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SC_n-moves and the $(n+1)$-st Coefficients of the Conway Polynomials of Links

Haruko Aida MIYAZAWA

Tsuda College

Abstract. A local move called a C_n-move is related to Vassiliev invariants. It is known that two knots are related by C_n-moves if and only if they have the same values of Vassiliev invariants of order less than n. In the link case, it is shown that a C_n-move does not change the values of any Vassiliev invariants of order less than n. It is also known that, if two links can be transformed into each other by a C_n-move, then the n-th coefficients of the Conway polynomials of them, which are Vassiliev invariants of order n, are congruent to each other modulo 2. An SC_n-move is defined as a special C_n-move. It is shown that an SC_n-move does not change the values of any Vassiliev invariants of links of order less than $n+1$. In this paper, we consider the difference of the $(n+1)$-st coefficients of the Conway polynomials of two links which can be transformed into each other by an SC_n-move.

1. Introduction

In 1990, V. A. Vassiliev introduced a knot invariant called a Vassiliev invariant. It is proved that many invariants derived from polynomial invariants are Vassiliev invariants. For example, the n-th coefficient of the Conway polynomial and the n-th derivative of the Jones polynomial at $t=1$ are Vassiliev invariants of order n ([1]). We can define a Vassiliev invariant of links as the same way as that of knots. If two links cannot be distinguished by any Vassiliev invariants of order less than or equal to n, then they are said to be *V_n-equivalent* ([11]).

K. Habiro defined a new local move called a *C_n-move* as indicated in Figure 1.1.

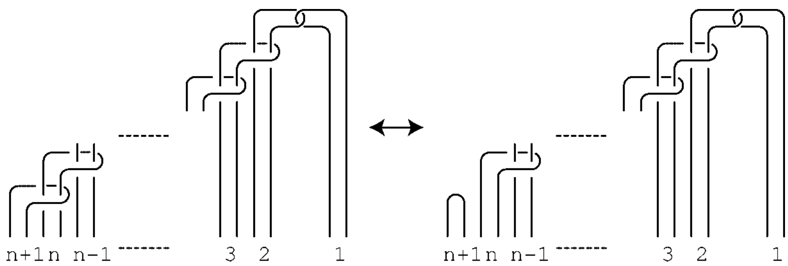

FIGURE 1.1

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A C_1-move is defined as a crossing change. He also obtained the result that shows the relationship between Vassiliev invariants and C_n-moves. The following theorem was proved by M. N. Goussarov and Habiro independently:

THEOREM 1.1 ([4, 7]). *Two oriented knots in S^3 can be transformed into each other by a finite sequence of C_{n+1}-moves if and only if they are V_n-equivalent.*

In the case of links, the following result is known:

THEOREM 1.2 ([3, 12, 16]). *If two oriented links in S^3 can be transformed into each other by a finite sequence of C_{n+1}-moves, then they are V_n-equivalent.*

In [8], the author discussed the relationship between C_n-moves and polynomial invariants which are Vassiliev invariants of order n. We take the n-th coefficient of the Conway polynomial of a link L and the n-th derivative of the Jones polynomial of L at $t = 1$, denoted by $a_n(L)$ and $V^{(n)}(L)$ respectively, as Vassiliev invariants of order n. Then we can obtain the following theorem:

THEOREM 1.3 ([8]). *If two oriented links L and L' in S^3 can be transformed into each other by a finite sequence of C_n-moves, then*

$$a_n(L) - a_n(L') \equiv 0 \pmod 2$$

and

$$V^{(n)}(L) - V^{(n)}(L') \equiv 0 \pmod{6 \cdot n!}$$

for any integer $n > 2$.

Recently Y. Ohyama and H. Yamada obtained the precise result for the change of the n-th coefficient of the Conway polynomial under a C_n-move for a knot.

THEOREM 1.4 ([15]). *If two oriented knots K and K' in S^3 can be transformed into each other by a C_n-move, then*

$$a_n(K) - a_n(K') = 0 \ \text{or} \ \pm 2$$

for any integer $n > 2$.

We define a special C_n-move which is called an *SC_n-move* as follows: Let $\alpha_1, \ldots, \alpha_{n+1}$ be the arcs shown in the tangle applied a C_n-move and $c(\alpha_i)$ denote the component of the link which contains α_i for each i with $i = 1, 2, \ldots, n + 1$. If there is an arc α_k such that $c(\alpha_k) \neq c(\alpha_i)$ for all i with $i \neq k$, we call the C_n-move an SC_n-move. We can describe the necessary and sufficient condition for that two links are V_2-equivalent or V_3-equivalent to each other in terms of C_n-moves and SC_n-moves ([9]). With respect to SC_n-moves, the following result is also shown:

THEOREM 1.5 ([9, 13]). *If two oriented links L and L' in S^3 can be transformed into each other by a finite sequence of SC_n-moves, then they are V_n-equivalent.*

Comparing Theorems 1.2 and 1.5, an SC_n-move seems to be similar to a C_{n+1}-move. In this paper, we will consider a relationship between an SC_n-move and the $(n+1)$-st coefficient of the Conway polynomial of a link which is a Vassiliev invariant of order $n+1$ and prove the following result:

THEOREM 1.6. *If two oriented links L and L' in S^3 can be transformed into each other by a finite sequence of SC_n-moves, then*

$$a_{n+1}(L) - a_{n+1}(L') \equiv 0 \pmod 2$$

for any integer $n > 2$.

2. Proof of Theorem 1.6

Let n be an integer more than 2 and L and L' links which are transformed into each other by an SC_n-move. We can suppose that the difference of the diagrams between L and L' is illustrated in Figure 2.1.

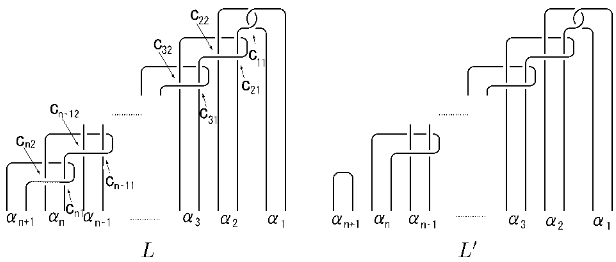


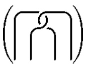
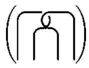
FIGURE 2.1

LEMMA 2.1. *Let v be a Vassiliev invariant of order k. Then*

$$v(L) - v(L') = \prod_{i=1}^{n} s_i \sum_{\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_n=\pm 1} \prod_{i=2}^{n} \varepsilon_i v(L\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & \varepsilon_2 & \cdots & \varepsilon_n \end{pmatrix}),$$

where s_1 is the sign of the crossing c_1 and s_i $(i = 2, 3, \ldots, n)$ is the sign of the crossing c_{i1} of L, and $L\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & \varepsilon_2 & \cdots & \varepsilon_n \end{pmatrix}$ is the singular link with n double points that is obtained from L by the following: Collapse the crossing to a double point at c_1. If $\varepsilon_i = 1$, collapse the crossing at c_{i1} and if $\varepsilon_i = -1$, switch the crossing at c_{i1} and collapse the crossing to a double point at c_{i2} $(i = 2, 3, \ldots, n)$.

PROOF. Fix a natural number k. If $n > k$, then the equation holds because for any $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n = \pm 1$, $v(L\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & \varepsilon_2 & \cdots & \varepsilon_n \end{pmatrix}) = 0$ and a C_n-move does not change the value of any Vassiliev invariants of order less than n. If $n = k$, then the equation was proved in [14]. Suppose $n < k$. We show the equation by induction on n. If $n = 1$, then we have

$$v(L) - v(L') = v\Big($$  $$\Big) - v\Big($$  $$\Big) = s_1 v\Big($$  $$\Big)$$

by the Vassiliev skein relation. We suppose that the equation holds for $n = l$. Suppose $n = l + 1$ and let L and L' be two links as shown in Figure 2.1. Let M and M' be links which are obtained from L and L' respectively by a C_{n-1}-move (see Figure 2.2).

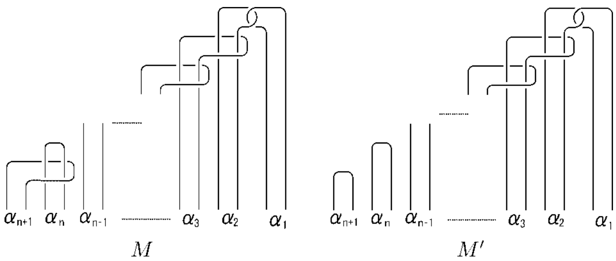


FIGURE 2.2

From the assumption of the induction,

$$v(L) - v(M) = \prod_{i=1}^{n-1} s_i \sum_{\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{n-1}=\pm 1} \prod_{i=2}^{n-1} \varepsilon_i v(L \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ 1 & \varepsilon_2 & \cdots & \varepsilon_{n-1} \end{pmatrix})$$

and

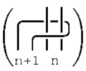
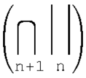
$$v(L') - v(M') = \prod_{i=1}^{n-1} s_i \sum_{\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{n-1}=\pm 1} \prod_{i=2}^{n-1} \varepsilon_i v(L' \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ 1 & \varepsilon_2 & \cdots & \varepsilon_{n-1} \end{pmatrix}),$$

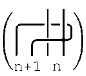
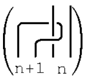
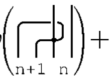
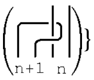
where $L' \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ 1 & \varepsilon_2 & \cdots & \varepsilon_{n-1} \end{pmatrix}$ is the singular link with $n-1$ double points obtained from L' as the same way as $L \begin{pmatrix} 1 & 2 & \cdots n-1 \\ 1 & \varepsilon_2 & \cdots \varepsilon_{n-1} \end{pmatrix}$ obtained from L. On the other hand, we can easily see that M and M' are ambient isotopic to each other. Hence we obtain

$$\begin{aligned} &v(L) - v(L') \\ &= \prod_{i=1}^{n-1} s_i \sum_{\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{n-1}=\pm 1} \prod_{i=2}^{n-1} \varepsilon_i \{v(L \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ 1 & \varepsilon_2 & \cdots & \varepsilon_{n-1} \end{pmatrix}) - v(L' \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ 1 & \varepsilon_2 & \cdots & \varepsilon_{n-1} \end{pmatrix})\}. \end{aligned}$$

Here we have

$$v(L \begin{pmatrix} 1 & 2 & \cdots n-1 \\ 1 & \varepsilon_2 & \cdots \varepsilon_{n-1} \end{pmatrix}) - v(L' \begin{pmatrix} 1 & 2 & \cdots n-1 \\ 1 & \varepsilon_2 & \cdots \varepsilon_{n-1} \end{pmatrix})$$

$= v\Big($  $\Big) - v\Big($  $\Big)$

$= s_n v\Big($  $\Big) + v\Big($  $\Big) - \{s_n v\Big($  $\Big) + v\Big($  $\Big)\}$

$$= s_n\{v(L\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & \varepsilon_2 & \cdots & \varepsilon_{n-1} & 1 \end{pmatrix} - v(L\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & \varepsilon_2 & \cdots & \varepsilon_{n-1} & -1 \end{pmatrix})\}$$

by the Vassiliev skein relation. Therefore we obtain

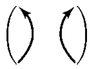
$$v(L) - v(L') = \prod_{i=1}^{n} s_i \sum_{\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n = \pm 1} \prod_{i=2}^{n} \varepsilon_i v(L\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & \varepsilon_2 & \cdots & \varepsilon_n \end{pmatrix}). \qquad \square$$

We remark that Lemma 2.1 holds for L and L' which are related by a C_n-move (it does not need to be an SC_n-move).

Using Lemma 2.1, we have

$$a_{n+1}(L) - a_{n+1}(L') \equiv \sum_{\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n = \pm 1} a_{n+1}(L\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & \varepsilon_2 & \cdots & \varepsilon_n \end{pmatrix}) \pmod 2 .$$

By the Vassiliev skein relation and the definition of the Conway polynomial,

$a_k\Big($  $\Big) = a_k\Big($  $\Big) - a_k\Big($  $\Big) = a_{k-1}\Big($  $\Big).$

Applying the above relation to all singular points of $L\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & \varepsilon_2 & \cdots & \varepsilon_n \end{pmatrix}$, we obtain

$$a_{n+1}(L) - a_{n+1}(L') \equiv \sum_{\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n = \pm 1} a_1(L\begin{pmatrix} 1 & 2 & \cdots & n \\ \overline{1} & \overline{\varepsilon_2} & \cdots & \overline{\varepsilon_n} \end{pmatrix}) \pmod 2 ,$$

where $L\begin{pmatrix} 1 & 2 & \cdots & n \\ \overline{1} & \overline{\varepsilon_2} & \cdots & \overline{\varepsilon_n} \end{pmatrix}$ is a link obtained from $L\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & \varepsilon_2 & \cdots & \varepsilon_n \end{pmatrix}$ by smoothing at all double points.

We recall that L is obtained from L' by an SC_n-move. Therefore there is an arc α_k such that $c(\alpha_k) \neq c(\alpha_i)$ for all i with $i \neq k$. Here we consider the cases (i) $k = 1$, (ii) $k = 2$, (iii) $k = 3$ and (iv) $k \geq 4$.

Case (i). We already have

$$\begin{aligned} & a_{n+1}(L) - a_{n+1}(L') \\ & \equiv \sum_{\varepsilon_2, \ldots, \varepsilon_n = \pm 1} a_1(L\begin{pmatrix} 1 & 2 & \cdots & n \\ \overline{1} & \overline{\varepsilon_2} & \cdots & \overline{\varepsilon_n} \end{pmatrix}) \pmod 2 \\ & = \sum_{\varepsilon_4, \ldots, \varepsilon_n = \pm 1} \{a_1(L\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \overline{1} & \overline{1} & \overline{1} & \overline{\varepsilon_4} & \cdots & \overline{\varepsilon_n} \end{pmatrix}) + a_1(L\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \overline{1} & \overline{1} & \overline{-1} & \overline{\varepsilon_4} & \cdots & \overline{\varepsilon_n} \end{pmatrix}) \end{aligned}$$

$$+a_1(L\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \overline{1} & \overline{-1} & \overline{1} & \overline{\varepsilon_4} & \cdots & \overline{\varepsilon_n}\end{pmatrix}) + a_1(L\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \overline{1} & \overline{-1} & \overline{-1} & \overline{\varepsilon_4} & \cdots & \overline{\varepsilon_n}\end{pmatrix})\}\,.$$

Fix $\varepsilon_4, \dots, \varepsilon_n = \pm 1$ and set

$$L_1 = L\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \overline{1} & \overline{1} & \overline{1} & \overline{\varepsilon_4} & \cdots & \overline{\varepsilon_n}\end{pmatrix}, \quad L_2 = L\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \overline{1} & \overline{1} & \overline{-1} & \overline{\varepsilon_4} & \cdots & \overline{\varepsilon_n}\end{pmatrix},$$

$$L_3 = L\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \overline{1} & \overline{-1} & \overline{1} & \overline{\varepsilon_4} & \cdots & \overline{\varepsilon_n}\end{pmatrix}, \quad L_4 = L\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \overline{1} & \overline{-1} & \overline{-1} & \overline{\varepsilon_4} & \cdots & \overline{\varepsilon_n}\end{pmatrix}.$$

Then L_1, L_2, L_3 and L_4 are identical except for the part corresponding to the arcs α_2, α_3 and α_4 in L. The difference of them depends on the orientation of the arcs α_1, α_2, α_3 and α_4. For example if L is oriented as in the left of Figure 2.3, then $L_1, \dots, L_4$ are like in the figure. We show the theorem in the case that L is oriented as in Figure 2.3. In the case that L is oriented differently, we can prove similarly. Let T_i be a tangle of L_i as shown in Figure 2.3 and we set a tangle $S = L_i - T_i$ (remark that $L_i - T_i = L_j - T_j$ $(i, j = 1,\dots,4)$).

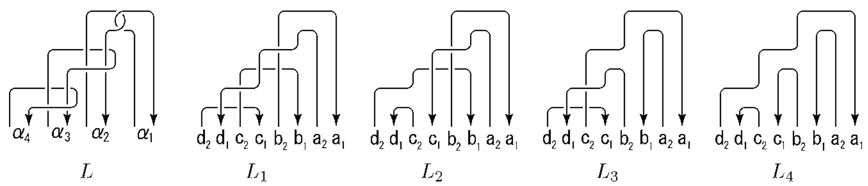


FIGURE 2.3

The first coefficient of the Conway polynomial of a μ-component link is equal to the linking number of the link if $\mu = 2$ and zero otherwise. We consider possible connections of arcs in the tangle S and calculate the linking number of L_i $(i = 1,\dots,4)$ if $\sharp L_i = 2$, where $\sharp L$ denotes the number of the components of a link L. The points a_1 and a_2 in T_i are connected by an arc in S because this C_n-move is an SC_n-move and $k = 1$ (we describe this situation as $a_1 \to a_2$). On the connection of b_1, b_2, c_1, c_2, d_1 and d_2 in S, we can consider the several cases and define a type of S in the following:

$$\begin{aligned}
&\text{type } A : \{b_1 \to b_2, c_1 \to c_2, d_1 \to d_2\}\\
&\text{type } B : \{b_1 \to b_2, c_1 \to d_2, d_1 \to c_2\}\\
&\text{type } C : \{b_1 \to c_2, c_1 \to b_2, d_1 \to d_2\}\\
&\text{type } D : \{b_1 \to c_2, c_1 \to d_2, d_1 \to b_2\}\\
&\text{type } E : \{b_1 \to d_2, c_1 \to b_2, d_1 \to c_2\}\\
&\text{type } F : \{b_1 \to d_2, c_1 \to c_2, d_1 \to b_2\}\,.
\end{aligned}$$

For each type of S, we have

$$\sharp L_1 = \sharp L_3 = \begin{cases} 1+m & \text{if } S \text{ is type } A \text{ or } E \\ 2+m & \text{if } S \text{ is type } B, C \text{ or } F \\ 3+m & \text{if } S \text{ is type } D \end{cases}$$

and

$$\sharp L_2 = \sharp L_4 = \begin{cases} 1+m & \text{if } S \text{ is type } A \text{ or } D \\ 2+m & \text{if } S \text{ is type } B, C \text{ or } F\,, \\ 3+m & \text{if } S \text{ is type } E \end{cases}$$

where m denotes the number of the components which are completely contained in S.

If S is type B, C or F and $m = 0$, then

$$\begin{aligned} a_1(L_1) + a_1(L_2) + a_1(L_3) + a_1(L_4) &\equiv a_1(L_1) - a_1(L_2) - a_1(L_3) + a_1(L_4) \\ &\equiv 0 \pmod 2 \end{aligned}$$

by (A5), (A1) and (A3) in §4. If S is type A, D or E and $m = 1$, then the linking number of L_i does not depend on T_i $(i = 1, 2, 3, 4)$. Hence we can obtain

$$a_1(L_1) = a_1(L_3), a_1(L_2) = a_1(L_4)\,,$$

and

$$a_1(L_1) + a_1(L_2) + a_1(L_3) + a_1(L_4) \equiv 0 \pmod 2\,.$$

Therefore the proof is completed.

Case (ii). Fix $\varepsilon_4, \ldots, \varepsilon_n = \pm 1$ and set L_1, L_2, L_3 and L_4 as same as in Case (i). We also use the notation T_i $(i = 1,\ldots,4)$ and S as in Case (i). We prove in the case that L is oriented as in Figure 2.3. The points b_1 and b_2 in T_i are connected in S because this C_n-move is an SC_n-move and $k = 2$. We define types of S as follows:

$$\begin{aligned} &\text{type } A: \{a_1 \to a_2, c_1 \to c_2, d_1 \to d_2\} \\ &\text{type } B: \{a_1 \to a_2, c_1 \to d_2, d_1 \to c_2\} \\ &\text{type } C: \{a_1 \to c_2, c_1 \to a_2, d_1 \to d_2\} \\ &\text{type } D: \{a_1 \to c_2, c_1 \to d_2, d_1 \to a_2\} \\ &\text{type } E: \{a_1 \to d_2, c_1 \to a_2, d_1 \to c_2\} \\ &\text{type } F: \{a_1 \to d_2, c_1 \to c_2, d_1 \to a_2\}\,. \end{aligned}$$

For each type of S, we have

$$\sharp L_1 = \sharp L_3 = \begin{cases} 1+m & \text{if } S \text{ is type } A \text{ or } E \\ 2+m & \text{if } S \text{ is type } B, C \text{ or } F \\ 3+m & \text{if } S \text{ is type } D \end{cases}$$

and

$$\sharp L_2 = \sharp L_4 = \begin{cases} 1+m & \text{if } S \text{ is type } A \text{ or } D \\ 2+m & \text{if } S \text{ is type } B, C \text{ or } F \\ 3+m & \text{if } S \text{ is type } E \end{cases},$$

where m denotes the number of the components which are completely contained in S.

If S is type B, C or F and $m = 0$, then $a_1(L_1) + a_1(L_2) + a_1(L_3) + a_1(L_4)$ is even by (A5), (A2) and (A4) in §4. If S is type A, D or E and $m = 1$, then

$$a_1(L_1) = a_1(L_3), a_1(L_2) = a_1(L_4)$$

as the cases of type A, D or E in Case (i).

Case (iii). We use the notations $L_1, \ldots, L_4$, $T_1, \ldots, T_4$ and S as in Case (i). We prove in the case that L is oriented as in Figure 2.3. The points c_1 and c_2 in T_i are connected in S because this C_n-move is an SC_n-move and $k = 3$. We define types of S as follows:

$$\begin{aligned} \text{type } A &: \{a_1 \to a_2, b_1 \to b_2, d_1 \to d_2\} \\ \text{type } B &: \{a_1 \to a_2, b_1 \to d_2, d_1 \to b_2\} \\ \text{type } C &: \{a_1 \to b_2, b_1 \to a_2, d_1 \to d_2\} \\ \text{type } D &: \{a_1 \to b_2, b_1 \to d_2, d_1 \to a_2\} \\ \text{type } E &: \{a_1 \to d_2, b_1 \to a_2, d_1 \to b_2\} \\ \text{type } F &: \{a_1 \to d_2, b_1 \to b_2, d_1 \to a_2\}. \end{aligned}$$

For each type of S, we have

$$\sharp L_1 = \sharp L_2 = \begin{cases} 1+m & \text{if } S \text{ is type } A \text{ or } E \\ 2+m & \text{if } S \text{ is type } B, C \text{ or } F \\ 3+m & \text{if } S \text{ is type } D \end{cases}$$

and

$$\sharp L_3 = \sharp L_4 = \begin{cases} 1+m & \text{if } S \text{ is type } A \text{ or } D \\ 2+m & \text{if } S \text{ is type } B, C \text{ or } F \\ 3+m & \text{if } S \text{ is type } E \end{cases},$$

where m denotes the number of the components which are completely contained in S.

If S is type B, C or F and $m = 0$, then $a_1(L_1) + a_1(L_2) + a_1(L_3) + a_1(L_4)$ is even by (A3), (A6) and (A4) in §4. If S is type A, D or E and $m = 1$, then

$$a_1(L_1) = a_1(L_2), a_1(L_3) = a_1(L_4)$$

as the cases of type A, D or E in Case (i).

Case (iv). We have

$$
\begin{aligned}
& a_{n+1}(L) - a_{n+1}(L') \\
& \equiv \sum_{\varepsilon_2,\ldots,\varepsilon_n=\pm1} a_1(L\begin{pmatrix} 1 & 2 & \cdots & n \\ \overline{1} & \overline{\varepsilon_2} & \cdots & \overline{\varepsilon_n} \end{pmatrix}) \pmod 2 \\
& = \sum_{\varepsilon_2,\ldots,\varepsilon_{k-2},\varepsilon_{k+1},\ldots,\varepsilon_n=\pm1} \{a_1(L\begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & n \\ \overline{1} & \cdots & \overline{1} & \overline{1} & \cdots & \overline{\varepsilon_n} \end{pmatrix}) \\
& \quad + a_1(L\begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & n \\ \overline{1} & \cdots & \overline{1} & \overline{-1} & \cdots & \overline{\varepsilon_n} \end{pmatrix}) \\
& \quad + a_1(L\begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & n \\ \overline{1} & \cdots & \overline{-1} & \overline{1} & \cdots & \overline{\varepsilon_n} \end{pmatrix}) + a_1(L\begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & n \\ \overline{1} & \cdots & \overline{-1} & \overline{-1} & \cdots & \overline{\varepsilon_n} \end{pmatrix})\}\,.
\end{aligned}
$$

Fix $\varepsilon_2, \ldots, \varepsilon_{k-2}, \varepsilon_{k+1}, \ldots, \varepsilon_n = \pm1$ and set

$$
L_1 = L\begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & n \\ \overline{1} & \cdots & \overline{1} & \overline{1} & \cdots & \overline{\varepsilon_n} \end{pmatrix}, \quad L_2 = L\begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & n \\ \overline{1} & \cdots & \overline{1} & \overline{-1} & \cdots & \overline{\varepsilon_n} \end{pmatrix},
$$

$$
L_3 = L\begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & n \\ \overline{1} & \cdots & \overline{-1} & \overline{1} & \cdots & \overline{\varepsilon_n} \end{pmatrix}, \quad L_4 = L\begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & n \\ \overline{1} & \cdots & \overline{-1} & \overline{-1} & \cdots & \overline{\varepsilon_n} \end{pmatrix}.
$$

Then they are identical except for the part corresponding to the arcs α_{k-2}, α_{k-1} and α_k in L. The difference of them is illustrated in Figure 2.4 for example.

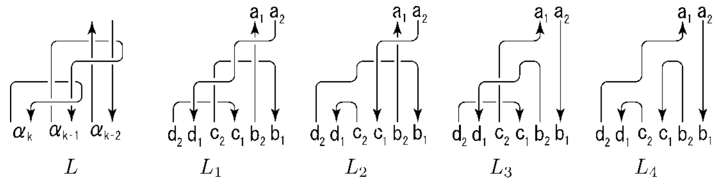


FIGURE 2.4

Let T_i be a tangle of L_i as shown in Figure 2.4 and we set a tangle $S = L_i - T_i$. We prove in the case that L is oriented as in Figure 2.4. The points d_1 and d_2 in T_i are connected in S because this C_n-move is an SC_n-move and $k \geq 4$. We define types of S as follows:

$$
\begin{aligned}
&\text{type } A : \{a_1 \to a_2, b_1 \to b_2, c_1 \to c_2\} \\
&\text{type } B : \{a_1 \to a_2, b_1 \to c_2, c_1 \to b_2\} \\
&\text{type } C : \{a_1 \to b_2, b_1 \to a_2, c_1 \to c_2\} \\
&\text{type } D : \{a_1 \to b_2, b_1 \to c_2, c_1 \to a_2\} \\
&\text{type } E : \{a_1 \to c_2, b_1 \to a_2, c_1 \to b_2\} \\
&\text{type } F : \{a_1 \to c_2, b_1 \to b_2, c_1 \to a_2\}\,.
\end{aligned}
$$

For each type of S, we have

$$\sharp L_1 = \sharp L_2 = \begin{cases} 1+m & \text{if } S \text{ is type } A \text{ or } E \\ 2+m & \text{if } S \text{ is type } B,\, C \text{ or } F \\ 3+m & \text{if } S \text{ is type } D \end{cases}$$

and

$$\sharp L_3 = \sharp L_4 = \begin{cases} 1+m & \text{if } S \text{ is type } A \text{ or } D \\ 2+m & \text{if } S \text{ is type } B,\, C \text{ or } F \\ 3+m & \text{if } S \text{ is type } E \end{cases},$$

where m denotes the number of the components which are completely contained in S.

If S is type B, C or F and $m = 0$, then $a_1(L_1) + a_1(L_2) + a_1(L_3) + a_1(L_4)$ is even by (A1), (A6) and (A2) in §4. If S is type A, D or E and $m = 1$, then

$$a_1(L_1) = a_1(L_2), a_1(L_3) = a_1(L_4)$$

as the cases of type A, D or E in Case (i) and the proof is completed.

3. Remark and Examples

In this section we make a few remarks on Theorem 1.6.

An SC_1-move is a crossing change between mutually distinct components. For an integer k, let $L_{1,1}(k)$ and $L_{1,2}(k)$ be two links as shown in Figure 3.1. The sign of the integer k is equal to the sign of a crossing in the tangle.

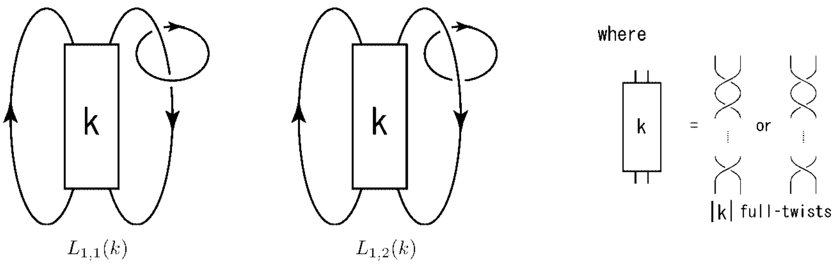


FIGURE 3.1

Then we can see that $L_{1,1}(k)$ and $L_{1,2}(k)$ are transformed into each other by an SC_1-move and

$$a_2(L_{1,1}(k)) - a_2(L_{1,2}(k)) = k\,.$$

For an integer k, let $L_{2,1}(k)$ and $L_{2,2}(k)$ be two links as shown in Figure 3.2. Then we can see that $L_{2,1}(k)$ and $L_{2,2}(k)$ are transformed into each other by an SC_2-move and

$$a_3(L_{2,1}(k)) - a_3(L_{2,2}(k)) = k\,.$$

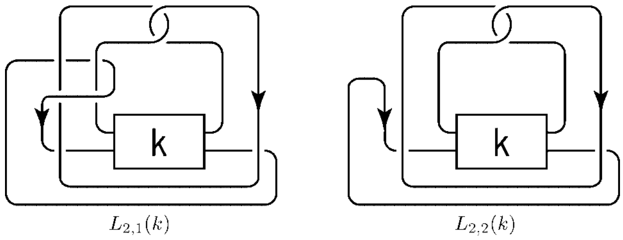


FIGURE 3.2

The above examples show that SC_n-moves and the $(n+1)$-st coefficients of the Conway polynomials have no relation for $n = 1$ and 2.

For an integer k, let $L_{n,1}(k)$ and $L_{n,2}(k)$ $(n = 3, 4, 5)$ be links as shown in Figure 3.3.

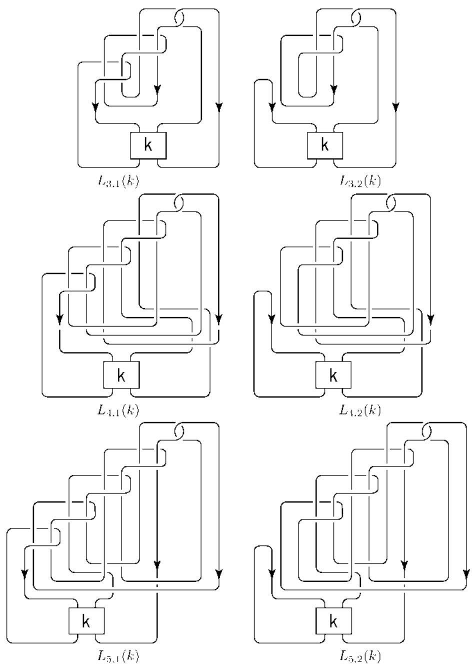


FIGURE 3.3

Then we can see that $L_{n,1}(k)$ and $L_{n,2}(k)$ are transformed into each other by an SC_n-move and

$$|a_{n+1}(L_{n,1}(k)) - a_{n+1}(L_{n,2}(k))| = 2|k|\,.$$

The above examples show that Theorem 1.5 is best possible for $n = 3, 4$ and 5.

4. Table

In this section, we give a table which we need for the proof of Theorem 1.6.

L_1, L_2, L_3 and L_4 in each row of the table indicate four links which are identical except for a neighborhood of one point. Non-identical part is illustrated by solid arcs and the connection outside the tangles by dotted arcs. Let x, y, z and w be oriented arcs indicated by dotted arcs in the table. We give the sum of the signs of the crossing which is made from oriented arcs x and y by $Lk(x, y)$. Then for example, in the case of (A1), we have

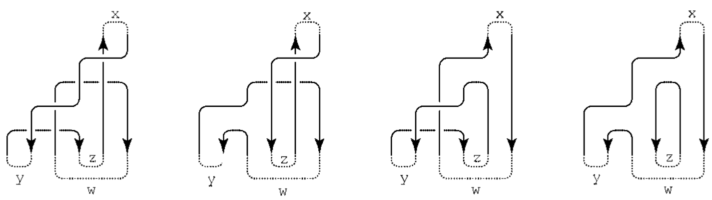
$$a_1(L_1) = \begin{cases} \frac{1}{2}\{Lk(x, w) + Lk(y, w) + Lk(z, w) - 2\} & \text{if } \sharp L_1 = 2 \\ 0 & \text{otherwise} \end{cases} .$$

By similar calculation for $a_1(L_2)$,$a_1(L_3)$ and $a_1(L_4)$, we obtain

$$a_1(L_1) - a_1(L_2) - a_1(L_3) + a_1(L_4) = \begin{cases} -Lk(x, y) & \text{if } \sharp L_1 = 2 \\ 0 & \text{otherwise} \end{cases} .$$

This table is a list of $a_1(L_1) - a_1(L_2) - a_1(L_3) + a_1(L_4)$.

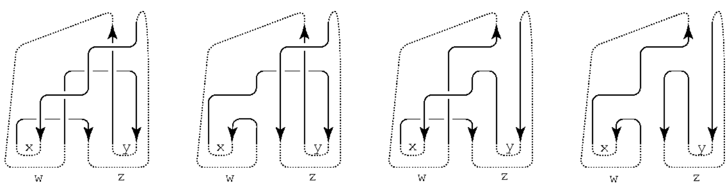
(A1)



L_1 L_2 L_3 L_4

$$a_1(L_1) - a_1(L_2) - a_1(L_3) + a_1(L_4) = \begin{cases} -Lk(x, y) & \text{if } \sharp L_1 = 2 \\ 0 & \text{otherwise} \end{cases} .$$

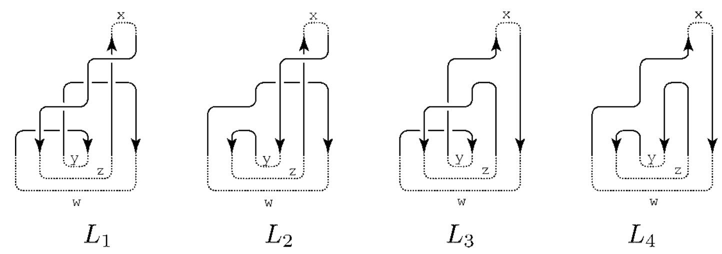
(A2)



L_1 L_2 L_3 L_4

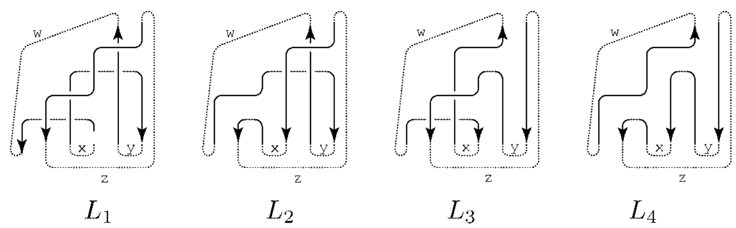
$$a_1(L_1) - a_1(L_2) - a_1(L_3) + a_1(L_4) = \begin{cases} Lk(x, y) & \text{if } \sharp L_1 = 2 \\ 0 & \text{otherwise} \end{cases} .$$

(A3)



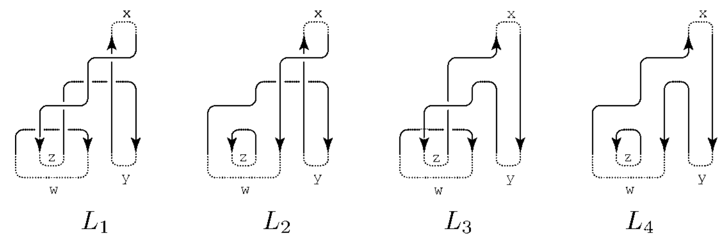
$$a_1(L_1) - a_1(L_2) - a_1(L_3) + a_1(L_4) = \begin{cases} Lk(x, y) & \text{if } \sharp L_1 = 2 \\ 0 & \text{otherwise} \end{cases} .$$

(A4)



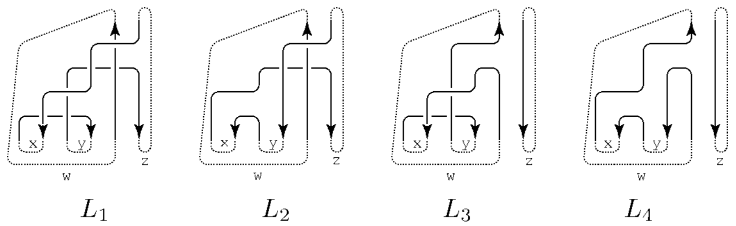
$$a_1(L_1) - a_1(L_2) - a_1(L_3) + a_1(L_4) = \begin{cases} -Lk(x, y) & \text{if } \sharp L_1 = 2 \\ 0 & \text{otherwise} \end{cases} .$$

(A5)



$$a_1(L_1) - a_1(L_2) - a_1(L_3) + a_1(L_4) = 0\,.$$

(A6)



$$a_1(L_1) - a_1(L_2) - a_1(L_3) + a_1(L_4) = 0\,.$$

References

[1] BAR-NATAN, D., On the Vassiliev knot invariants, Topology **34** (1995), 423–472.

[2] BIRMAN, J. S., LIN, X,-S., Knot polynomials and Vassiliev's invariants, Invent. Math. **111** (1993), 225–270.

[3] GOUSSAROV, M. N., On n-equivalence of knots and invariants of finite degree, in *Topology of Manifold and Varieties* (VIRO, O. ed.), Amer. Math. Soc., Province (1994), 173–192.

[4] GOUSSAROV, M. N., Knotted graphs and a geometrical tequnique of n-equivalence, POMI Sankt Petersburg preprint, circa (1995), in Russian.

[5] HABIRO, K., Master Thesis, University of Tokyo (1994).

[6] HABIRO, K., Clasp-pass moves on knots, preprint.

[7] HABIRO, K., Claspers and finite type invariants of links, Geom. Topol. **4** (2000), 1–83.

[8] MIYAZAWA, H. A., C_n-moves and polynomial invariants for links, Kobe J. Math. **17** (2000), 99–117.

[9] MIYAZAWA, H. A., C_n-moves and V_n-equivalence for links, Tokyo J. Math. (to appear).

[10] MURAKAMI, H., NAKANISHI, Y., On a certain move generating link-homology, Math. Ann. **284** (1989), 75–89.

[11] NG, K. Y., Stanford, T., On Gusarov's groups of knots, Math. Proc. Camb. Phil. Soc. **126** (1999), 63–76.

[12] OHYAMA, Y., Vassiliev invariants and similarity of knots, Proc. Amer. Math. Soc. **123** (1995), 287–291.

[13] OHYAMA, Y., Remarks on C_n-moves for links and Vassiliev invariants of order n, J. Knot Theory Ramifications **11** (2002), 507–514.

[14] OHYAMA, Y., TSUKAMOTO, T., On Habiro's C_n-moves and Vassiliev invariants, J. Knot Theory Ramifications **8** (1999), 15–26.

[15] OHYAMA, Y., YAMADA, H., A C_n-move for a knot and the coefficients of the Conway polynomial, J. Knot Theory Ramifications **17** (2008), 771–785.

[16] STANFORD, T., Braid commutators and Vassiliev invariants, Pacific J. Math. **174** (1996), 269–276.

[17] VASSILIEV, V. A., Cohomology of knot space, in *Theory of Singularities and its Applications* (ed. ARNOLD, V. I.), Adv. Soviet Math., Vol.1, Amer. Math. Soc. (1990).

Present Address:
RESEARCH INSTITUTE FOR MATHEMATICS AND COMPUTER SCIENCE,
TSUDA COLLEGE,
KODAIRA, TOKYO, 187–8577 JAPAN,
e-mail: aida@tsuda.ac.jp