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# Compact Quotient Manifolds of Domains in a Complex 3-Dimensional Projective Space and the Lebesgue Measure of Limit Sets

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**Abstract.** We shall consider compact complex manifolds of dimension three which have subdomains in a three dimensional projective space as their unramified even coverings. Assume that the subdomains contain projective lines. Then any pair of such manifolds can be connected together complex analytically by an analogue of Klein combinations [K1]. In this paper, we shall prove a (weak) analogue of two results of B. Maskit [M] on the Lebesgue measures of the limit set of Kleinian groups (Theorems A and B).

In section 1, we shall give definitions of terms and precise statement of our result. In section 2, we shall analyze the limit set by the same method as that of Maskit. Section 3 is devoted to proving Theorem 2.1. As a corollary to Theorem 2.1, we obtain Theorem A, which is an analogue of Combination Theorem I of [M]. In section 4, we shall prove Theorem B, which is an analogue of Combination Theorem II of [M].

## 1. Introduction.

A complex manifold $X$, $\dim X=3$, is of Class $L$ by definition, if $X$ contains a subdomain onto which there is a biholomorphic map from a neighborhood of a projective line in a three dimensional complex projective space $\mathbf{P}^3$. The biholomorphic image in $X$ of the projective line is called a *line*.

*Klein combination* of Class $L$ manifolds is defined as follows ([K2]). Let $X_\nu$, $\nu=1, 2$, be manifolds of Class $L$. Let $\Sigma$ be a connected and simply connected smooth real hypersurface in $\mathbf{P}^3$, and $W$ a tubular neighborhood of $\Sigma$. Let $W_1'$ and $W_2'$ be the connected components of $\mathbf{P}^3 \setminus \Sigma$. Put $W_1=W_1' \cup W$ and $W_2=W_2' \cup W$. Suppose that there are open holomorphic embeddings $j_\nu\colon W_\nu \to X_\nu$. Then the Klein combination $Kl(X_1, X_2, j_1, j_2, \Sigma)$ of $X_1$ and $X_2$ is the union $X_1^\# \cup X_2^\#$, $X_\nu^\#=X_\nu \setminus j_\nu(W_\nu \setminus W)$, where $j_1(x) \in j_1(W)$, $x \in W$, is identified with $j_2(x) \in j_2(W)$. It is easy to check that a Klein combination can be defined for a pair $X_1$, $X_2$ of Class $L$ manifolds if both $W_1'$ and $W_2'$ are of Class $L$. Let $\Sigma_0$ denote the hypersurface in $\mathbf{P}^3$ defined by

$$|z_0|^2+|z_1|^2=|z_2|^2+|z_3|^2\,,$$

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where $[z_0 : z_1 : z_2 : z_3]$ is a system of homogeneous coordinates. The Klein combination is called a *connected sum* denoted by $\mathrm{Sum}(X_1, X_2, j_1, j_2, \Sigma)$ or $\mathrm{Sum}(X_1, X_2)$ for short, if the domain $W$ is biholomorphic to a tubular neighborhood of $\Sigma_0$. We can consider a connected sum for any pair of Class $L$ manifolds and the resulting manifold is also of Class $L$.

Let $\Omega$ be a domain in $\mathbf{P}^3$. Assume that $\Omega$ is of Class $L$. Then any holomorphic automorphism of $\Omega$ extends to an element of $PGL_4(\mathbf{C})$ by [K1, Lemma 3.2]. Let $\Gamma$ be a group of holomorphic automorphism of $\Omega$. We can assume that $\Gamma$ is a subgroup of $PGL_4(\mathbf{C})$ without loss of generality. Assume further that the action of $\Gamma$ on $\Omega$ is free and properly discontinuous and that the quotient manifold $\Gamma\backslash\Omega$ is compact. It is easy to see that $\Gamma\backslash\Omega$ is also of Class $L$. In the following, a complex manifold $X$ which can be described as a quotient of a subdomain of $\mathbf{P}^3$ is said to be *covered by a subdomain of* $\mathbf{P}^3$.

It is easy to see that a Klein combination of two manifolds of Class $L$ which are covered by subdomains in $\mathbf{P}^3$ is also covered by a subdomain in $\mathbf{P}^3$ (see [M] or [K2]).

Since a manifold of Class $L$ admits at most one holomorphically flat projective structure ([K1, Proposition 5.1]), the domain $\Omega$ can be recovered, up to compositions of elements of $PGL_4(\mathbf{C})$, from the quotient $\Gamma\backslash\Omega$ as the image of the development of the projective structure. Therefore the complement $\Lambda = \mathbf{P}^3 \setminus \Omega$ is defined canonically by $\Gamma\backslash\Omega$ up to translations by $PGL_4(\mathbf{C})$. It is an easy consequence of a theorem of H. Hopf that the cardinality of the connected component of $\Lambda$ is 0, 1, 2, or that of continuum.

Now we state our main result, which is a (weak) analogue of a part of Combination Theorem I of B. Maskit [M].

THEOREM A. *Let $X_\nu$, $\nu = 1, 2$ be compact manifolds of Class L which are covered by subdomains $\Omega_\nu$ of $\mathbf{P}^3$. Assume that the complements in $\mathbf{P}^3$ of $\Omega_\nu$ have Lebesgue measure zero for $\nu = 1, 2$. Then* $\mathrm{Sum}(X_1, X_2)$ *is covered by a subdomain whose complement in $\mathbf{P}^3$ has also Lebesgue measure zero.*

The theorem above is a corollary to a little more general result (Theorem 2.1) of Klein combination version of Theorem A, which will be proved in section 3.

## 2. Decomposition of limit sets.

Let $X_1 = (\Omega_1, \Gamma_1)$, $X_2 = (\Omega_2, \Gamma_2)$ be compact manifolds of Class $L$ which are covered by subdomains $\Omega_1$, $\Omega_2$ of $\mathbf{P}^3$ and let $X = Kl(X_1, X_2, j_1, j_2, \Sigma)$ be a Klein combination of these two manifolds. Since $X$ is also covered by a subdomain, we can write $X = (\Omega, \Gamma)$. Set $\Lambda = \mathbf{P}^3 \setminus \Omega$, $\Lambda_1 = \mathbf{P}^3 \setminus \Omega_1$ and $\Lambda_2 = \mathbf{P}^3 \setminus \Omega_2$. Let $\mathrm{Vol}_{\mathbf{P}^3}$ denote the volume (Lebesgue measure) defined by a Riemannian metric on $\mathbf{P}^3$. Now we shall describe the condition under which $\mathrm{Vol}_{\mathbf{P}^3}(\Lambda_\nu) = 0$, $\nu = 1, 2$ implies $\mathrm{Vol}_{\mathbf{P}^3}(\Lambda) = 0$. If either $\Gamma_1$ or $\Gamma_2$ is a trivial group, then the implication above holds obviously. Thus it is enough to consider the case where both $\Gamma_1$ and $\Gamma_2$ are non-trivial groups.

First we shall make the following

ASSUMPTION A.

A1. $W$ contains projective lines.

By Assumption A1, $\Omega$ contains a line. Since the action of $\Gamma$ on $\Omega$ is properly discontinuous, we see that the limit of a line in $\Omega$ has no common points with $\Omega$. Hence $\Lambda$ contains a line. Thus let $l_0$ be a line such that

$$l_0 \text{ is contained in } \Lambda . \tag{1}$$

We use the notation of section 1. Let $\check{j}_\nu : W_\nu \to \Omega_\nu \subset \mathbf{P}^3$ be a lift of $j_\nu$. Since $W$ contains projective lines, $\check{j}_\nu$ extends to an element of $PGL_4(\mathbf{C})$ by [K1, Lemma 3.2]. Put $\check{W}_\nu = \check{j}_\nu(W_\nu)$ and $\check{\Sigma}_\nu = \check{j}_\nu(\Sigma)$. Let $F_\nu$ be a fundamental region for $\Gamma_\nu$ on $\Omega_\nu$ which contains $\check{W}_\nu$. Here we can assume that the $F_\nu$ are compact simplicial complexes embedded in $\Omega_\nu$, by considering triangulations of $X_\nu$. By $\check{j}_\nu^{-1}$, we regard $F_\nu$ as a subset in $\mathbf{P}^3$ which contains $W_\nu$, and $\check{\Sigma}_\nu$ as $\Sigma$. Put $F = (F_1 \setminus W_1') \cup (F_2 \setminus W_2')$ and $\Omega = \bigcup_{g \in \Gamma} g(F)$, where $\Gamma$ is the subgroup of $PGL_4(\mathbf{C})$ generated by $\check{j}_\nu^{-1} \Gamma_\nu \check{j}_\nu$, $\nu = 1, 2$. Then it is easy to see that $\Omega$ is an unramified even covering of $X$. Moreover, we can show easily that $F$ is a fundamental region for $\Gamma$ and that $\Gamma$ is the free product of $\check{j}_\nu^{-1} \Gamma_\nu \check{j}_\nu$, $\nu = 1, 2$ by the same argument as the proof of Maskit [M, Proposition 1]. We follow an argument of [M]. Our case is simpler than that of [M]. For a group $G$, we indicate the set $G \setminus \{1\}$ by $G^*$. Every element of $\Gamma^*$ can be written in the *normal form*

$$g = g_n \circ \cdots \circ g_1 , \tag{2}$$

where either $g_{2i} \in \Gamma_1^*$, $g_{2i+1} \in \Gamma_2^*$, or $g_{2i} \in \Gamma_2^*$, $g_{2i+1} \in \Gamma_1^*$. It is well-known that the number of factors $n$ in the right hand side of (2) is determined by the element $g$. We call $n$ the *length* of $g$, and denote the length by $|g|$. We set $|1| = 0$. Furthermore, writing $g$ in the normal form (2), we say that $g$ is *positive* ($g > 0$) if $g_1 \in \Gamma_1^*$, and we say that $g$ is *negative* ($g < 0$) if $g_1 \in \Gamma_2^*$.

By Assumption A1, there is a line $l_\infty$ in $W$. We consider the following sets of lines

$$\mathscr{P}_1 = \{ l \subset \mathbf{P}^3 : g(l) = l_\infty, g > 0 \} , \qquad \mathscr{P}_2 = \{ l \subset \mathbf{P}^3 : g(l) = l_\infty, g < 0 \} ,$$

and their supports

$$|\mathscr{P}_\nu| = \bigcup_{l \in \mathscr{P}_\nu} \operatorname{Supp} l .$$

It is easy to see that the set $|\mathscr{P}_\nu|$ is contained in $\mathbf{P}^3 \setminus W_\nu$.

In what follows, throughout this paper, we let $U$ denote the domain in $\mathbf{P}^3$ defined by

$$U = \{ z = [z_0 : z_1 : z_2 : z_3] : |z_0|^2 + |z_1|^2 < |z_2|^2 + |z_3|^2 \} .$$

We shall assume the following condition on $\Sigma$.

ASSUMPTION A (continued). There are subdomains $W''_\nu$, $\nu=1,2$, which admit open coverings $\{U^1_\nu, \cdots, U^{r_\nu}_\nu\}$ such that

A2. each $W''_\nu$ contains $W'_\nu \cup \Sigma$,

A3. each $U^j_\nu$ is biholomorphic to $U$,

A4. the closure of each $U^j_\nu$ does not intersect the closure of $|\mathscr{P}_\nu|$,

A5. each $U^j_\nu$ satisfies either $U^j_\nu \cap U(l_0)=\varnothing$ or $U^j_\nu \cap U(l_\infty)=\varnothing$, where $U(l_0)$ (resp. $U(l_\infty)$) is a small neighborhood of $l_0$ (resp. $l_\infty$).

It is easy to see that if there are open coverings which satisfy A2, $\cdots$, A4, we can replace them by those which satisfy also A5. Note that Assumptions A1, $\cdots$, A5 are obviously fulfilled in the case of connected sum. We shall prove the following

THEOREM 2.1. *If $\Sigma$ satisfies Assumptions* A1, $\cdots$, A5, *then* $\mathrm{Vol}_{\mathbf{P}^3}(\Lambda_1)=\mathrm{Vol}_{\mathbf{P}^3}(\Lambda_2)=0$ *implies* $\mathrm{Vol}_{\mathbf{P}^3}(\Lambda)=0$.

This section and the next are devoted to proving the theorem above. To prove the theorem, we decompose $\Gamma$ into the sets of positive elements, negative elements and the identity element, and write

$$\Gamma=\{1\}+\sum_{n,m} p_{nm}+\sum_{n,m} q_{nm}\,, \tag{3}$$

where $|p_{nm}|=|q_{nm}|=n$, $p_{nm}>0$, and $q_{nm}<0$. For fixed $n>0$, we set

$$T_n=\bigcup_m p_{nm}(W'_1)\cup\bigcup_m q_{nm}(W'_2)\,. \tag{4}$$

LEMMA 2.1. *For $n\geq 1$, $T_{n-1}\supset T_n$ holds.*

PROOF. Let $x\in T_n$ be any element. Suppose $x\in p_{nm}(W'_1)$ for some $p_{nm}>0$. We write $p_{nm}=g_n\circ\cdots\circ g_1$ in the normal form, where $g_1\in\Gamma^*_1$. Since $g_1(W'_1)\subset W'_2$, we have

$$p_{nm}(W'_1)\subset g_n\circ\cdots\circ g_2(W'_2)\,.$$

Since $|g_n\circ\cdots\circ g_2|=n-1$ and $g_n\circ\cdots\circ g_2$ is negative, we conclude $x\in T_{n-1}$. We can settle the case $x\in q_{nm}(W'_2)$ for some $q_{nm}<0$ in the same manner. □

LEMMA 2.2. *For any $g\in\Gamma$ with $|g|\leq n$, $n\geq 1$, $g(T_n)\subset T_{n-|g|}$ holds.*

PROOF. Take any $x\in T_n$. We write $g=g_k\circ\cdots\circ g_1$ in the normal form, $k=|g|$. Then we see easily that $g(x)\in\bigcup_{l=n-k}^{n+k}T_l$. Since $\{T_n\}$ is a descending sequence by Lemma 2.1, we have $g(x)\in T_{n-k}$. □

We set

$$T=\bigcap_{n\geq 1}T_n\,.$$

As a corollary to Lemma 2.2, we have immediately

LEMMA 2.3. *For any $g \in \Gamma$, $g(T) \subset T$ holds and hence $T$ is $\Gamma$-invariant.*

LEMMA 2.4. *For any $g \in \Gamma$, $T_n \cap g(F) = \emptyset$ holds for $n \geq |g| + 1$.*

PROOF. It is clear by the definition that $T_1 \cap F = \emptyset$. Since $\{T_n\}$ is a descending sequence by Lemma 2.1, we have

$$T_n \cap F = \emptyset \tag{5}$$

for $n \geq 1$. We shall prove the lemma by induction on $k = |g|$. The lemma holds for $k = 0$ by (5). Suppose that $k \geq 1$. Put $g = g_k \circ h$, $|h| = k - 1$, $g_k \in \Gamma_\nu^*$, where $\nu = 1$ or 2. Then we have

$$T_n \cap g(F) = T_n \cap g_k \circ h(F) = g_k(g_k^{-1}(T_n) \cap h(F)) . \tag{6}$$

For $n \geq 1$, we have

$$g_k^{-1}(T_n) \subset T_{n-1}$$

by Lemma 2.1. By the induction assumption, we have

$$T_{n-1} \cap h(F) = \emptyset .$$

Hence by (6), we have

$$T_n \cap g(F) = g_k(g_k^{-1}(T_n) \cap h(F)) \subset g_k(T_{n-1} \cap h(F)) = \emptyset . \qquad \square$$

LEMMA 2.5. $T \subset \Lambda$.

PROOF. We shall prove the lemma by absurdity. Suppose there were a point $x \in T$ which were contained in $\Omega = \mathbf{P}^3 \setminus \Lambda$. Let $K$ be a compact neighborhood of $x$ in $\Omega$. By the definition of $T$ and the fact that $\{T_n\}$ is a descending sequence,

$$x \in K \cap T_n \qquad \text{for all } n \geq 1 . \tag{7}$$

Since $K$ and $F$ are compact, and since $F$ is a fundamental region, there is a finite number of elements, $g_1, g_2, \cdots, g_r \in \Gamma$ such that

$$K \subset \bigcup_{j=1}^{r} g_j(F) \qquad \text{and}$$

$$K \cap g(F) = \emptyset \qquad \text{for all } g \in \Gamma \setminus \{g_1, g_2, \cdots, g_r\} .$$

Hence, by (7),

$$\bigcup_{j=1}^{r} g_j(F) \cap T_n \supset K \cap T_n \supset \{x\} \neq \emptyset$$

for all $n \geq 1$. On the other hand, by Lemma 2.4, we have $T_n \cap g_j(F) = \emptyset$ for all integers $n$ with

$$n \geq \max_{j=1,\cdots,r}\{|g_j| + 1\} .$$

Thus we have a contradiction for such $n$. □

We set

$$S_n=\mathbf{P}^3\setminus T_n\,,\qquad S=\bigcup_n S_n\,.$$

Then we have

LEMMA 2.6. $\Lambda=(\Lambda\cap S)\cup T$.

PROOF. Since $\mathbf{P}^3=S\cup T$, the lemma follows immediately from Lemma 2.5. □

Following three lemmas follow immediately from Lemmas 2.1, 2.2 and 2.3.

LEMMA 2.7. *For* $n\geq 1$, $S_{n-1}\subset S_n$.

LEMMA 2.8. *For any* $g\in\Gamma$ *and* $n\geq 1$, $g(S_n)\supset S_{n-|g|}$ *holds.*

LEMMA 2.9. *For any* $g\in\Gamma$, $g(S)\subset S$ *holds and hence* $S$ *is* $\Gamma$*-invariant.*

It is easy to verify the following.

LEMMA 2.10. *If* $x\in\mathbf{P}^3$ *satisfies*

$$x\notin\bigcup_{g\in\Gamma}g(\Lambda_1)\cup\bigcup_{g\in\Gamma}g(\Lambda_2)\,, \tag{8}$$

*then*

$$g(x)\in\Omega_1\cap\Omega_2 \qquad \textit{for all } g\in\Gamma\,. \tag{9}$$

LEMMA 2.11. $S\subset\Omega\cup\bigcup_{g\in\Gamma}g(\Lambda_1)\cup\bigcup_{g\in\Gamma}g(\Lambda_2)$.

PROOF. Take $x\in S$ satisfying (8). It is enough to show that $x\in\Omega$. Since $x\in S$ and $\{S_n\}$ is ascending, there is an integer $n_0$ such that $x\in S_{n_0}\setminus S_{n_0-1}$. Then $x\in T_{n_0-1}=\mathbf{P}^3\setminus S_{n_0-1}$. Therefore either there is a point $w_1\in W_1'$ and an element $p_{n_0-1\,m}>0$ in $\Gamma$ with $x=p_{n_0-1\,m}(w_1)$, or there is a point $w_2\in W_2'$ and an element $q_{n_0-1\,m}<0$ in $\Gamma$ with $x=q_{n_0-1\,m}(w_2)$. Suppose the former case holds; the proof of the latter case is similar. If $w_1\in T_1$, then there is a point $v_2\in W_2'$ and $q\in\Gamma_2^*$ such that $w_1=q(v_2)$. Hence we have $x=p_{n_0-1\,m}\circ q(v_2)$. Since $p_{n_0-1\,m}\circ q$ is negative and has length $n_0$, we see that $x\in T_{n_0}$. But this contradicts $x\in S_{n_0}$. Hence $w_1\notin T_1$, i.e., $w_1\in S_1$. Namely, for every element $x\in S_n$ with $n>1$, there is an element $g\in\Gamma$ such that $g(x)\in S_1$. On the other hand, by the choice of $x$ and by Lemma 2.10, we also have $g(x)\in\Omega_1\cap\Omega_2$. If $g(x)\notin W_1'\cup W_2'$, i.e., $g(x)\in\Sigma$, then $g(x)\in F$. Hence $x\in\Omega$. If $g(x)\in W_2'$, then $g(x)\notin W_1'$. Since $g(x)\in\Omega_1$, $g_1\circ g(x)\in F_1$ holds for some $g_1\in\Gamma_1$. Suppose that $g_1\circ g(x)\in W_1'$. Then we have $g_1\neq 1$ and $g(x)=g_1^{-1}(g_1\circ g(x))\in g_1^{-1}(W_1')\subset T_1$. This contradicts $g(x)\in S_1$. Therefore $g_1\circ g(x)\in F_1\setminus W_1'=F$. Hence $x\in\Omega$. The remaining case $g(x)\in W_1'$ can be settled similarly. □

LEMMA 2.12. $\Lambda\supset\bigcup_{g\in\Gamma}g(\Lambda_1)\cup\bigcup_{h\in\Gamma}h(\Lambda_2)$.

PROOF. Since $\Lambda$ is $\Gamma$-invariant, it is enough to show that $\Omega \subset \Omega_\nu$. We consider the case $\nu=1$. The proof works also for $\nu=2$. Recall that $\Omega = \bigcup_{g \in \Gamma} g(F)$ and $\Omega_1 = \bigcup_{g \in \Gamma_1} g(F_1)$. The lemma follows immediately from the following.

SUBLEMMA 2.1. *Let $g$ be an element of $\Gamma$ and put $k = |g|$. If $g$ is positive, then $g(F) \subset F_2 \cap \Omega_1$ for odd $k$, and $g(F) \subset F_1$ for even $k$. If $g$ is negative, then $g(F) \subset F_1$ for odd $k$, and $g(F) \subset F_2 \cap \Omega_1$ for even $k$.*

PROOF. Consider the case $g$ positive. We shall prove the sublemma by induction on $k$. If $k=1$, then $g \in \Gamma_1^*$. Hence $g(F) \subset g(F_1) \subset F_2 \cap \Omega_1$. If $k=2$, then $g = g_2 \circ g_1$ for some $g_1 \in \Gamma_1^*$ and $g_2 \in \Gamma_2^*$. Hence

$$g(F) \subset g_2 \circ g_1(F_1) \subset g_2(F_2 \cap \Omega_1) \subset g_2(F_2) \subset F_1 .$$

Suppose that $k \geq 3$. Write $g$ as $g = g_k \circ h$, $h = g_{k-1} \circ \cdots \circ g_1$. If $k$ is odd, then $g_k \in \Gamma_1^*$ and $h(F) \subset F_1$ by induction assumption. Hence $g(F) = g_k \circ h(F) \subset g_k(F_1) \subset F_2 \cap \Omega_1$. If $k$ is even, then $g_k \in \Gamma_2^*$ and $h(F) \subset F_2 \cap \Omega_1$ by induction assumption. Hence

$$g(F) = g_k \circ h(F) \subset g_k(F_2 \cap \Omega_1) \subset g_k(F_2) \subset F_1 .$$

The case $g$ negative can be settled similarly. □

PROPOSITION 2.1. $\Lambda = \bigcup_{g \in \Gamma} g(\Lambda_1) \cup \bigcup_{h \in \Gamma} h(\Lambda_2) \cup T.$

PROOF. By Lemmas 2.6 and 2.11, we have

$$\Lambda \subset \left( \Omega \cup \bigcup_{g \in \Gamma} g(\Lambda_1) \cup \bigcup_{h \in \Gamma} h(\Lambda_2) \right) \cup T .$$

Hence, by $\Omega \cap \Lambda = \varnothing$, we obtain

$$\Lambda \subset \bigcup_{g \in \Gamma} g(\Lambda_1) \cup \bigcup_{h \in \Gamma} h(\Lambda_2) \cup T .$$

The other implication follows from Lemmas 2.12 and 2.5. □

REMARK 2.1. By Assumption A1 and the definition of $T$, $T$ contains lines.

## 3. Volume of limit sets.

We use the notation in the earlier sections. Let $[z_0 : z_1 : z_2 : z_3]$ be a system of homogeneous coordinates on $\mathbf{P}^3$ such that the two lines $l_0$, $l_\infty$ in Assumptions A2, $\cdots$, A5 are given by

$$l_0 : z_0 = z_1 = 0 , \qquad l_\infty : z_2 = z_3 = 0 .$$

Recall that

$$(10) \qquad l_0 \text{ is contained in } \Lambda ,$$

(11) $$l_\infty \text{ is contained in } W.$$

By a small displacement of $l_\infty$ if necessary, we can assume without loss of generality that

(12) $$l_\infty \text{ is mapped bijectively into } X=\Gamma\backslash\Omega$$

by the canonical projection. Every line in $\mathbf{P}^3\setminus l_\infty$ can be written uniquely in the form

$$z'=Xz''$$

where $z'={}^t(z_0, z_1)$, $z''={}^t(z_2, z_3)$ and $X$ is a $2\times 2$ matrix with complex components, $X\in M_2(\mathbf{C})$. Consider the subgroup in $SL_4(\mathbf{C})$ defined by

$$\tilde{\Gamma}=\{\tilde{g}\in SL_4(\mathbf{C}) : \tilde{g} \text{ represents an element } g\in\Gamma\}.$$

Set

$$\tilde{\Gamma}^*=\{\tilde{g}\in\tilde{\Gamma} : \tilde{g} \text{ is } \textit{not} \text{ a scalar matrix}\}.$$

Note that the equality

(13) $$\det(AC^{-1}DC-BC)=1$$

holds for any $\tilde{g}=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in SL_4(\mathbf{C})$ with $\det C\neq 0$, where $A$, $B$, $C$, $D\in M_2(\mathbf{C})$. We identify the set of quaternions with the set of $2\times 2$ matrices

$$\mathbf{H}=\left\{X=\begin{pmatrix} x_1 & -\bar{x}_2 \\ x_2 & \bar{x}_1\end{pmatrix} : x_1, x_2\in\mathbf{C}\right\}.$$

On $M_2(\mathbf{C})$, we define a norm by

$$\|X\|=\left(\sum_{1\leq i,j\leq 2}|x_{ij}|^2\right)^{1/2},$$

where $x_{ij}$ is the $(i, j)$-component of $X\in M_2(\mathbf{C})$. Define a diffeomorphism

(14) $$\begin{aligned}\phi: \quad \mathbf{P}^3\setminus l_\infty &\longrightarrow \mathbf{H}\times\mathbf{P}^1 \\ [z_0:z_1:z_2:z_3] &\longmapsto (X, z'')\end{aligned}$$

by

$$X=\begin{pmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0\end{pmatrix}\begin{pmatrix} z_2 & -\bar{z}_3 \\ z_3 & \bar{z}_2\end{pmatrix}^{-1}, \qquad z''=[z_2:z_3].$$

Let

$$\pi: \mathbf{H}\times\mathbf{P}^1\to\mathbf{H}$$

be the projection to the first component. By the assumption (10), there is a positive number $R$ such that the set

$$V_R=\pi^{-1}(\{\|X\|>R\})\cup l_\infty$$

is contained in $\Omega$. Note that $V_R$ is biholomorphic to the domain $U$. Further, if $R$ is sufficiently large, then

$$g(V_R)\cap V_R=\varnothing \tag{15}$$

holds for any $g\in\Gamma^*$ by the assumption (12). For later use, we put

$$V_R^*=\pi^{-1}(\{\|X\|>R\}), \qquad U_R=\pi^{-1}(\{\|X\|<R\}).$$

LEMMA 3.1. *For any $\tilde g=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\tilde\Gamma^*$, we have the following.*

$$\det C\neq 0 \tag{16}$$

$$\det D\neq 0 \tag{17}$$

$$\|AC^{-1}\|\leq R_0 \tag{18}$$

$$\|C^{-1}D\|\leq R_0 \tag{19}$$

*where $R_0$ is a positive number which is independent of $\tilde g$.*

PROOF. We fix an $R_0=R$ which satisfies (15).

Proof of (16). Suppose that $\det C=0$. Then there is a point $z$ on $l_\infty$ such that $g(z)\in l_\infty$. Thus $g(l_\infty)\cap l_\infty\neq\varnothing$. Since $g\neq 1$ by assumption, this contradicts (15).

Proof of (17). Suppose that $\det D=0$. Then there is a point $z$ on $l_0$ such that $g(z)\in l_\infty$. Thus $g(l_0)\cap l_\infty\neq\varnothing$. Since $g(\Omega)=\Omega$, this contradicts (10).

Proof of (18). The equation of the line $g(l_\infty)$ is given by $z'=AC^{-1}z''$. Since $g(l_\infty)\cap V_{R_0}=\varnothing$ by (15), we have $\|AC^{-1}\|\leq R_0$.

Proof of (19). The equation of the line $g^{-1}(l_\infty)$ is given by $z'=A'C'^{-1}z''$, where $g^{-1}=\begin{pmatrix} A' & B'\\ C' & D'\end{pmatrix}$. Since $A'C'^{-1}=-C^{-1}D$, we have $\|C^{-1}D\|\leq R_0$ by the argument above. □

For an element $\tilde g=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in SL_4(\mathbf{C})$, the composition $\bar g=\phi\circ\tilde g\circ\phi^{-1}$ is defined on $(\mathbf{H}\times\mathbf{P}^1)\setminus\mathscr{P}_g$, where

$$\mathscr{P}_g=\{(X,z'')\in\mathbf{H}\times\mathbf{P}^1 : (CX+D)z''=0\}.$$

In particular, every $\bar g$ with $\tilde g\in\tilde\Gamma$ is defined on $V_R^*$. In the following, we write $\tilde g$ instead of $\bar g$ to avoid abuse of symbols. On $\mathbf{H}\times\mathbf{P}^1$, we introduce a volume form $dV$ by

$$dV=dv(X)\wedge\omega_{\mathbf{P}^1}(z), \tag{20}$$

where

$$dv(X)=\left(\frac{\sqrt{-1}}{2}\right)^2 dx_1\wedge d\bar{x}_1\wedge dx_2\wedge d\bar{x}_2\,,\qquad X=\begin{pmatrix} x_1 & -\bar{x}_2\\ x_2 & \bar{x}_1\end{pmatrix},$$

and $\omega_{\mathbf{P}^1}=\omega_{\mathbf{P}^1}(z)$, $z=[z_0:z_1]$, is the Fubini-Kähler form

$$i\bar{\partial}\partial\log(|z_0|^2+|z_1|^2)$$

on $\mathbf{P}^1$. For a 2-vector $w=\begin{pmatrix} w_1\\ w_2\end{pmatrix}$, we put $\|w\|=(|w_1|^2+|w_2|^2)^{1/2}$.

LEMMA 3.2. *For* $\tilde{g}=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in SL_4(\mathbf{C})$ *with* $\det C\neq 0$, *the pull-back of* $dV$ *is given by*

$$\tilde{g}^*dV=\left(\frac{\|z''\|}{\|(X+C^{-1}D)z''\|}\right)^4\left(\frac{\|z''\|}{\|(CX+D)z''\|}\right)^8|\det(X+C^{-1}D)|^2 dv(X)\wedge\omega_{\mathbf{P}^1}(z'')$$

*on* $(\mathbf{H}\times\mathbf{P}^1)\setminus\mathscr{P}_g$, *where the homogeneous coordinates* $z''$ *on* $\mathbf{P}^1$ *are regarded as a* 2-*vector.*

PROOF. Any element $\tilde{g}$ with $\det C\neq 0$ can be written as a composition $\tilde{g}=\tilde{g}_5\circ\tilde{g}_4\circ\tilde{g}_3\circ\tilde{g}_2\circ\tilde{g}_1$ of the following;

$$\tilde{g}_1=\begin{pmatrix} I & C^{-1}D\\ 0 & I\end{pmatrix},\quad \tilde{g}_2=\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix},\quad \tilde{g}_3=\begin{pmatrix} B-AC^{-1}D & 0\\ 0 & I\end{pmatrix},$$

$$\tilde{g}_4=\begin{pmatrix} I & 0\\ 0 & C\end{pmatrix},\quad \tilde{g}_5=\begin{pmatrix} I & AC^{-1}\\ 0 & I\end{pmatrix}.$$

It is easy to check that the volume form $dV$ is invariant by the translations $\tilde{g}_1$ and $\tilde{g}_5$. Let $\gamma(\tilde{g})=\gamma(\tilde{g}, X)$ denote the automorphism of $\mathbf{P}^1$ induced by the linear map

$$z\mapsto(CX+D)z\,,\qquad z\in\mathbf{C}^2\,.$$

Then, by somewhat tedious calculations, we have

$$\tilde{g}_2^*dV=\|X\|^{-8}dv(X)\wedge\gamma(\tilde{g}_2)^*\omega_{\mathbf{P}^1}\,,$$

$$\tilde{g}_3^*dV=|\det(B-AC^{-1}D)|^2 dv(X)\wedge\gamma(\tilde{g}_3)^*\omega_{\mathbf{P}^1}\,,$$

$$\tilde{g}_4^*dV=\left(\frac{\|z''\|}{\|Cz''\|}\right)^4 dv(X)\wedge\gamma(\tilde{g}_4)^*\omega_{\mathbf{P}^1}\,,$$

$$\tilde{g}_1^*(\|X\|)=\|z''\|^{-1}\|(X+C^{-1}D)z''\|\,.$$

By composition of these equations and (13), we obtain

$$\tilde{g}^*dV=|\det C|^{-2}\left(\frac{\|z''\|^2}{\|(X+C^{-1}D)z''\|\,\|(CX+D)z''\|}\right)^4 dv(X)\wedge\gamma(\tilde{g})^*\omega_{\mathbf{P}^1}(z'')\,.$$

Since

$$\gamma(\tilde{g})^*\omega_{\mathbf{P}^1}(z'')=|\det(CX+D)|^2\left(\frac{\|z''\|}{\|(CX+D)z''\|}\right)^4\omega_{\mathbf{P}^1}(z'')$$

we have the lemma. $\square$

PROPOSITION 3.1. $\sum_{\tilde{g}\in\tilde{\Gamma}^*}\|C(g)\|^4|\det C(g)|^{-6}<\infty$.

PROOF. For simplicity, put $A=A(g)$, $B=B(g)$, $C=C(g)$ and $D=D(g)$. By (15), all $g(V_R)$, $g\in\Gamma^*$, are mutually disjoint and contained in the closure $[U_R]$ of $U_R$. Therefore we have

$$\sum_{\tilde{g}\in\tilde{\Gamma}^*}\int_{g(V_R)}dV<\infty \tag{21}$$

for any sufficiently large $R>0$. Now we shall estimate the value $\int_{g(V_R)}dV$ from below. Letting $y'=\gamma(\tilde{g})z''$ and $Y=X^{-1}=\begin{pmatrix} y_1 & -\bar{y}_2 \\ y_2 & \bar{y}_1\end{pmatrix}$, we have by Lemma 3.2 that

$$\tilde{g}^*dV=|\det C|^{-2}\left(\frac{M(Y,y')}{\|C^{-1}y'\|\,\|y'\|}\right)^4 dv(Y)\wedge\omega_{\mathbf{P}^1}(y')\,, \tag{22}$$

where

$$M(Y,y')=\|Y(I+C^{-1}DY)^{-1}C^{-1}y'\|^2\|Y\|^{-2}\,. \tag{23}$$

For $\|Y\|<R^{-1}$, $R>2R_0$, it follows from Lemma 3.1 that

$$\|(I+C^{-1}DY)^{-1}-I\|\le(1-R_0R^{-1})^{-1}R_0R^{-1}\,.$$

Hence, using the fact that $Y\|Y\|^{-1}\in SU(2)$, we have

$$\begin{aligned} M(Y,y')&\ge(\|YC^{-1}y'\|-\|Y((I+C^{-1}DY)^{-1}-I)C^{-1}y'\|)^2\|Y\|^{-2}\\ &\ge(\|C^{-1}y'\|-(1-R_0R^{-1})^{-1}R_0R^{-1}\|C^{-1}y'\|)^2\\ &=(1-R_0R^{-1})^{-1}(1-2R_0R^{-1})\|C^{-1}y'\|^2\,.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\int_{g(V_R)}dV&=\int_{g(V_R^*)}dV=\int_{V_R^*}\tilde{g}^*dV\\ &\ge M_1|\det C|^{-2}\int_{0<\|Y\|<R^{-1},\,y'\in\mathbf{P}^1}\left(\frac{\|C^{-1}y'\|}{\|y'\|}\right)^4 dv(Y)\wedge\omega_{\mathbf{P}^1}(y')\\ &=M_1|\det C|^{-6}\int_{\|Y\|<R^{-1}}dv(Y)\int_{y'\in\mathbf{P}^1}\left(\frac{\|\tilde{C}y'\|}{\|y'\|}\right)^4\omega_{\mathbf{P}^1}(y')\,,\end{aligned}$$

where $M_1>0$ is a constant independent of $g$ and $\tilde{C}$ is the cofactor matrix of $C$. By an elementary calculation, we have

$$\int_{y'\in\mathbf{P}^1}\left(\frac{\|\tilde{C}y'\|}{\|y'\|}\right)^4\omega_{\mathbf{P}^1}(y')\geq\frac{\pi}{24}\left(6-\frac{9\pi^2}{16}\right)\|C\|^4 .$$

Thus we have

$$\int_{g(V_{\mathbf{R}})}dV\geq M_2|\det C(g)|^{-6}\|C(g)\|^4 ,$$

where $M_2>0$ is a constant independent of $g$. Combining this inequality with (21), we obtain the proposition. □

Now recall Assumptions A2, $\cdots$, A5. Put

$$V_\nu=\bigcup_{j=1}^{r_\nu}U_\nu^j , \tag{24}$$

$$V_{(n)}=\bigcup_{m,\,p_{nm}>0}p_{nm}(V_1)\cup\bigcup_{m,q_{nm}<0}q_{nm}(V_2) , \tag{25}$$

$$V^{(n)}=\bigcap_{k=1}^{n}V_{(k)} , \tag{26}$$

$$V=\lim_n V^{(n)} . \tag{27}$$

Obviously, we have $V_\nu\supset[W_\nu]$. Note that, the set $g(V_1)$ (resp. $g(V_2)$) is contained in $\mathbf{P}^3\setminus l_\infty$ for $g\in\Gamma^*$ with $g>0$ (resp. $g<0$) by Assumption A4.

LEMMA 3.3. *The inequality*

$$\int_{g(V_\nu)}dV\leq M\|C(g)\|^4|\det C(g)|^{-6} \tag{28}$$

*holds for any* $g\in\Gamma^*$ *with* $g>0$ *if* $\nu=1$, *and for any* $g\in\Gamma^*$ *with* $g<0$ *if* $\nu=2$, *where* $M$ *is a positive constant which is independent of* $g$.

PROOF. Since $V_\nu$ is covered by finitely many $U_\nu^j$'s, it is enough to prove the case $V_\nu=U_\nu^j$;

$$\int_{g(U_\nu^j)}dV\leq M\|C(g)\|^4|\det C(g)|^{-6} . \tag{29}$$

First we consider the case where $U_\nu^j\cap U(l_\infty)=\varnothing$. There exists an element $P=\begin{pmatrix}P_1 & P_2\\ P_3 & P_4\end{pmatrix}\in SL_4(\mathbf{C})$ which sends $U$ onto $U_\nu^j$ bijectively ([K1, Lemma 3.2]). We introduce a system of coordinates on $U$ by the restriction of the differeomorphism $\phi$

(see (14)) to $U$. Note that $U=\pi^{-1}(\{\|X\|<1\})$. The image $(X, x'')\in U_\nu^j$ of $(Z, z'')\in U$ by the map $P$ is determined by

$$\begin{cases} Xx''=(P_1Z+P_2)z'' \\ x''=(P_3Z+P_4)z'' , \end{cases}$$

where $X$, $Z\in\mathbf{H}$. Letting $x'=Xx''$, from Lemma 3.2 we get

$$\begin{aligned}\int_{g(U_\nu^j)} dV &= \int_U P^*g^*dV \\ &= \int_U \left(\frac{\|x''\|}{\|x'+C^{-1}Dx''\|}\right)^4 \left(\frac{\|(P_3Z+P_4)z''\|}{\|(C(P_1Z+P_2)+D(P_3Z+P_4))z''\|}\right)^8 \\ &\quad \times|\det(X+C^{-1}D)|^2\rho(Z, z'')dv(Z)\wedge\omega_{\mathbf{P}^1}(z'') ,\end{aligned}$$

where $\rho(Z, z'')$ is a differentiable function defined by

$$dv(X)\wedge\omega_{\mathbf{P}^1}(x'')=\rho(Z, z'')dv(Z)\wedge\omega_{\mathbf{P}^1}(z'') .$$

Obviously, $\rho(Z, z'')$ is determined independently of $\tilde{g}$ and satisfies

$$m_1\le\rho(Z, z'')\le m_2 \tag{30}$$

on $U$ for some positive numbers $m_1$, $m_2$. Now we shall prepare two sublemmas.

SUBLEMMA 3.1. *The inequality*

$$\|x'+C(g)^{-1}D(g)x''\|\ge\delta_1\|x''\|$$

*holds on $U_\nu^j$ for any $g\in\Gamma^*$ with $g>0$ if $\nu=1$, and for any $g\in\Gamma^*$ with $g<0$ if $\nu=2$, where $\delta_1$ is a positive constant which is independent of $g$.*

PROOF. Suppose that, for each $n>0$, there are $g_n\in\Gamma^*$ with $g_n>0$ (resp. $g_n<0$) if $\nu=1$ (resp. $\nu=2$), and a point $[x'_n, x''_n]$ in $U_\nu^j$ such that

$$\|x'_n+C(g_n)^{-1}D(g_n)x''_n\|<(1/n)\|x''_n\| . \tag{31}$$

Since $U_\nu^j$ does not intersect $U(l_\infty)$, there is a positive number $R$ such that $U_\nu^j\subset\pi^{-1}(\{\|X\|\le R\})$. Hence, choosing a subsequence, we can assume that the sequence $\{[x'_n, x''_n]\}$ converges to a point $x_0=[x'_0, x''_0]\in[U_\nu^j]$, and $\{C(g_n)^{-1}D(g_n)\}$ converges to some matrix $Q$ by Lemma 3.1. Then, in the limit, we have

$$x'_0+Qx''_0=0 .$$

Since the line $x'+C(g_n)^{-1}D(g_n)x''=0$ is a member of $\mathscr{P}_\nu$, the equality above implies that the limit point $x_0$ is in $[|\mathscr{P}_\nu|]$ and consequently $[|\mathscr{P}_\nu|]\cap[U_\nu^j]\ne\varnothing$. This contradicts Assumption A4. $\square$

SUBLEMMA 3.2. *The inequality*

$$|\det((P_1Z+P_2)+C(g)^{-1}D(g)(P_3Z+P_4))|\geq\delta_2$$

*holds on $\pi(U)$ for any $g\in\Gamma^*$ with $g>0$ if $\nu=1$, and for any $g\in\Gamma^*$ with $g<0$ if $\nu=2$, where $\delta_2$ is a positive constant which is independent of $g$.*

PROOF. Suppose that, for each $n>0$, there are $g_n\in\Gamma^*$ with $g_n>0$ (resp. $g_n<0$) if $\nu=1$ (resp. $\nu=2$), and a point $Z_n\in\pi(U)$ such that

$$|\det((P_1Z_n+P_2)+C(g_n)^{-1}D(g_n)(P_3Z_n+P_4))|\leq 1/n\,. \tag{32}$$

Choosing a subsequence, we can assume that the sequence $\{Z_n\}$ converges to a point $Z_0\in[\pi(U)]$, and $\{C(g_n)^{-1}D(g_n)\}$ converges to some matrix $Q$ by Lemma 3.1. Then in the limit, we have

$$\det((P_1Z_0+P_2)+Q(P_3Z_0+P_4))=0\,.$$

Hence there is a point $z_0''\in\mathbf{P}^1$ such that

$$((P_1Z_0+P_2)+Q(P_3Z_0+P_4))z_0''=0\,.$$

Put $x_n'=(P_1Z_n+P_2)z_0''$ and $x_n''=(P_3Z_n+P_4)z_0''$. Since the limit line $x'+Qx''=0$ is contained in $[|\mathscr{P}_\nu|]$ and since the limit point $x_0$ of the sequence $\{x_n\}$, $x_n=[x_n', x_n'']$, is on the limit line, $x_0$ is in $[|\mathscr{P}_\nu|]$. On the other hand, each $x_n$ is contained in the image of $U$, i.e., $x_n\in U_\nu^j$. Consequently, $\{x_0\}\subset[|\mathscr{P}_\nu|]\cap[U_\nu^j]\neq\varnothing$. This contradicts Assumption A4. □

Now we shall continue proving (29). Put

$$F=(P_1Z+P_2)+C^{-1}D(P_3Z+P_4)\,.$$

By Sublemma 3.1, it follows from (30) that

$$\int_{g(U_\nu^j)}dV\leq\int_U\delta_1^{-4}\left(\frac{\|(P_3Z+P_4)z''\|}{\|CFz''\|}\right)^8|\det(X(Z,z'')+C^{-1}D)|^2 \tag{33}$$
$$\times\rho(Z,z'')dv(Z)\wedge\omega_{\mathbf{P}^1}(z'')\,,$$

where $X(Z,z'')$ indicates the $X$-coordinate of the point in $U_\nu^j$ which corresponds to $(Z,z'')\in U$. Put $u''=CFz''$. Then the right-hand side of the inequality above is equal to

$$\int_U\delta_1^{-4}\left(\frac{\|(P_3Z+P_4)(CF)^{-1}u''\|}{\|u''\|}\right)^8|\det(X(Z,(CF)^{-1}u'')+C^{-1}D)|^2$$
$$\times\rho(Z,(CF)^{-1}u'')\left(\frac{\|u''\|}{\|(CF)^{-1}u''\|}\right)^4|\det(CF)|^{-2}dv(Z)\wedge\omega_{\mathbf{P}^1}(u'')\,.$$

Since $U_\nu^j$ does not intersect a neighborhood of $l_\infty$, there is a positive number $R$ such that $U_\nu^j\subset\pi^{-1}(\{\|X\|\leq R\})$. Therefore by Lemma 3.1, there is a positive constant $M_3$

which is independent of $g$ such that

$$|\det(X(Z,(CF)^{-1}u'')+C^{-1}D)|\leq M_3\,. \tag{34}$$

By the same reason, it follows that

$$\|F\|=\|\tilde{F}\|\leq M_4 \tag{35}$$

holds for some $M_4>0$. Hence, by Sublemma 3.2, we have

$$|\det F|^{-1}\leq \delta_2^{-1}\,, \tag{36}$$

$$\|F^{-1}\|\leq M_4\delta_2^{-1}\,. \tag{37}$$

It is clear that

$$\|P_3Z+P_4\|\leq M_6 \tag{38}$$

holds for some $M_6>0$. Thus combining the inequalities (30) and from (34) to (38), we obtain from (33) that

$$\int_{g(U_\nu^j)} dV\leq M_7|\det C|^{-2}\int_U\left(\frac{\|C^{-1}u''\|}{\|u''\|}\right)^4 dv(Z)\wedge\omega_{\mathbf{P}^1}(u'')\leq M_8\|C\|^4|\det C|^{-6}$$

holds, where $M_7$, $M_8$ are positive constants which are independent of $g$. This proves our lemma in the case where $U_\nu^j\cap U(l_\infty)=\varnothing$.

The other case where $U_\nu^j\cap U(l_0)=\varnothing$ can be settled by the same manner by using coordinates $(Y,z')$ with $Y=X^{-1}$ and $z'=[z_0:z_1]$ on $U_\nu^j$ instead of $(X,z'')$. □

We define the volume form on $\mathbf{P}^3$ by

$$dv_{\mathbf{P}^3}=(\omega_{\mathbf{P}^3})^3\,,$$

and for a measurable set $E$, we put

$$\mathrm{Vol}_{\mathbf{P}^3}(E)=\int_{\mathbf{P}^3}\chi_E dv_{\mathbf{P}^3}\,,$$

where $\chi_E$ is a characteristic function of $E$.

LEMMA 3.4. $\mathrm{Vol}_{\mathbf{P}^3}(V)=0$.

PROOF. For simplicity, we put

$$v(E)=\int_{\mathbf{P}^3\setminus l_\infty}\chi_E\phi^*(dV(X)\wedge\omega_{\mathbf{P}^1})\,,$$

where $\chi_E$ is the characteristic function of $E$. Since $V\subset\mathbf{P}^3\setminus l_\infty$, it is enough to show

$$\int_{\mathbf{P}^3\setminus l_\infty}\chi_V\phi^*(dV(X)\wedge\omega_{\mathbf{P}^1})=0\,.$$

By Proposition 3.1 and Lemma 3.3, we have

$$\sum_n v(V_{(n)}) \leq \sum_{n,\, p_{nm}>0} v(p_{nm}(V_1)) + \sum_{n,\, q_{nm}<0} v(q_{nm}(V_2))$$

$$\leq M \sum_{\tilde{g} \in \tilde{\Gamma}^*} \|C(g)\|^4 |\det C(g)|^{-6} < \infty .$$

Therefore we infer that

$$\lim_n v(V_{(n)}) = 0 .$$

Hence

$$\lim_n v(V^{(n)}) = 0 ,$$

and consequently, we get $v(V)=0$. □

PROPOSITION 3.2. $\mathrm{Vol}_{\mathbf{P}^3}(T)=0$.

PROOF. Since $T \subset V$, this follows from the lemma above immediately. □

PROOF OF THEOREM 2.1. It is enough to show the equality $\mathrm{Vol}_{\mathbf{P}^3}(\Lambda)=0$ under the assumption

$$\mathrm{Vol}_{\mathbf{P}^3}(\Lambda_1) = \mathrm{Vol}_{\mathbf{P}^3}(\Lambda_2) = 0 .$$

By Proposition 2.1, we have

$$\mathrm{Vol}_{\mathbf{P}^3}(\Lambda) \leq \sum_{g \in \Gamma} \mathrm{Vol}_{\mathbf{P}^3}(g(\Lambda_1)) + \sum_{h \in \Gamma} \mathrm{Vol}_{\mathbf{P}^3}(h(\Lambda_2)) + \mathrm{Vol}_{\mathbf{P}^3}(T) .$$

Hence, by the assumption and by Proposition 3.2, we obtain $\mathrm{Vol}_{\mathbf{P}^3}(\Lambda)=0$. □

PROOF OF THEOREM A. Since Assumptions A1, $\cdots$, A5, are satisfied for connected sum, Theorem A follows from Theorem 2.1 immediately. □

REMARK 3.1. There are many examples of compact manifolds of Class $L$ covered by a domain in $\mathbf{P}^3$ whose complement is of volume zero. All examples treated in [K2] have this property. There are compact non-singular quotient of a complement in $\mathbf{P}^3$ of a real 3 or 4-dimensional differentiable manifold (e.g. [K3]), which are twistor spaces over a conformally flat 4-manifolds. Note also that connected sum in differential topology between conformally flat 4-manifolds induces our (complex analytic) connected sum between twistor spaces. But converse is not true.

## 4. Handle Attachment.

In this section, we consider volumes of limits in the case of handle attachments, which corresponds to Combination Theorem II in [M]. Let $X=(\Omega_0, \Gamma_0)$ be a compact manifold of Class $L$ which is covered by a subdomain $\Omega_0$ of $\mathbf{P}^3$. Let $\Sigma$ be a connected and simply connected smooth real hypersurface in $\mathbf{P}^3$, and $W$ a tubular neighborhood of $\Sigma$. Let $W'_1$ and $W'_2$ be the connected components of $\mathbf{P}^3 \setminus \Sigma$. Put $W_1 = W'_1 \cup W$ and $W_2 = W'_2 \cup W$. Suppose that there are open holomorphic embeddings $j_\nu \colon W_\nu \to X$ such that $[j_1(W_1)] \cap [j_2(W_2)] = \varnothing$. We consider the quotient space $X^\#/\sim$, where $X^\# = X \setminus (j_1(W_1 \setminus W) \cup j_2(W_2 \setminus W))$, and $j_1(x) \in j_1(W)$, $x \in W$, is identified with $j_2(x) \in j_2(W)$. The quotient space $X^\#/\sim$ is indicated by $Kl(X, j_1, j_2, \Sigma)$ and called a *Klein combination of the second type* (The Klein combination $Kl(X_1, X_2, j_1, j_2, \Sigma)$ stated in section 1 is said to be of *first type*, if there is any danger of confusion). If $W$ is biholomorphic to a tubular neighborhood of $\Sigma_0$, the Klein combination of the second kind is called a *handle attachment* and denoted by $\mathrm{Ha}(X, j_1, j_2, \Sigma)$, or $\mathrm{Ha}(X)$ for short. In this section, we shall prove the following.

THEOREM B. *Let $X$ be a compact manifold of Class $L$ which is covered by a subdomain $\Omega_0$ of $\mathbf{P}^3$. Assume that the complement in $\mathbf{P}^3$ of $\Omega_0$ has Lebesgue measure zero. Then* $\mathrm{Ha}(X)$ *is covered by a subdomain whose complement in $\mathbf{P}^3$ has also Lebesgue measure zero.*

Similarly to the case of Klein combinations of the first type, there is a little more general result, which we shall explain.

Let $\check{j}_\nu \colon W_\nu \to \Omega_0 \subset \mathbf{P}^3$ be a lift of $j_\nu$.

ASSUMPTION B.

B1. $W$ contains projective lines.

By Assumption B1, $\Lambda$ contains a line as in the beginning of section 2. We let $l_0$ be a line such that

$$l_0 \text{ is contained in } \Lambda\,. \tag{39}$$

Put $\check{W} = \check{j}_1(W)$, $\check{W}'_\nu = \check{j}_\nu(W'_\nu)$, $\check{W}_\nu = \check{j}_\nu(W_\nu)$, and $\check{\Sigma}_\nu = \check{j}_\nu(\Sigma)$. Since $\Sigma$ is simply connected, there is a fundamental region $F_0$ for $\Gamma_0$ on $\Omega_0$ which contains both $\check{W}_1$ and $\check{W}_2$. Here we can assume that $F_0$ is a compact simplicial complex embedded in $\Omega_0$ by considering a triangulation of $X$. Since $\check{W}$ contains projective lines, the map $\check{j}_2 \circ \check{j}_1^{-1}$ defined on $\check{W}$ extends to an element $f$ of $PGL_4(\mathbf{C})$ by [K1, Lemma 3.2], which sends $\check{W}'_1$ onto the complement of the closure of $\check{W}'_2$, and $\check{\Sigma}_1$ onto $\check{\Sigma}_2$. Put $F = F_0 \setminus (\check{W}'_1 \cup \check{W}'_2)$ and $\Omega = \bigcup_{g \in \Gamma} g(F)$, where $\Gamma$ is the subgroup of $PGL_4(\mathbf{C})$ generated by $\Gamma_0$ and $f$. Then it is easy to see that $\Omega$ is an unramified even covering of $\mathrm{Ha}(X) = \mathrm{Ha}(X, j_1, j_2, \Sigma)$. Moreover, we can show easily that $F$ is a fundamental region for $\Gamma$ and that $\Gamma$ is the free product of $\Gamma_0$ and $f$ by the same argument as the proof of Maskit [M, Proposition 12]. Put $\Lambda_0 = \mathbf{P}^3 \setminus \Omega_0$ and $\Lambda = \mathbf{P}^3 \setminus \Omega$.

We follow the argument of [M, page 310–313]. Every element of $\Gamma^*$ can be written in the *normal form*

(40) $$g=f^{a_n}\circ g_n\circ\cdots\circ f^{a_1}\circ g_1\,,$$

where $g_1\in\Gamma_0$, $g_2, \cdots, g_n\in\Gamma_0^*$, and $a_1, \cdots, a_{n-1}\neq 0$.

The number $\sum_{i=1}^n|a_i|$ determined by $g$ is called the *length of* $g$ and is denoted by $|g|$. We set $|1|=0$. Writing $g$ in the normal form (40), we say that $g$ is *positive*, $g>0$, (resp. *non-negative*, $g\geq 0$) if $g_1=1$ and $a_1>0$ (resp. $g_1\neq 1$ or $a_1\geq 0$), and *negative*, $g<0$, (resp. *non-positive*, $g\leq 0$) if $g_1=1$ and $a_1<0$ (resp. $g_1\neq 1$ or $a_1\leq 0$). Note that $\Gamma_0^*$ coincides with the set of elements which are non-positive and non-negative, but neither positive nor negative, and with the set of non-unit elements whose length are zero.

By Assumption B1, there is a line $l_\infty$ in $W$. We consider the following sets of lines

$$\mathcal{Q}_1=\{l\subset\mathbf{P}^3 : g(l)=l_\infty, g\leq 0\}\,,\qquad \mathcal{Q}_2=\{l\subset\mathbf{P}^3 : g(l)=l_\infty, g\geq 0\}\,,$$

and their supports

$$|\mathcal{Q}_\nu|=\bigcup_{l\in\mathcal{Q}_\nu}\operatorname{Supp} l\,.$$

It is easy to see that the set $|\mathcal{Q}_\nu|$ is contained in $\check{W}'_{3-\nu}$.

ASSUMPTION B (continued). There are subdomains $\check{W}''_\nu$, $\nu=1, 2$, which admit open coverings $\{U_\nu^1, \cdots, U_\nu^{r_\nu}\}$ such that

B2. each $\check{W}''_\nu$ contains $\check{W}'_\nu\cup\check{\Sigma}_\nu$,

B3. each $U_\nu^j$ is biholomorphic to $U$,

B4. the closure of each $U_\nu^j$ does not intersect the closure of $\mathcal{Q}_\nu$,

B5. each $U_\nu^j$ satisfies either $U_\nu^j\cap U(l_0)=\varnothing$ or $U_\nu^j\cap U(l_\infty)=\varnothing$, where $U(l_0)$ (resp. $U(l_\infty)$) is a small neighborhood of $l_0$ (resp. $l_\infty$).

It is easy to see that if there are open coverings which satisfy B2, $\cdots$, B4, we can replace them by those which satisfy also B5. Note that Assumptions B1, $\cdots$, B5 are obviously fulfilled in the case of handle attachment. We shall prove the following.

THEOREM 4.1. *If $\Sigma$ satisfies Assumptions* B1, $\cdots$, B5, *then* $\mathrm{Vol}_{\mathbf{P}^3}(\Lambda_0)=0$ *implies* $\mathrm{Vol}_{\mathbf{P}^3}(\Lambda)=0$.

To prove the theorem above, as in the section 2, we decompose $\Gamma$ into the set of positive elements, negative elements and the identity element, and write

(41) $$\Gamma=\{1\}\sum_{n,m}p_{nm}+\sum_{n,m}q_{nm}\,,$$

where $|p_{nm}|=|q_{nm}|=n$, $p_{nm}\geq 0$, and $q_{nm}\leq 0$.

For fixed $n\geq 0$, we set

(42) $$T_n=\bigcup_m q_{nm}(\check{W}'_1)\cup\bigcup_m p_{nm}(\check{W}'_2)\,.$$

LEMMA 4.1. *For* $n \geq 2$, $T_{n-1} \supset T_n$ *holds.*

PROOF. Let $x \in T_n$ be any element. Suppose $x \in q_{nm}(\check{W}'_1)$ for some $q_{nm} \leq 0$. We write $q_{nm} = f^{a_k} \circ g_k \circ \cdots \circ f^{a_1} \circ g_1$ in the normal form. If $a_1 > 0$, put $q^* = f^{a_k} \circ g_k \circ \cdots \circ f^{a_1 - 1}$. Then, since $q_1 = 1$, we have $f \circ g_1(\check{W}'_1) \subset \check{W}'_2$ and $q_{nm}(\check{W}'_1) \subset q^*(\check{W}'_2)$. Since $|q^*| = n-1$ and $q^* \geq 0$, this implies that $x \in T_{n-1}$. If $a_1 < 0$, put $q^* = f^{a_k} \circ g_k \circ \cdots \circ f^{a_1 + 1}$. Then, we have $q_{nm}(\check{W}'_1) \subset q^*(\check{W}'_1)$. Since $|q^*| = n-1$ and $q^* \leq 0$, again we have $x \in T_{n-1}$. We can settle the case $x \in p_{nm}(\check{W}'_2)$ for some $p_{nm} > 0$ in the same manner. □

We set

$$T = \bigcap_{n \geq 1} T_n ,$$

and

$$S_n = \mathbf{P}^3 \setminus T_n , \qquad S = \bigcup_n S_n .$$

All lemmas 2.2, $\cdots$, 2.9 hold true also in this case, which can be proved by exactly the same manner as before.

It is easy to verify the following.

LEMMA 4.2. *If* $x \in \mathbf{P}^3$ *satisfies*

$$x \notin \bigcup_{g \in \Gamma} g(\Lambda_0) , \tag{43}$$

*then*

$$g(x) \in \Omega_0 \qquad \text{for all } g \in \Gamma . \tag{44}$$

LEMMA 4.3. $\Lambda \supset \bigcup_{g \in \Gamma} g(\Lambda_0)$.

PROOF. Since $\Lambda$ is $\Gamma$-invariant, it is enough to show that $\Omega \subset \Omega_0$. Take any element $g \in \Gamma$ and write $g$ in the normal form (40). Note that $f^a \circ g_1(F) \subset F_0$ for non-zero $a$, $f^a \circ g_1(F_0) \subset F_0$ for non-zero $a$ and $g_1 \in \Gamma_0^*$, and $f^a(\Omega_0 \setminus F_0) \subset F_0$ for non-zero $a$. From these three facts the implication $\Omega \subset \Omega_0$ follows immediately. □

LEMMA 4.4. $S \subset \Omega \cup \bigcup_{g \in \Gamma} g(\Lambda_0)$.

PROOF. Take $x \in S$ satisfying (43). It is enough to show that $x \in \Omega$. Since $x \in S$ and $\{S_n\}$ is ascending, there is an integer $n_0$ such that $x \in S_{n_0} \setminus S_{n_0 - 1}$. Then $x \in T_{n_0 - 1} = \mathbf{P}^3 \setminus S_{n_0 - 1}$. Suppose that $x = q_{n_0 - 1\, m}(w_1)$ holds for some point $w_1 \in \check{W}'_1$ and an element $q_{n_0 - 1\, m} \leq 0$ in $\Gamma$. The other case that $x = p_{n_0 - 1\, m}(w_2)$ for some point $w_2 \in \check{W}'_2$ with an element $p_{n_0 - 1\, m} \geq 0$ in $\Gamma$ can be settled similarly. For some $a > 0$, $y := f^a(w_1) \in \mathbf{P}^3 \setminus (\check{W}'_1 \cup \check{W}'_2)$. If $y \in T_1$, then either there are a point $v_1 \in \check{W}'_1$ and $h \in \Gamma^*$ such that $y = h(v_1)$ with $h \leq 0$, or there are a point $v_2 \in \check{W}'_2$ and $h \in \Gamma^*$ such that $y = h(v_2)$

with $h \geq 0$. In the former case, $g^* := q_{n_0-1\,m} \circ f^{-a} \circ h$ is non-positive with $|q^*| \geq n_0$, and satisfies $x = g^*(v_1)$. Thus $x \in T_{n_0}$. In the latter case, we also have $x \in T_{n_0}$ by the similar argument. This contradicts $x \in S_{n_0}$. Hence $w_1 \notin T_1$, i.e., $w_1 \in S_1$. Namely, for every element $x \in S_n$ with $n > 1$, there is an element $g \in \Gamma$ such that $g(x) \in S_1$. By Lemma 4.2, we also have $g(x) \in \Omega_0$. If $g(x) \notin \check{W}_1' \cup \check{W}_2'$, then $g(x) \in F$. If $g(x) \in W_2'$, then by Lemma 4.2, $f^{-1} \circ g(x) \in \Omega_0 \setminus \check{W}_1'$. Suppose $f^{-1} g(x) \in \check{W}_2'$, then $g(x) = f(f^{-1} \circ g(x)) \in f(\check{W}_2') \subset T_1$. This contradicts $g(x) \in S_1$. Therefore $f^{-1} \circ g(x) \in \Omega \setminus (\check{W}_1' \cup \check{W}_2') = F$. Hence $x \in \Omega$. The remaining case $g(x) \in \check{W}_1'$ can be settled similarly. □

PROPOSITION 4.1. $\Lambda = \bigcup_{g \in \Gamma} g(\Lambda_0) \cup T$.

PROOF. By Lemmas 2.6 and 4.4, we have

$$\Lambda \subset \Omega \cup \bigcup_{g \in \Gamma} g(\Lambda_0) \cup T .$$

Since $\Omega \cap \Lambda = \emptyset$, we obtain

$$\Lambda \subset \bigcup_{g \in \Gamma} g(\Lambda_0) \cup T .$$

The other implication follows easily from Lemmas 4.3 and 2.5. □

REMARK 4.1. By Assumption B1 and the definition of $T$, $T$ contains lines.

PROOF OF THEOREM 4.1. As in section 2, we choose a system of homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ on $\mathbf{P}^3$ such that the two lines $l_\infty \subset \check{W}$ and $l_0 \subset f(\check{W})$ are given by

$$l_0 : z_0 = z_1 = 0 , \qquad l_\infty : z_2 = z_3 = 0 .$$

Then the whole argument in section 3 works without essential changes: The definition of $V_{(n)}$ of (25) should be replaced by

$$V_{(n)} = \bigcup_{m,\, q_{nm} < 0} q_{nm}(V_1) \cup \bigcup_{m,\, p_{nm} > 0} p_{nm}(V_2) , \tag{45}$$

and $\mathscr{P}_\nu$ should be replaced by $\mathscr{Q}_\nu$. □

PROOF OF THEOREM B. Theorem B follows immediately form Theorem 4.1.

REMARK 4.2. A connected sum of a Class $L$ manifold $M$ with an $L$-Hopf manifold [K2, page 372] is a handle attachment of $M$. Converse seems to be false.

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