $(C, Y) = c_2$, $Z \subset C$. In this case the residual curve R is contained in a singular fibre of π.

Let $\boldsymbol{U}$ be a set of these triple (H, P, R), where H is an unstable plane, $P = L$ in case (5.4.1), $P = L_1 \cup L_2$ in case (5.4.2), $P = C$ in case (5.4.3) and R is the residual curve in the corresponding case. We define the following equivalence relation $\sim$ between the elements of $\boldsymbol{U}$;

$$(H_1, P_1, R_1) \sim (H_2, P_2, R_2) \quad \text{if and only if} \quad P_1 = P_2 \ (\text{as a set}).$$

Set $\boldsymbol{J} := \boldsymbol{U}/\sim$ the quotient set of $\boldsymbol{U}$ and denote by $[(H, P, R)] \in \boldsymbol{J}$ the equivalence class of $(H, P, R) \in \boldsymbol{U}$.

DEFINITION 5.5. An element $h = [(H, P, R)] \in \boldsymbol{J}$ is called a *jumping plane of type* J_ξ, if $P = L$ (as in (5.4.1)) and $\xi = c_2 - (Y, L)$, in this case $0 \leq \xi \leq 4$ and $\lambda | \xi$; *of type* J_ξ, if $P = L_1 \cup L_2$ (as in (5.4.2)) and $\xi = \min\{(L_1, Y), (L_2, L)\}$, in this case $5 \leq \xi \leq [c_2/2]$ (integral part of $c_2/2$) and $\lambda | \xi$, and *of type* J_{c_2}, if $P = C$ (as in (5.4.3)).

Set $\boldsymbol{Ja} = \{h \in \boldsymbol{J}; h$ is a jumping plane of type J_ξ and $0 \leq \xi \leq 4\}$, $\alpha = \#\boldsymbol{Ja}$,
$\boldsymbol{Jb} = \{h \in \boldsymbol{J}; h$ is a jumping plane of type J_ξ and $5 \leq \xi \leq [c_2/2]\}$, $\beta = \#\boldsymbol{Jb}$,
$\boldsymbol{Jc} = \{h \in \boldsymbol{J}; h$ is a jumping plane of type $J_{c_2}\}$ and $\gamma = \#\boldsymbol{Jc}$,
where $\#A$ stands for the number of elements of a finite set A.

REMARK 5.6. (1) In general $\boldsymbol{U}$ is an infinite set, but $\boldsymbol{J}$ is always finite.
(2) By (4.4), the following conditions are equivalent.
(i) $c_2/\lambda = 3$. (ii) There exists one and only one jumping plane of type J_0.
(3) By (4.8) if $c_2/\lambda \geq 4$ and $\rho(S) = 2$, then $\boldsymbol{U} = \boldsymbol{J} = \varnothing$ *i.e.* if $\boldsymbol{J} \neq \varnothing$ then $\rho(S) \geq 3$.
(4) If $\lambda \geq 5$, then $\boldsymbol{Ja} = \{$a jumping plane of type $J_0\}$.

5.7. Intersection of jumping planes. Let $(H_1, P_1, R_1), (H_2, P_2, R_2) \in \boldsymbol{U}$ and assume $[(H_1, P_1, R_1)] \neq [(H_2, P_2, R_2)]$ in $\boldsymbol{J}$. Note that $(P_i, P_j) + (P_i, R_j) = 4 - (R_i, P_j) - (R_i, R_j) =$ degree of P_i in H_i for $i, j = 1, 2$. A possible common component of $P_1 + R_1$ and $P_2 + R_2$ is the line $L = H_1 \cap H_2$. Moreover if L is their component, then $Supp(P_1 + R_1 - L) \cap Supp(P_2 + R_2 - L) = \varnothing$ and $(L, L) = -2$. By these facts we can show the following:

(1) Assume that both P_1 and P_2 are smooth conics. Then $(P_1, P_2) = (R_1, R_2) = 0, 1$ or 2. More precisely, $P_1 \cap P_2 = \varnothing \Leftrightarrow R_1 \cap R_2 = \varnothing$ or a line, $\deg(P_1 \cap P_2) = r \Leftrightarrow \deg(R_1 \cap R_2) = r (r = 1, 2)$.

(2) Assume that P_1 is a smooth conic and that P_2 is a singular conic. Then $(P_1, P_2) = (R_1, R_2) = 0, 1$ or 2 and the same as (1).

(3) Assume that both P_1 and P_2 are singular conics. Then $(P_1, P_2) = (R_1, R_2) = 0, 1$ or 2. More precisely, $P_1 \cap P_2 =$ a line $\Rightarrow R_1 \cap R_2 = \varnothing$, $\deg(P_1 \cap P_2) = r \Leftrightarrow \deg(R_1 \cap R_2) =$

r ($r=1, 2$), $R_1 \cap R_2 = \varnothing$ or a line $\Rightarrow P_1 \cap P_2 = \varnothing$.

(4) Assume that P_1 is a smooth conic and that P_2 is a line. Then $(P_1, P_2)=0$ or 1. Namely, $P_1 \cap P_2 = \varnothing \Rightarrow R_1 \cap R_2 =$ a single point or a line; in this case $(R_1, P_2)=(R_1, R_2)=1$. $\deg(P_1 \cap P_2)=1 \Rightarrow \deg(R_1 \cap R_2)=2$; in this case $(R_1, R_2)=2$, $(R_1, P_2)=0$.

(5) Assume that P_1 is a singular conic and that P_2 is a line. If $(P_1, P_2)>0$, then there exists a plane $H_0 \subset \boldsymbol{P}^3$ such that $S \cap H_0$ contains P_2 and one of the irreducible component L_1 of P_1. But this leads a contradiction: $c_2=(H_0, Y) \geq (L_1, Y)+(P_2, Y)>c_2$. In this case $P_1 \cap P_2 = \varnothing$, $(R_1, P_2)=1$ and $R_1 \cap R_2$ is a single point or a line.

(6) Assume that both P_1 and P_2 are lines. Then, the same as in (5), $P_1 \cap P_2 = \varnothing$, $(R_1, R_2)=2$ and $R_1 \cap R_2$ is a finite scheme of degree two or a line.

If $c_2/\lambda=3$, the unstable planes for $\mathscr{E}$ is completely described. Because of the following proposition and (4.8), we have to assume $c_2/\lambda \geq 4$ for Theorem 2.

PROPOSITION 5.8. *If $c_2/\lambda=3$, then $\boldsymbol{J}$ consists of only one element of type J_0 and $\boldsymbol{U}$ is parametrized by $\boldsymbol{P}^1$.*

PROOF. By (4.4) and 5.6 (2), there exists a jumping plane of type J_0, $h_0=[(H_0, L_0, R_0)]$ and the fibre Y_1 of π is a plane cubic curve. Now $L_0+Y_1 \sim H$ (linearly equivalent to the hyperplane section).

(1) Assume that there exists $h_1=[(H_1, L_1, R_1)] \in \boldsymbol{Ja}$. Now $\lambda \geq 3$ by assumption $c_2 \geq 9$. Combining with 5.6 (4), $\lambda=3$ or 4. Let $\lambda=3$, then $c_2=9$. By 5.7 (6), $L_0 \cap L_1 = \varnothing$ so $(L_0, L_1)=0$. Hence $1=(H, L_1)=(L_0, L_1)+(Y_1, L_1)=(Y_1, L_1)$. On the other hand, since h_1 is of type J_3, $3=c_2-(Y, L_1)=9-(Y, L_1)$ so $(Y, L_1)=6$. By $6=(Y, L_1)=(\lambda Y_1, L_1)$, $(Y_1, L_1)=2$. This is a contradiction $1=(Y_1, L_1)=2$. The same argument applies to the case $\lambda=4$ and $c_2=12$.

(2) Assume that there exists $h_2=[(H_2, P_2, R_2)] \in \boldsymbol{Jb} \cup \boldsymbol{Jc}$. R_2 is contained in some singular fibre Y_1 of π. Noting that R_2 is a reduced conic, there is a line L on S such that $Y_1=R_2+L$. Then $3=(L_0, Y_1)=(L_0, R_2+L)=(L_0, R_2)+(L_0, L)$. By 5.7 (4) and (5), $(L_0, R_2)=0$ or 1, so we have $(L_0, L)=2$ or 3. This is impossible. Q.E.D.

§6. Picard numbers of determinant surfaces.

In this section we will prove Theorem 1.2. The proof consists of some lemmas. Throughout this section we will use the same notations as in §5 and the following.

$$\alpha=\#\boldsymbol{Ja}, \quad \beta=\#\boldsymbol{Jb}, \quad \gamma=\#\boldsymbol{Jc}, \quad \#\boldsymbol{J}=\alpha+\beta+\gamma \qquad \text{(see (5.5))}.$$

$\langle D_1, D_2, \cdots, D_n \rangle$: the intersection matrix of divisors $D_1, D_2, \cdots, D_n$ on S.

DEFINITION 6.1. Let $D_i \cong \boldsymbol{P}^1$ ($i=1, 2, \cdots, n$) be curves on S with $D_i \neq D_j$ if $i \neq j$. By a *cyclic chain* contained in $\bigcup_{i=1}^n D_i$, we shall mean a curve composed of some of components say $D_1, D_2, \cdots, D_b$ such that $(D_1, D_2)=(D_2, D_3)=\cdots=(D_{b-1}, D_b)=(D_b, D_1)=1$ and $(D_i, D_j)=0$ for other pairs (i, j).

Theorem 1.2 follows easily from the following two propositions.

PROPOSITION 6.2. *The following inequalities* (1)–(3) *hold.*
(1) $\rho(S)\geq 1+\alpha$. (2) $\rho(S)\geq 3+\alpha$ *if* $\beta>0$. (3) $\rho(S)\geq 2+\alpha$ *if* $\gamma>0$.

PROPOSITION 6.3. *The following inequalities* (1) *and* (2) *hold.*
(1) $\rho(S)\geq 2+\frac{1}{2}(\beta+\gamma)$. (2) $\rho(S)\geq 2+\alpha+\gamma$ *if* $\alpha>0$.

PROOF OF THEOREM 1.2. We may assume that $\#\boldsymbol{J}\geq 3$ by 3.2 (3) and 5.6 (3).
(1) If $\alpha=0$ then $\rho(S)\geq 2+\frac{1}{2}(\beta+\gamma)=2+\frac{1}{2}\#\boldsymbol{J}$ by 6.3 (1).
(2) If $\beta+\gamma=0$ then $\rho(S)\geq 1+\alpha>2+\frac{1}{2}\#\boldsymbol{J}$ by 6.2 (1).
(3) If $\frac{1}{2}\#\boldsymbol{J}\leq\beta+\gamma<\#\boldsymbol{J}$ then $\rho(S)\geq 2+\beta+\gamma\geq 2+\frac{1}{2}\#\boldsymbol{J}$ by 6.3 (2).
(4) If $\frac{1}{2}\#\boldsymbol{J}<\alpha<\#\boldsymbol{J}$ then $\rho(S)\geq 2+\alpha>2+\frac{1}{2}\#\boldsymbol{J}$ by 6.2 (2) and (3). Q.E.D.

PROOF OF 6.2. Let $h_i=[(H_i, L_i, R_i)]\in\boldsymbol{Ja}$ $(i=1, 2, \cdots, \alpha)$. Note that $(L_i, L_j)=0$ for $i\neq j$ by 5.7 (6).
(1) Then

$$\langle H, L_1, L_2, \cdots, L_\alpha\rangle=\begin{pmatrix} 4 & 1 & 1 & \cdots & 1 \\ 1 & -2 & & & \\ 1 & & -2 & & \mathbf{0} \\ \vdots & & & \ddots & \\ 1 & \mathbf{0} & & & -2 \end{pmatrix}$$

has maximum rank.
(2) Let $h_\beta=[(H_\beta, L_{\beta 1}\cup L_{\beta 2}, R_\beta)]\in\boldsymbol{Jb}$. $(L_i, L_{\beta_j})=0$ for any $1\leq i\leq\alpha$ and $j=1, 2$ by 5.6 (5). Then

$$\langle H, L_{\beta 1}, L_{\beta 2}, L_1, \cdots, L_\alpha\rangle=\begin{pmatrix} 4 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 1 & 1 & -2 & 0 & \cdots & 0 \\ 1 & 0 & 0 & -2 & & \mathbf{0} \\ \vdots & \vdots & \vdots & & \ddots & \\ 1 & 0 & 0 & \mathbf{0} & & -2 \end{pmatrix}$$

has maximum rank.
(3) Let $h_\gamma=[(H_\gamma, C_\gamma, R_\gamma)]\in\boldsymbol{Jc}$. Now $(L_i, C_\gamma)=0$ or 1 by 5.6 (4). So we may assume that $(L_i, C_\gamma)=1$ for $1\leq i\leq k$, $=0$ for $k<i\leq\alpha$ for some k $(0\leq k\leq\alpha)$. Then by elementary transformations of rows and columns,

$$\langle H, C_\gamma, L_1, \cdots, L_k, L_{k+1}, \cdots, L_\alpha\rangle$$

$$= \begin{pmatrix} 4 & 2 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 2 & -2 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & -2 & & & & & \mathbf{0} \\ \vdots & \vdots & & \ddots & & & & \\ 1 & 1 & & & -2 & & & \\ 1 & 0 & & & & -2 & & \\ \vdots & \vdots & & & & & \ddots & \\ 1 & 0 & \mathbf{0} & & & & & -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 8 & 4 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 4 & -4 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 2 & 2 & -2 & & & & & \mathbf{0} \\ \vdots & \vdots & & \ddots & & & & \\ 2 & 2 & & & -2 & & & \\ 2 & 0 & & & & -2 & & \\ \vdots & \vdots & & & & & \ddots & \\ 2 & 0 & \mathbf{0} & & & & & -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} k+l+8 & k+4 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ k+4 & k-4 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & -2 & & & & & \mathbf{0} \\ \vdots & \vdots & & \ddots & & & & \\ 0 & 0 & & & -2 & & & \\ 0 & 0 & & & & -2 & & \\ \vdots & \vdots & & & & & \ddots & \\ 0 & 0 & \mathbf{0} & & & & & -2 \end{pmatrix}$$

and

$$\det\begin{pmatrix} k+l+8 & k+4 \\ k+4 & k-4 \end{pmatrix} = (k-4)(l-4)-64\,,$$

where $l=\alpha-k$. Since S is a K-3 surface, $\rho(S)\leq 20$. By (1) $1+\alpha\leq\rho(S)\leq 20$, so $\alpha=k+l\leq 19$ and $(k-4)+(l-4)\leq 11$. Then $(k-4)(l-4)-64<0$. Hence $\operatorname{rank}\langle H, C_\gamma, L_1, \cdots, L_\alpha\rangle = 2+\alpha$. Q.E.D.

For Proposition 6.3, we have to investigate more carefully the intersections of jumping planes of type $\boldsymbol{Jb}$ and of type $\boldsymbol{Jc}$.

6.4. Let $h_i=[(H_i, P_i, R_i)]\in \boldsymbol{Jb}\cup\boldsymbol{Jc}$, $i=1, 2, \cdots, n$. Assume that $\bigcup_{i=1}^n R_i\subset\pi^{-1}(p)$ for some $p\in\boldsymbol{P}^1$. Set $N=$the number of the irreducible components of $\bigcup_{i=1}^n R_i$, and let $\sum_{i=1}^n R_i=\sum_{i=1}^N f_iF_i$ ($f_i\in\boldsymbol{N}$) be the irreducible decomposition. Note that R_i is either a smooth conic or a union of two lines with normal crossing at one point. We consider the following five cases.

(6.4.1) Every R_i is a smooth conic and $\bigcup_{i=1}^n R_i\subsetneq\pi^{-1}(p)$. Then $N=n$ and $\langle F_1, F_2, \cdots, F_n\rangle$ has maximum rank by the following well known Lemma 6.6.

(6.4.2) Every R_i is a smooth conic and $\bigcup_{i=1}^n R_i=\pi^{-1}(p)$. Then $N=n$ and $\langle F_1, F_2, \cdots, F_{n-1}\rangle$ has maximum rank by Lemma 6.6. Moreover, $\pi^*(p)=\sum_{i=1}^n m_iR_i$ for some $m_i\in\boldsymbol{N}$ and $c_2/\lambda=\deg\pi^*(p)=\deg\sum_{i=1}^n m_iR_i=2\sum_{i=1}^n m_i$.

(6.4.3) There is a singular conic and a smooth conic. We may assume that R_1 is a singular conic and R_2 is a smooth conic. By induction on n, we get $N\geq n+1$. By Lemma 6.6, $\langle F_1, F_2, \cdots, F_n\rangle$ has maximum rank.

(6.4.4) Every R_i is a singular conic and $N\geq n+1$. By Lemma 6.6, $\langle F_1, F_2, \cdots, F_n\rangle$ has maximum rank.

(6.4.5) Every R_i is a singular conic and $N\leq n$. Then by the following Lemma 6.5, we can see that $\bigcup_{i=1}^n R_i$ is a cyclic chain and $N=n$. Then by the classification of singular fibres of elliptic surfaces [Ko], $\pi^*(p)$ is a singular fibre of type ${}_m\mathrm{I}_n$ ($m=$multiplicity≥ 1, $n=$the number of the irreducible components≥ 3) or of type IV. Then $\sum_{i=1}^n R_i=2\operatorname{red}(\sum_{i=1}^n R_i)$, $\pi^*(p)=m\operatorname{red}(\sum_{i=1}^n R_i)=(m/2)\sum_{i=1}^n R_i$ and $c_2/\lambda=\deg\pi^*(p)=(m/2)\sum_{i=1}^n \deg R_i=mn$. By Lemma 6.6, $\langle F_1, F_2, \cdots, F_{n-1}\rangle$ has maximum rank.

LEMMA 6.5. *Under the situation of* (6.4.5), $\bigcup_{i=1}^n R_i$ *is a cyclic chain and* $N=n$.

PROOF. (1) Assume that every f_i is greater than 1. By chasing the component R_i, we can see $\bigcup_{i=1}^n R_i$ has a cyclic chain. By the classification of singular fibres of elliptic surfaces [Ko], $\bigcup_{i=1}^n R_i=\pi^{-1}(p)=$the cyclic chain. If $f_1\geq 3$, we may assume that $F_1\subset R_1\cap R_2\cap R_3$. We can write $R_i=F_1+R_i'$ ($i=1, 2, 3$) for some line R_i' ($i=1, 2, 3$), note that $F_1\neq R_i'$ ($i=1, 2, 3$) and $R_i'\neq R_j'$ if $i\neq j$. Then $(F_1, R_i')=1$ ($i=1, 2, 3$), this is impossible, since $\bigcup_{i=1}^n R_i=$the cyclic chain. Therefore we have $f_i=2$ for any $1\leq i\leq N$, and $N=n$.

(2) Assume that $f_N=1$. We want to show that $N\geq n+1$ by induction on n. We may assume that $F_N\subset R_n$ and $F_N\not\subset R_i$ for any $1\leq i\leq n-1$. Let $\sum_{i=1}^{n-1} R_i=\sum_{i=1}^{N-1} f_i'F_i$ ($f_i'\geq 0$) be the irreducible decomposition. If $f_i'\geq 2$ for any $f_i'\neq 0$, then $\bigcup_{i=1}^{n-1} R_i=\pi^{-1}(p)$ by (1) as above. Since $F_N\not\subset\bigcup_{i=1}^{n-1} R_i$, there exists i such that $f_i'=1$. By the induction hypothesis, (the number of the irreducible components of $\sum_{i=1}^{n-1} R_i)\geq n$. Also by $F_N\not\subset\bigcup_{i=1}^{n-1} R_i$, $N\geq n+1$. By the assumption $N\leq n$, every f_i must be greater than 1. Q.E.D.

LEMMA 6.6 (cf. [Be, Corollary VIII. 4]). *Let X be a smooth projective surface, B a smooth projective curve and $g: X\to B$ a surjective morphism with connected fibres. Let $b\in B$, $g^*(b)=\sum_i m_iX_i$ be the irreducible decomposition and let $D=\sum_i r_iX_i$ ($r_i\in\boldsymbol{Z}$). Then*

$D^2 \leq 0$, *with equality if and only if* $D = rg^*(b)$ *for some* $r \in \boldsymbol{Q}$.

LEMMA 6.7. *Let* $D_1, D_2, \cdots, D_k$ *be divisors on S. Suppose that* $(Y, D_i) = 0$ *for each* $1 \leq i \leq k$. *Then* $\det\langle H, Y, D_1, D_2, \cdots, D_k\rangle = -c_2^2 \det\langle D_1, D_2, \cdots, D_k\rangle$.

PROOF. Since $(H, H) = \deg S = 4$, $(H, Y) = \deg Y = c_2$ and $(Y, Y) = 0$,

$$\langle H, Y, D_1, D_2, \cdots, D_k\rangle = \begin{pmatrix} 4 & c_2 & * & \cdots & * \\ c_2 & 0 & 0 & \cdots & 0 \\ * & 0 & & & \\ \vdots & \vdots & & \langle D_1, D_2, \cdots, D_k\rangle & \\ * & 0 & & & \end{pmatrix}.$$

At first, expand along the first row. By expanding each cofactor along the first column, we get the assertion. Q.E.D.

PROOF OF 6.3 (1). Let $h_i = [(H_i, P_i, R_i)] \in \boldsymbol{Jb} \cup \boldsymbol{Jc}$ $(i = 1, 2, \cdots, \beta + \gamma)$. Since $c_2/\lambda = \deg \pi^*(p) \geq 4$ and $\deg R_i = 2$, $R_i \subsetneq \pi^{-1}(p)$ for any $1 \leq i \leq \beta + \gamma$. Hence we need at least two h_i and h_j so that $R_i \cup R_j = \pi^{-1}(p)$. By (6.4.1)–(6.4.5), we can find k-irreducible curves $F_1, F_2, \cdots, F_k$ contained in singular fibres such that $\operatorname{rank}\langle F_1, F_2, \cdots, F_k\rangle = k$, where $k = \beta + \gamma - [(\beta + \gamma)/2]$. By (6.7), $\langle H, Y, F_1, F_2, \cdots, F_k\rangle$ has maximum rank. So $\rho(S) \geq 2 + \frac{1}{2}(\beta + \gamma)$. Q.E.D.

LEMMA 6.8. *Assume that there exists an* $h_0 = [(H_0, L_0, R_0)] \in \boldsymbol{Ja}$. *Then both of* (6.4.2) *and* (6.4.5) *do not occur.*

PROOF. (1) Assume that (6.4.2) does occur. Then $\pi^*(p) = \sum_{i=1}^n m_i R_i$ for some $m_i \in \boldsymbol{N}$ and $c_2/\lambda = 2\sum_{i=1}^n m_i$. Note that $(L_0, R_i) = 0$ or 1 by 5.7 (4), (5). Now $c_2 - 4 \leq (L_0, Y) = (L_0, \lambda\pi^*(p)) = \lambda(L_0, \sum_{i=1}^n m_i R_i) = \lambda\sum_{i=1}^n m_i(L_0, R_i) \leq \lambda\sum_{i=1}^n m_i = c_2/2$. We get $c_2 \leq 8$. This contradicts our assumption $c_2 \geq 9$.

(2) Assume that (6.4.5) does occur. Then $\pi^*(p) = (m/2)\sum_{i=1}^n R_i$ for some $m \in \boldsymbol{N}$ and $c_2/\lambda = \deg \pi^*(p) = (m/2)\sum_{i=1}^n \deg R_i = mn$. Note that $(L_0, R_i) = 0$ or 1 by 5.7 (4), (5). Now $c_2 - 4 \leq (L_0, Y) = (L_0, \lambda\pi^*(p)) = \lambda(L_0, (m/2)\sum_{i=1}^n R_i) = (\lambda m/2)\sum_{i=1}^n (L_0, R_i) \leq (\lambda/2)mn = c_2/2$. We get $c_2 \leq 8$. This contradicts our assumption $c_2 \geq 9$. Q.E.D.

PROOF OF 6.3 (2). By Lemma 6.8, (6.4.1), (6.4.3) or (6.4.4) occurs. So we can find $(\beta + \gamma)$-irreducible curves $F_1, F_2, \cdots, F_{\beta+\gamma}$ contained in singular fibres such that $\operatorname{rank}\langle F_1, F_2, \cdots, F_{\beta+\gamma}\rangle = \beta + \gamma$. By Lemma 6.7, $\langle H, Y, F_1, F_2, \cdots, F_{\beta+\gamma}\rangle$ has maximum rank. So $\rho(S) \geq 2 + \beta + \gamma$. Q.E.D.

References

[Ba] W. BARTH, Some properties of stable rank-2 bundles on $\boldsymbol{P}_n$, Math. Ann. **226** (1977), 125–150.

[Be] A. BEAUVILLE, *Complex Algebraic Surfaces*, London Math. Soc. Lecture Note Ser. **68** (1983), Cambridge Univ. Press.

[Ha 1] R. HARTSHORNE, *Algebraic Geometry*, Graduate Texts in Math. **52** (1977), Springer.

[Ha 2] R. HARTSHORNE, Stable vector bundles of rank 2 on $\boldsymbol{P}^3$, Math. Ann. **238** (1978), 229–280.

[Ko] K. KODAIRA, On compact analytic surfaces; II, Ann. of Math. **77** (1963), 563–626.

[Lo] A. F. LOPEZ, *Noether-Lefschetz Theory and the Picard Group of Projective Surface*, Mem. Amer. Math. Soc. **438** (1991).

[Ma] M. MARUYAMA, On a family of algebraic vector bundles, *Number Theory, Algebraic Geometry and Commutative Algebra*, Kinokuniya (1973), 95–149.

[Mo] S. MORI, On degrees and genera of curves on smooth quartic surfaces on $\boldsymbol{P}^3$, Nagoya Math. J. **96** (1984), 127–132.

Present Address:
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE, NAGOYA UNIVERSITY,
CHIKUSA-KU, NAGOYA, 464–01 JAPAN.
e-mail: takayama@math.nagoya-u.ac.jp