TOKYO J. MATH.
VOL. 30, NO. 2, 2007

The Turaev-Viro Invariants of All Orientable Closed Seifert Fibered Manifolds

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(Communicated by M. Kurihara)

Abstract. The Turaev-Viro invariants are topological invariants of closed 3-manifolds. In this paper, we give a formula of the Turaev-Viro invariants of all orientable closed Seifert fibered manifolds. Our formula is based on a new construction of special spines of all orientable closed Seifert fibered manifolds and the "gluing lemma" of topological quantum field theory. By using our formula, we get sufficient conditions of coincidence of the Turaev-Viro invariants of orientable closed Seifert fibered manifolds.

1. Introduction

In 1992, Turaev and Viro [13] defined topological invariants of closed 3-manifolds by using the quantum group $U_q(sl(2,\mathbf{C}))$ at q a root of unity. The Turaev-Viro invariants of closed 3-manifolds are parameterized by an integer $r \geq 3$ called a level. So, we denote by $\mathrm{TV}^{(r)}(M)$ the Turaev-Viro invariant of a closed 3-manifold M at a level r. It is defined by using a triangulation of M. We take an arbitrary triangulation T_M of M, and we consider a map $\sigma : \{\text{edges of } T_M\} \to \{0, 1, \ldots, r-2\}$ called a coloring of T_M. For a coloring σ, a complex number $6j^{(r)}(\tau, \sigma)$ is assigned to each tetrahedron τ of T_M by the function "$6j$-symbol". Then, roughly speaking, the Turaev-Viro invariant of M at a level r is defined by

$$(\text{normalization}) \times \sum_{\sigma \in \{\text{coloring}\}} \left(\prod_{\tau \in T_M} 6j^{(r)}(\tau, \sigma) \right).$$

It does not depend on the choice of triangulation of M. Kauffman [3] defined a state sum type invariant of closed 3-manifolds by using special spines, and Piunikhin [9] showed that the Kauffman invariant coincides with the Turaev-Viro invariant. So, we call the Kauffman invariant as the Turaev-Viro invariant[1].

In this paper, we give a formula of the Turaev-Viro invariants of all orientable closed Seifert fibered manifolds [7]. In [10], a formula of the Turaev-Viro invariants of all closed

Received April 21, 2006; revised November 21, 2006
This work is supported in part by the Ministry of Education, Culture, Sport, Science and Technology in Japan under a grant in Aid of the 21st Century Center of Excellence for Integrative Mathematical Sciences: Progress in Mathematics Motivated by Social and Natural Sciences.

[1] Kauffman said in [3] that "Kauffman invariant is our version of the Turaev-Viro invariant".

Seifert fibered manifolds is given. It is based on the surgery presentation of 3-manifold. Our formula is based on a new construction of special spines of all orientable closed Seifert fibered manifolds shown in [11]. The outline of the construction is as follows. We define five 2-dimensional polyhedra P_ϕ, P_L, P_R, P_J and P_W with non-empty boundaries embedded in compact oriented 3-manifolds $D^2 \times S^1$, $S^1 \times S^1 \times [0, 1]$, $S^1 \times S^1 \times [0, 1]$, $\bigl(T^2 - \mathrm{Int}(D^2)\bigr) \times S^1$ and $\bigl(S^2 - \amalg_{i=1}^3 \mathrm{Int}(D_i^2)\bigr) \times S^1$ respectively. Then, any orientable closed Seifert fibered manifold and its special spine can be obtained by gluing these compact manifolds with polyhedra. These pairs of a manifold and a polyhedron can be regarded as cobordisms between closed surfaces in which 3-regular graphs are embedded. By a similar method shown in [13], we assign to each boundary component Σ of these cobordisms a vector space $V(\Sigma)$, and assign to each cobordism W a $\mathbf{C}$-linear map Z_W by using the Turaev-Viro invariants of compact 3-manifolds. When we restrict ourselves to these cobordisms the assignment satisfies an axiom of $(2+1)$-dimensional topological quantum field theory by Atiyah [1]. It induces the "gluing lemma" to calculate the invariant. Thus, we have a formula of the Turaev-Viro invariants of all orientable closed Seifert fibered manifolds.

Our formula gives sufficient conditions of coincidence of the Turaev-Viro invariants of orientable closed Seifert fibered manifolds. They are related to the continued fraction of β_i/α_i, where α_i and β_i are indices of the Seifert presentation of $M := S(F_g, b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_n, \beta_n))$. Also, we make a computer program by Mathematica to calculate the Turaev-Viro invariant $TV^{(r)}(M)$. The input of our program is a level r, the genus g of base space of M, the obstruction class b and the indices (α_i, β_i).

This paper is organized as follows. In Section 2, we introduce the definition of special spines and the Turaev-Viro invariants of closed 3-manifolds. In Section 3, we define DS-spines and linear maps obtained from DS-spines by using the Turaev-Viro invariants of compact 3-manifolds. In Section 4, we consider linear maps Z_ϕ, Z_L, Z_R, Z_J and $Z_{W(n)}$ obtained from DS-spines P_ϕ, P_L, P_R, P_J and $P_{W(n)}$. We note that a special spine of any orientable closed Seifert fibered manifold can be obtained by gluing these DS-spines. In Section 5, we give a formula of the Turaev-Viro invariant $TV^{(r)}(M)$ at the level r of all orientable closed Seifert fibered manifolds by using presentation matrices of the linear maps Z_ϕ, Z_L, Z_R, Z_J and $Z_{W(n)}$.

2. The Turaev-Viro invariants

For an integer $r \geq 3$, the Turaev-Viro invariant $TV^{(r)}(M)$ of 3-manifold M at a level r [13] is originally defined by using a triangulation of M. In this section, we describe a definition of the invariant in terms of special spines of 3-manifolds.

2.1. The Special spine. A 2-dimensional polyhedron P is called *simple* if each point x in P has a regular neighborhood $N(x)$ which is homeomorphic either to (i), (ii) or (iii) shown in Figure 1. A simple polyhedron has a natural stratification. A simple polyhedron P is called *special* if each i-stratum of P is an open i-cell, where $i = 1, 2$. We call a 0-cell, a

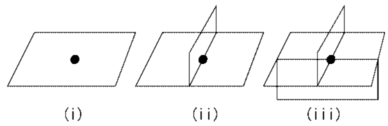


FIGURE 1. Neighborhood of a point of simple polyhedron.

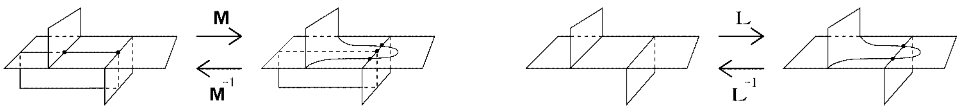


FIGURE 2. M-move and L-move of special spines.

1-cell and a 2-cell of P as a *vertex*, an *edge* and a *face* of P, and denote by $V(P)$, $E(P)$ and $F(P)$ the set of all vertices, edges and faces of P respectively.

Let M be a compact connected 3-manifold and P be a simple polyhedron which is embedded in M. When ∂M is non-empty, P is called *a spine of M* if M collapses to P. When ∂M is empty, that is M is closed, P is called *a spine of M* if $M - \mathrm{Int}(B^3)$ collapses to P, where B^3 is a 3-ball in the interior of M.

THEOREM 2.1 (Casler[2]). *Any compact* 3*-manifold possesses a special spine.*

THEOREM 2.2 (Matveev[6], Piergallini[8]). *Any two special spines of a closed* 3*-manifold can be transformed one to another by a sequence of moves* $\mathrm{M}^{\pm 1}$ *and* $\mathrm{L}^{\pm 1}$ *shown in Figure* 2 *with intermediate results also being special spines.*

2.2. The Turaev-Viro invariants for closed 3-manifolds. Let M be a closed 3-manifold. Then, the Turaev-Viro invariant of M at a level r, denoted by $\mathrm{TV}^{(r)}(M)$, is defined by the following three steps.

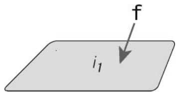
$\boxed{\text{Step 1}}$ Take a special spine P of M.

By a coloring of P, we mean an arbitrary mapping $\sigma : F(P) \to \{0, 1, 2, \dots, r-2\}$. We call three integers $a, b, c \in \{0, 1, \dots, r-2\}$ form *r-admissible* if the following conditions hold.

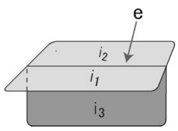
$$a+b+c \equiv 0 \ (\mathrm{mod}\ 2)\,, \quad a+b+c \leq 2r-4\,, \quad -a+b+c \geq 0\,,$$
$$a-b+c \geq 0\,, \quad a+b-c \geq 0\,.$$

A coloring σ is called *r-admissible* if for any edge $e \in E(P)$, three integers assigned to faces which are adjacent to the edge e form r-admissible. We denote by $\mathrm{Adm}^{(r)}(P)$ the set of all r-admissble colorings of P.

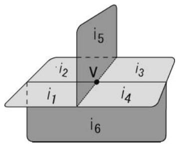
$\boxed{\text{Step 2}}$ Let $\sigma \in \mathrm{Adm}^{(r)}(P)$ be an r-admissible coloring. Then, we assign a complex number to each face f, edge e and vertex v of P as follows.

$\Delta^{(r)}$: {face of P} $\ni$


$\longmapsto \Delta^{(r)}(f,\sigma) := \delta^{(r)}(i_1) \in \mathbf{C}\,,$

$\Theta^{(r)}$: {edge of P} $\ni$


$\longmapsto \Theta^{(r)}(e,\sigma) := \theta^{(r)}(i_1, i_2, i_3) \in \mathbf{C}\,,$

$\mathrm{TET}^{(r)}$: {vertex of P} $\ni$


$\longmapsto \mathrm{TET}^{(r)}(v,\sigma) := \mathrm{Tet}^{(r)}\begin{bmatrix} i_1 & i_2 & i_5 \\ i_3 & i_4 & i_6 \end{bmatrix} \in \mathbf{C}\,,$

where the functions δ, θ and Tet are shown in Section 5.2 and $i_k \in \{0, 1, \ldots, r-2\}$ $(k = 1, \ldots, 6)$.

$\boxed{\text{Step 3}}$ The Turaev-Viro invariant $TV^{(r)}(M)$ of M at a level r is defined by

$$TV^{(r)}(M) := \sum_{\sigma \in \mathrm{Adm}^{(r)}(P)} \frac{\prod_{v \in V(P)} \mathrm{TET}^{(r)}(v,\sigma) \prod_{f \in F(P)} \Delta^{(r)}(f,\sigma)}{\prod_{e \in E(P)} \Theta^{(r)}(e,\sigma)}\,.$$

The topological invariance comes from the following equations [13].

$$(1) \qquad \sum_{c=0}^{r-2} \begin{Bmatrix} d & i & c \\ b & i & a \end{Bmatrix} \begin{Bmatrix} i & d & a' \\ j & b & c \end{Bmatrix} = \begin{cases} 0 & a = a'\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

$$(2) \qquad \sum_{m=0}^{r-2} \begin{Bmatrix} a & i & m \\ d & e & j \end{Bmatrix} \begin{Bmatrix} b & c & l \\ d & m & i \end{Bmatrix} \begin{Bmatrix} b & l & k \\ e & a & m \end{Bmatrix} = \begin{Bmatrix} b & c & k \\ j & a & i \end{Bmatrix} \begin{Bmatrix} k & c & l \\ d & e & j \end{Bmatrix},$$

where $\{\ \}$ is called *6j-symbol* defined by $\begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix} := \dfrac{\mathrm{Tet}\begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix}\delta(i)}{\theta(a,i,d)\theta(b,i,c)}$. These equations are corresponding to invariance under an M-move and an L-move for special spines respectively shown in Figure 3. (for detail see [13])

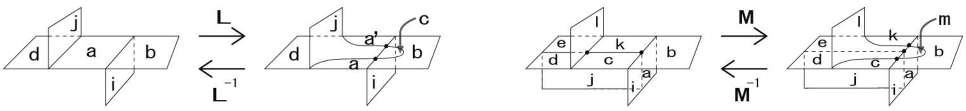


FIGURE 3. (1) L-move and (2) M-move.

3. DS-spines and linear maps obtained from DS-spines

In this section, we define *DS-spines* of compact 3-manifolds with non-empty boundaries, and assign a **C**-linear map to a DS-spine by using the Turaev-Viro invariants of compact 3-manifolds. We will show that the assignment satisfies an axiom of topological quantum field theory [1] under some conditions in Section 5.1.

A 2-dimentional polyhedron P is called *simple polyhedron with non-empty boundary* if each point x in P has a regular neighborhood $N(x)$ which is homeomorphic either to (i), (ii), (iii), (iv) or (v) shown in Figure 4. The set $\{x \mid x \in P$ such that $N(x) \cong$ (iv) or (v)$\}$ is called the boundary of P, denoted by ∂P. A simple polyhedron with non-empty boundary ha a natural stratification. A simple polyhedron with non-empty boundary P is called *special* if each i-stratum of P is an open i-cell for $i = 1, 2$.

Let Σ be a closed surface and G be a 3-regular graph. Then, for an embedding $\varphi : G \to \Sigma$, the surface Σ is called *completely marked by* $\varphi(G)$ [13] if each component of $\Sigma - \varphi(G)$ is homeomorphic to an open 2-disc.

DEFINITION 3.1. Let M be a compact 3-manifold with non-empty boundary and P be a simple polyhedron with non-empty boundary. Then, P is *a DS-spine* of M if P is properly embedded in M, that is, $(\partial P, P) \subset (\partial M, M)$ and ∂M is completely marked by ∂P and $M - \mathrm{Int}(B^3)$ collapses to $P \cup \partial M$, where B^3 is a 3-ball in the interior of M.

We can prove the following proposition by using the method called "arch construction" shown in [5].

PROPOSITION 3.2. *For a compact* 3*-manifold M such that ∂M is completely marked by $\varphi(G)$, there exists a special DS-spine P of M such that $\varphi(G) = P \cap \partial M$. We call such P as a special DS-spine of* (M, G, φ).

Now, we assign a **C**-linear map to a DS-spine by using a similar method shown in [13]. Let Σ be a connected oriented closed surface and G be a connected directed and labeled 3-regular graph, that is, all edges of G have orientations and different "names", and $\varphi : G \hookrightarrow \Sigma$ be an embedding such that Σ is completely marked by $\varphi(|G|)$, where $|G|$ is the underling space of the graph G.

Let us consider such objects $\Gamma = (\Sigma, G, \varphi)$ and $\Gamma' = (\Sigma', G', \varphi')$, and an orientation preserving homeomorphism $h : \Sigma \to \Sigma'$. We suppose that $G = G'$. Then, Γ and Γ' are *h-equivalent*, denoted by $\Gamma \overset{h}{\approx} \Gamma'$, if $\varphi' \cdot \mathrm{id} = h \cdot \varphi$, where $\mathrm{id} : G \to G'$ is the identity map

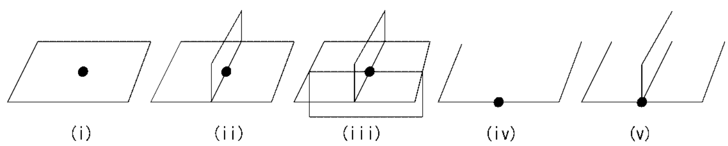


FIGURE 4. Neighborhood of a point of simple polyhedron with non-empty boundary.

on $G = G'$. Γ and Γ' are *h-quasi-equivalent*, denoted by $\Gamma \overset{h}{\sim} \Gamma'$, if $h \cdot \varphi(|G|) = \varphi'(|G'|)$. We note that when two objects Γ and Γ' are h-quasi-equivalent, there exists a canonical isomorphism $\hat{h} : G \to G'$ such that $\varphi' \cdot \hat{h} = h \cdot \varphi$.

DEFINITION 3.3. A pair of compact oriented 3-manifold M and 2-dimensional polyhedron P is a *cobordism* between $\Gamma = (\Sigma, G, \varphi)$ and $\Gamma' = (\Sigma', G', \varphi')$ if $\partial M = \Sigma \sqcup (-\Sigma')$ and P is a special DS-spine of $(M, G \sqcup G', \varphi \sqcup \varphi')$, where $-\Sigma'$ means Σ' with the opposite orientation and $\varphi \sqcup \varphi'$ is the embedding of $G \sqcup G'$ defined by φ and φ'. We denote such a cobordism by $W = (M, P; \Gamma, \Gamma')$, or briefly $W = (M, P)$.

NOTATION 3.4. For two objects $\Gamma = (\Sigma, G, \varphi)$ and $\Gamma' = (\Sigma', G', \varphi')$, $\mathrm{Hom}_1(\Gamma, \Gamma')$ is the set of all cobordisms between Γ and Γ'. $\mathrm{Hom}_2(\Gamma, \Gamma')$ is the set of all orientation preserving homeomorphisms $h : \Sigma \to \Sigma'$ such that $\Gamma \overset{h}{\sim} \Gamma'$.

The identity cobordism on $\Gamma = (\Sigma, G, \varphi)$ is given by $W = (\Sigma \times [0,1], \varphi(G) \times [0,1]; \Gamma, \Gamma)$. For two cobordisms $W_1 = (M_1, P_1\,;\, \Gamma, \Gamma')$ and $W_2 = (M_2, P_2\,;\, \Gamma', \Gamma'')$, the composition of W_1 and W_2 is defined by $W_2 \cdot W_1 := (M_1 \cup_{\mathrm{id}} M_2,\ P_1 \cup_{\mathrm{id}} P_2\ ;\ \Gamma,\ \Gamma'')$. Two cobordisms $W = (M, P\,;\, \Gamma_1, \Gamma_2)$ and $W' = (M', P'\,;\, \Gamma_1', \Gamma_2')$ are *equivalent* if there exists an orientation preserving homeomorphism $H : M \to M'$ such that $\Gamma_i \overset{h_i}{\sim} \Gamma_i'$, where $h_i := H|_{\Sigma_i}$ for $i = 1, 2$.

For each level $r \geq 3$, we will assign a $\mathbf{C}$-vector space $V^{(r)}(\Gamma)$ to an object $\Gamma = (\Sigma, G, \varphi)$, assign a $\mathbf{C}$-linear map $Z_W^{(r)}$ to a cobordism $W(M, P; \Gamma, \Gamma') \in \mathrm{Hom}_1(\Gamma, \Gamma')$, assign a $\mathbf{C}$-linear map $h_*^{(r)}$ to a homeomorphism $h \in \mathrm{Hom}_2(\Gamma, \Gamma')$. For simplicity, we denote $V(\Gamma)$, Z_W and h_* instead of $V^{(r)}(\Gamma)$, $Z_W^{(r)}$ and $h_*^{(r)}$.

At first, we define a vector space $V(\Gamma)$. A level $r \geq 3$ is fixed. Let G be a 3-regular graph. By a coloring of G, we mean an arbitrary mapping $\tau : E(G) \to \{0, 1, 2, \ldots, r-2\}$. A coloring τ is called *r-admissible* if for any vertex v of G, three integers of edges adjacent to the vertex v form r-admissible. We denote by $\mathrm{Adm}^{(r)}(G)$ the set of all r-admissble colorings of G. For simplicity, we denote Adm(G) instead of $\mathrm{Adm}^{(r)}(G)$.

DEFINITION 3.5. For an object $\Gamma = (\Sigma, G, \varphi)$, the vector space $V^{(r)}(\Gamma)$ is freely spanned by $\mathrm{Adm}^{(r)}(G)$ over $\mathbf{C}$. In the case where Σ is the empty surface $\emptyset$, we define $V^{(r)}(\emptyset) := \mathbf{C}$.

Since the vector space $V(\Gamma)$ is spanned by $\mathrm{Adm}(G)$, we denote $V(G)$ instead of $V(\Gamma)$.

Now, we define a $\mathbf{C}$-linear map Z_W for a cobordism $W = (M, P\,;\, \Gamma, \Gamma')$. Before defining the linear map, we prepare a notation $Z^{(r)}(P, \tau(\partial P))$. Let P be a special polyhedron with non-empty boundary. For a coloring $\tau \in \mathrm{Adm}(\partial P)$, the complex number $Z^{(r)}(P, \tau(\partial P))$ is defined by

$$C^{(r)}(\partial P, \tau) \sum_{\sigma \in \mathrm{Adm}^{(r)}(P,\tau)} \frac{\prod_{v \in V(P)-V(\partial P)} \mathrm{TET}^{(r)}(v, \sigma) \prod_{f \in F(P)} \Delta^{(r)}(f, \sigma)}{\prod_{e \in E(P)-E(\partial P)} \Theta^{(r)}(e, \sigma)},$$

where

$$C^{(r)}(\partial P, \tau) := \frac{\prod_{v \in V(\partial P)} \sqrt{\theta^{(r)}\left(\tau(e_v), \tau(e'_v), \tau(e''_v)\right)}}{\prod_{e \in E(\partial P)} \sqrt{\delta^{(r)}(\tau(e))}},$$

e_v, e'_v and e''_v are three edges adjacent to the vertex v, and $\mathrm{Adm}^{(r)}(P, \tau) := \{\sigma \in \mathrm{Adm}^{(r)}(P)$ such that $\sigma|_{\partial P} = \tau\}$.

DEFINITION 3.6. For a cobordism $W = (M, P\,;\, \Gamma, \Gamma')$ between two objects $\Gamma = (\Sigma, G, \varphi)$ and $\Gamma' = (\Sigma', G', \varphi')$, the linear map $Z_W^{(r)} : V^{(r)}(G) \to V^{(r)}(G')$ is defined by the following equation.

$$Z_W^{(r)}(\tau) := \sum_{\tau' \in \mathrm{Adm}^{(r)}(G')} Z^{(r)}\left(P, \tau(G) \sqcup \tau'(G')\right) \tau',$$

where $\tau \in \mathrm{Adm}^{(r)}(G)$.

At last, we define a $\mathbf{C}$-linear map f_* for a homeomorphism $f \in \mathrm{Hom}_2(\Gamma, \Gamma')$.

DEFINITION 3.7. For a homeomorphism $f \in \mathrm{Hom}_2(\Gamma, \Gamma')$, the linear map $f_*^{(r)}$: $V^{(r)}(G) \to V^{(r)}(G')$ is defined by the following equation.

$$f_*(\tau) := \sum_{\tau' \in \mathrm{Adm}^{(r)}(G')} Z^{(r)}\big(\varphi'(G') \times [0, 1], \hat{f}(\tau)(G') \sqcup \tau'(G')\big)\, \tau',$$

where $\tau \in \mathrm{Adm}^{(r)}(G)$ and $\hat{f}(\tau)$ is the coloring of G' defined by $\hat{f}(\tau)(e) = \tau\left(f^{-1}(e)\right)$ for an edge $e \in E(G')$.

We note that the assignments $\Gamma \mapsto V(\Gamma)$, $W \mapsto Z_W$ and $f \mapsto f_*$ do not necessarily satisfy the axiom of topological quantum field theory. But, the axiom is satisfied under the condition that the gluing maps between cobordisms keep the property that $M - \mathrm{Int}(B^3)$ collapses to $P \cup \partial M$. Since the special spine defined in [11] satisfies the condition, we apply the axiom to these cobordisms and get a formula of the Turaev-Viro invariants of orientable closed Seifert fibered manifolds in Section 5.

4. Linear maps obtained by cobordisms yielding a special spine of any orientable closed Seifert fibered manifold

In [11], we define five special DS-spines P_ϕ, P_L, P_R P_J and $P_{W(n)}$ of the compact fibered 3-manifolds $V := D^2 \times S^1$, $U := S^1 \times S^1 \times [0, 1]$, U, $J := \big(S^1 \times S^1 - \mathrm{Int}(D^2)\big) \times S^1$ and

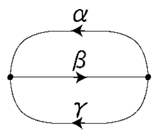


FIGURE 5. theta curve θ.

$W(n) := \left(S^2 - \coprod_{i=1}^{n} \mathrm{Int}(D_i^2)\right) \times S^1$ respectively. Then, we showed that any orientable closed Seifert fibered manifold and its special spine can be obtained by gluing these compact fibered manifolds equipped with special DS-spines.

These special DS-spines have the following property. Each component of the boundaries of them is the directed labeled theta-curve θ shown in Figure 5. So, we can regard these compact 3-manifolds with special DS-spines as cobordisms in the following way. The pair $W_\phi := (V, P_\phi)$ is regarded as a cobordism from the empty surface $\emptyset$ to the object $\Gamma = (T^2, \theta, \varphi_\phi)$. For $X = L, R$, the pair $W_X := (U, P_X)$ is regarded as a cobordism from $\Gamma_X^{(0)} = (T^2, \theta, \varphi_X^{(0)})$ to $\Gamma_X^{(1)} = (T^2, \theta, \varphi_X^{(1)})$. The pair $W_J := (J, P_J)$ is regarded as the cobordism from the empty surface $\emptyset$ to the object $\Gamma := (T^2, \theta, \varphi_J)$. The pair $W_{W(n)} := (W(n), P_{W(n)})$ is regarded as a cobordism from the empty surface $\emptyset$ to $\coprod_{i=1}^{n} \Gamma_W^{(i)}$, where $\Gamma_W^{(i)} := (T^2, \theta_i, \varphi_W)$.

In this section, we consider linear maps $Z_\phi^{(r)}$, $Z_L^{(r)}$, $Z_R^{(r)}$ $Z_J^{(r)}$ and $Z_{W(n)}^{(r)}$ obtained by applying Definition 3.6 to the cobordisms $W_\phi := (V, P_\phi)$, $W_L := (U, P_L)$, $W_R := (U, P_R)$, $W_J := (J, P_J)$ and $Z_{W(n)} := (W(n), P_{W(n)})$. By definition, domains and ranges of these linear maps are as follows.

$$Z_\phi^{(r)} : \mathbf{C} \to V^{(r)}(\theta)\,, \qquad Z_L^{(r)} : V^{(r)}(\theta) \to V^{(r)}(\theta) \qquad Z_R^{(r)} : V^{(r)}(\theta) \to V^{(r)}(\theta)$$
$$Z_J^{(r)} : \mathbf{C} \to V^{(r)}(\theta)\,, \qquad Z_{W(n)}^{(r)} : \mathbf{C} \to V^{(r)}(\amalg_{i=1}^{n} \theta_i)\,.$$

Throughout this section, a level $r \geq 3$ is fixed. For simplicity, we sometimes omit the character "r".

In Section 4.1, we give an order to the basis $\mathrm{Adm}(\theta)$ of the vector space $V(\theta)$. In Section 4.2 and Section 4.3, we calculate presentation matrices of the linear maps Z_ϕ, Z_L, Z_R and Z_J with respect to an ordered basis. In Section 4.4, we give an order to a basis of the vector space $V(\amalg_{i=1}^{n} \theta)$ and calculate presentation matrix of the linear map $Z_{W(n)}$.

4.1. An order of the element of $V^{(r)}(\theta)$. We give an order to the basis $\mathrm{Adm}^{(r)}(\theta)$ of the vector space $V^{(r)}(\theta)$ as follows. By $\mathrm{adm}(r)$, we mean the set of all ordered triple integers which form r-admissible.

EXAMPLE 4.1 ($r = 3$). $\mathrm{adm}(3) := \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$.

We give the dictionary-order to the set $\mathrm{adm}(r)$, and denote by $\mu_i^{(r)}$ the i-th element of $\mathrm{adm}(r)$, and denote by $a_i^{(r)}$, $b_i^{(r)}$ and $c_i^{(r)}$ the first, the second and the third element of $\mu_i^{(r)}$.

EXAMPLE 4.2 ($r = 3$).

$$\begin{array}{llll} \mu_1^{(3)} = (0,0,0)\,, & a_1^{(3)} = 0\,, & b_1^{(3)} = 0\,, & c_1^{(3)} = 0\,, \\ \mu_2^{(3)} = (0,1,1)\,, & a_2^{(3)} = 0\,, & b_2^{(3)} = 1\,, & c_2^{(3)} = 1\,, \\ \mu_3^{(3)} = (1,0,1)\,, & a_3^{(3)} = 1\,, & b_3^{(3)} = 0\,, & c_3^{(3)} = 1\,, \\ \mu_4^{(3)} = (1,1,0)\,, & a_4^{(3)} = 1\,, & b_4^{(3)} = 1\,, & c_4^{(3)} = 0\,. \end{array}$$

Then, we define an order to the set $\mathrm{Adm}^{(r)}(\theta)$ as follows.

DEFINITION 4.3. $\tau \in \mathrm{Adm}^{(r)}(\theta)$ is the i-th element if $\tau(\alpha) = a_i^{(r)}$, $\tau(\beta) = b_i^{(r)}$ and $\tau(\gamma) = c_i^{(r)}$, where α, β and γ are edges of the theta-curve θ.

4.2. A presentation matrix of the linear map Z_ϕ. We consider the presentation matrix of the linear map $Z_\phi : V(\emptyset) \to V(\theta)$ with respect to the ordered basis $\{\tau_i\} = \mathrm{Adm}^{(r)}(\theta)$. It can be regarded as a vector in $V(\theta)$ since $V(\emptyset) = \mathbf{C}$.

LEMMA 4.4. *The i-th element of the presentation matrix $v_\phi = (\phi_i)$ of the linear map $Z_\phi : \mathbf{C} \to V(\theta)$ with respect to the ordered basis $\{\tau_i\} = Adm(\theta)$ is given by*

$$\phi_i = \sum_{k=0}^{r-2} \frac{Tet\begin{bmatrix} k & b_i & b_i \\ a_i & b_i & b_i \end{bmatrix} \delta(k)\sqrt{\delta(a_i)}}{\theta(a_i, b_i, b_i)\theta(k, b_i, b_i)}\,,$$

where the sum is taken under the condition that three integers k, b_i and b_i form r-admissible.

PROOF. By definition, we have $\phi_i = Z^{(r)}(P_\phi, \tau_i(\theta))$, where the special DS-spine P_ϕ is obtained by the ϕ-diagram [11] shown in Figure 6. The coloring τ_i assign the integers a_i and b_i to the faces f_α and f_β, where $f_\alpha := \alpha\bar{Q}\bar{P}$ and $f_\beta := \beta P\bar{A}\bar{P}\gamma\bar{Q}\bar{A}Q$.

Suppose that an integer k is assigned to the face $f_A := A$. Since $P_\phi - \partial P_\phi$ has the only vertex w, and the neighborhood of it is shown in Figure 7, we have

$$\prod_{v \in V(P_\phi) - V(\partial P_\phi)} \mathrm{TET}(v, \tau_i) = \mathrm{TET}(w, \tau_i) = \mathrm{Tet}\begin{bmatrix} k & b_i & b_i \\ a_i & b_i & b_i \end{bmatrix}.$$

There are three edges A, P and Q in $P_\phi - \partial P_\phi$, and there are three faces f_A, f_α and f_β in P_ϕ. So, we get

$$\prod_{e \in E(P_\phi) - E(\partial P_\phi)} \Theta(e, \tau_i) = \Theta(A, \tau_i)\Theta(P, \tau_i)\Theta(Q, \tau_i) = \theta(k, b_i, b_i)\theta(a_i, b_i, b_i)\theta(a_i, b_i, b_i)\,,$$

$$\prod_{f \in F(P_\phi)} \Delta(f, \tau_i) = \Delta(f_A, \tau_i)\Delta(f_\alpha, \tau_i)\Delta(f_\beta, \tau_i) = \delta(k)\delta(a_i)\delta(b_i)\,.$$

Also, we have

$$\begin{aligned}
\prod_{v\in V(\partial P_\phi)} \sqrt{\theta(\tau_i(e_v),\tau_i(e'_v),\tau_i(e''_v))} &= \sqrt{\theta(\tau_i(e_v),\tau_i(e'_v),\tau_i(e''_v))}\sqrt{\theta(\tau_i(e_u),\tau_i(e'_u),\tau_i(e''_u))}\\
&= \sqrt{\theta(a_i,b_i,b_i)}\sqrt{\theta(a_i,b_i,b_i)}\\
&= \theta(a_i,b_i,b_i)\,,\\
\prod_{e\in E(\partial P_\phi)} \sqrt{\delta(\tau_i(e))} &= \sqrt{\delta(\tau_i(\alpha))}\sqrt{\delta(\tau_i(\beta))}\sqrt{\delta(\tau_i(\gamma))}\\
&= \sqrt{\delta(a_i)}\sqrt{\delta(b_i)}\sqrt{\delta(b_i)}\,.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\phi_i &= Z^{(r)}(P_\phi,\tau_i(\theta))\\
&= \frac{\theta(a_i,b_i,b_i)}{\sqrt{\delta(a_i)\delta(b_i)\delta(b_i)}}\sum_{k=0}^{r-2}\frac{\mathrm{Tet}\begin{bmatrix} k & b_i & b_i\\ a_i & b_i & b_i\end{bmatrix}\delta(k)\delta(a_i)\delta(b_i)}{\theta(k,b_i,b_i)\theta(a_i,b_i,b_i)\theta(a_i,b_i,b_i)}\\
&= \sum_{k=0}^{r-2}\frac{\mathrm{Tet}\begin{bmatrix} k & b_i & b_i\\ a_i & b_i & b_i\end{bmatrix}\delta(k)\sqrt{\delta(a_i)}}{\theta(a_i,b_i,b_i)\theta(k,b_i,b_i)}\,,
\end{aligned}$$

∎

4.3. Presentation matrices of the linear maps Z_L, Z_R and Z_J. Let X be L or R. We consider the presentation matrix M_X of the linear map $Z_X : V(\theta)\to V(\theta)$ with respect to the ordered basis $\{\tau_i\} = \mathrm{Adm}(\theta)$.

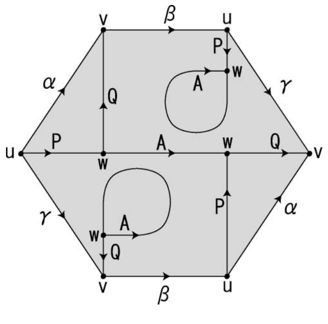


FIGURE 6. ϕ-diagram.

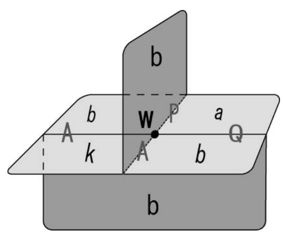


FIGURE 7. Neighborhood of the vertex w.

LEMMA 4.5. *The (i, j)-element of the presentation matrix $M_X = (X_{i,j})$ of the linear map $Z_X : V(\theta) \to V(\theta)$ with respect to the ordered basis $\{\tau_i\} = Adm^{(r)}(\theta)$ is given by*

$$L_{i,j} = \begin{cases} \dfrac{Tet\begin{bmatrix} a_i & b_i & c_i \\ a_i & a_j & c_i \end{bmatrix}\sqrt{\delta(a_j)}\sqrt{\delta(b_i)}}{\theta(a_i, a_j, c_i)\theta(a_i, b_i, c_i)} & \textit{if } a_i = b_j \textit{ and } c_i = c_j\,, \\ 0 & \textit{otherwise}\,. \end{cases}$$

$$R_{i,j} = \begin{cases} \dfrac{Tet\begin{bmatrix} a_i & b_i & c_i \\ a_i & c_j & c_i \end{bmatrix}\sqrt{\delta(c_j)}\sqrt{\delta(b_i)}}{\theta(a_i, c_j, c_i)\theta(a_i, b_i, c_i)} & \textit{if } a_i = a_j \textit{ and } c_i = b_j\,, \\ 0 & \textit{otherwise}\,. \end{cases}$$

PROOF. By definition, $X_{i,j} = Z^{(r)}\left(P_X, \tau_i(\theta) \sqcup \tau_j(\theta')\right)$ is the (i, j)-element of the presentation matrix of the linear map Z_X with respect to the ordered basis $\{\tau_i\} = \mathrm{Adm}(\theta)$.

At first, we consider the case $X = L$. The special DS-spine P_L is obtained by the L-diagram [11] shown in Figure 8. The coloring $\tau_i(\theta) \sqcup \tau_j(\theta')$ assigns the integers a_i, b_i, c_i, a_j, b_j and c_j to the faces $f_\alpha := \alpha\bar{Q}A\bar{\beta}'B\bar{P}$, $f_\beta := \beta PQ$, $f_\gamma := \gamma\bar{Q}\bar{B}\bar{\gamma}'\bar{A}\bar{P}$, $f_{\alpha'} := \alpha' BA$, f_α and f_γ respectively. So, in case that $a_i \neq b_j$ or $c_i \neq c_j$, the coloring $\tau_i(\theta) \sqcup \tau_j(\theta')$ doesn't rearized, that is, $L_{i,j} = 0$. In the other case, we have

$$\prod_{v \in V(P_L) - V(\partial P_L)} \mathrm{TET}(v, \tau_i(\theta) \sqcup \tau_j(\theta')) = \mathrm{TET}(w, (\tau_i(\theta), \tau_j(\theta'))) = \mathrm{Tet}\begin{bmatrix} a_i & b_i & c_i \\ a_i & a_j & c_i \end{bmatrix},$$

$$\prod_{f \in F(P_L)} \Delta\left(f, (\tau_i, \tau_j)\right) = \delta(a_i)\delta(b_i)\delta(c_i)\delta(a_j)\,,$$

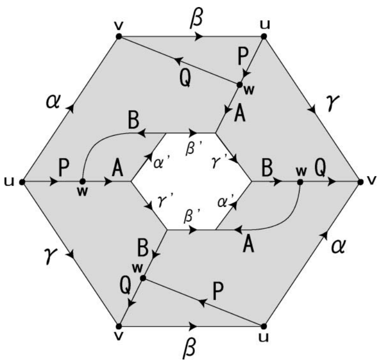


FIGURE 8. L-diagram.

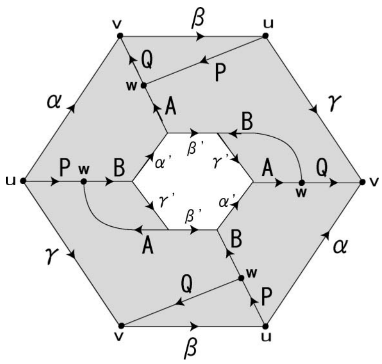


FIGURE 9. R-diagram.

$$\prod_{e\in E(P_L)-E(\partial P_L)} \Theta(e,\tau_i(\theta)\sqcup\tau_j(\theta'))$$
$$=\Theta(A,(\tau_j,\tau_i))\Theta(B,(\tau_j,\tau_i))\Theta(P,(\tau_j,\tau_i))\Theta(Q,(\tau_j,\tau_i))$$
$$=\theta(a_i,c_i,a_j)\theta(a_i,c_i,a_j)\theta(a_i,b_i,c_i)\theta(a_i,b_i,c_i)\,,$$

$$\prod_{v\in V(\partial P_L)} \sqrt{\theta(\tau(e_v),\tau(e'_v),\tau(e''_v))}$$
$$=\sqrt{\theta(\tau_j(e_{u'}),\tau_j(e'_{u'}),\tau_j(e''_{u'}))}\times\sqrt{\theta(\tau_j(e_{v'}),\tau_j(e'_{v'}),\tau_j(e''_{v'}))}$$
$$\times\sqrt{\theta(\tau_i(e_u),\tau_i(e'_u),\tau_i(e''_u))}\times\sqrt{\theta(\tau_i(e_v),\tau_i(e'_v),\tau_i(e''_v))}$$
$$=\sqrt{\theta(a_j,a_i,c_i)}\sqrt{\theta(a_j,a_i,c_i)}\sqrt{\theta(a_i,b_i,c_i)}\sqrt{\theta(a_i,b_i,c_i)}$$
$$=\theta(a_j,a_i,c_i)\theta(a_i,b_i,c_i)\,,$$

$$\prod_{e\in E(\partial P_L)} \sqrt{\delta(\tau(e))}$$
$$=\sqrt{\delta(\tau_j(\alpha'))}\sqrt{\delta(\tau_j(\beta'))}\sqrt{\delta(\tau_j(\gamma'))}\sqrt{\delta(\tau_i(\alpha))}\sqrt{\delta(\tau_i(\beta))}\sqrt{\delta(\tau_i(\gamma))}$$
$$=\sqrt{\delta(a_j)}\sqrt{\delta(a_i)}\sqrt{\delta(c_i)}\sqrt{\delta(a_i)}\sqrt{\delta(b_i)}\sqrt{\delta(c_i)}$$
$$=\sqrt{\delta(a_j)}\delta(a_i)\sqrt{\delta(b_i)}\delta(c_i)\,.$$

So, we get

$$L_{i,j}=Z^{(r)}(P_L,\tau_i(\theta)\sqcup\tau_j(\theta'))$$
$$=\frac{\theta(a_j,a_i,c_i)\theta(a_i,b_i,c_i)}{\sqrt{\delta(a_j)}\delta(a_i)\sqrt{\delta(b_i)}\delta(c_i)}\,\frac{\mathrm{Tet}\begin{bmatrix}a_i & b_i & c_i\\ a_i & a_j & c_i\end{bmatrix}\delta(a_j)\delta(a_i)\delta(b_i)\delta(c_i)}{\theta(a_i,c_i,a_j)\theta(a_i,c_i,a_j)\theta(a_i,b_i,c_i)\theta(a_i,b_i,c_i)}$$
$$=\frac{\mathrm{Tet}\begin{bmatrix}a_i & b_i & c_i\\ a_i & a_j & c_i\end{bmatrix}\sqrt{\delta(a_j)}\sqrt{\delta(b_i)}}{\theta(a_i,a_j,c_i)\theta(a_i,b_i,c_i)}\,.$$

In the same manner, we can prove the case $X=R$. ∎

Now, we consider the presentation matrix of the linear map $Z_J : V(\emptyset)\to V(\theta)$ with respect to the ordered basis $\{\tau_i\}=\mathrm{Adm}^{(r)}(\theta)$. It can be regarded as a vector in $V(\theta)$ since $V(\emptyset)=\mathbf{C}$.

The special DS-spine P_J is obtained by the DS-diagram Δ_J [11]. We name faces of D_J as shown in Figure 10. Then, we have the following Lemma. The proof is similar to Lemma 4.4.

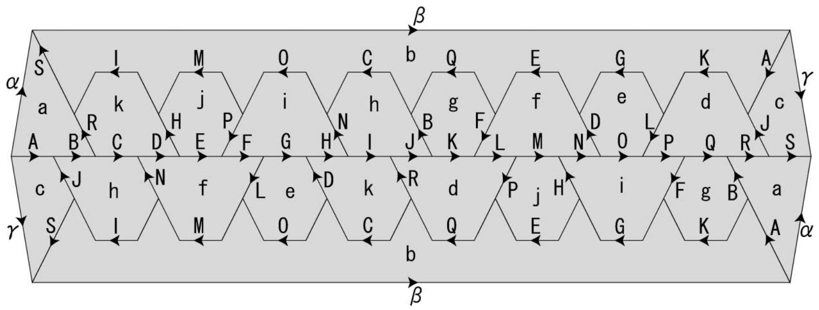


FIGURE 10. DS-diagram Δ_J.

LEMMA 4.6. *The l-th element of the presentation matrix $v_J = (J_l)$ of the linear map $Z_J : \mathbf{C} \to V(\theta)$ with respect to the basis $\{\tau_i\} = Adm^{(r)}(\theta)$ is given by*

$$J_l = \sum_{d,e,f,g,h,i,j,k=0}^{r-2} \left(\left(\prod_{t=1}^{9} Tet_t \right) \left(\prod_{t=1}^{11} \delta_t \right) \left(\prod_{t=1}^{18} \theta_t \right)^{-1} \right) \Bigg|\, a = a_l^{(r)} ,\ \ b = b_l^{(r)} ,\ \ c = c_l^{(r)} ,$$

where the sum is taken under the condition that the following triple integers are r-admissible (a, b, c), (d, c, h), (b, g, d), (f, g, i), (e, k, f), (b, e, i), (b, f, j), (i, j, d), (a, g, h), (f, h, i), (b, h, k), (e, d, f), (i, j, k), (a, d, k). The values Tet_i, δ_i and θ_i are given as follows.

$$Tet_1 = Tet\begin{bmatrix} e & b & d \\ g & f & i \end{bmatrix}, \quad Tet_2 = Tet\begin{bmatrix} b & c & d \\ h & g & a \end{bmatrix}, \quad Tet_3 = Tet\begin{bmatrix} b & e & f \\ k & j & i \end{bmatrix},$$

$$Tet_4 = Tet\begin{bmatrix} g & b & f \\ j & i & d \end{bmatrix}, \quad Tet_5 = Tet\begin{bmatrix} b & g & h \\ a & k & d \end{bmatrix}, \quad Tet_6 = Tet\begin{bmatrix} b & h & i \\ f & e & k \end{bmatrix},$$

$$Tet_7 = Tet\begin{bmatrix} j & b & i \\ e & d & f \end{bmatrix}, \quad Tet_8 = Tet\begin{bmatrix} b & j & k \\ i & h & f \end{bmatrix}, \quad Tet_9 = Tet\begin{bmatrix} b & k & a \\ d & c & h \end{bmatrix}.$$

$$\delta_1 = \sqrt{\delta(a)}, \quad \delta_2 = \sqrt{\delta(b)}, \quad \delta_3 = \sqrt{\delta(c)}, \quad \delta_4 = \delta(d), \quad \delta_5 = \delta(e), \quad \delta_6 = \delta(f),$$
$$\delta_7 = \delta(g), \quad \delta_8 = \delta(h), \quad \delta_9 = \delta(i), \quad \delta_{10} = \delta(j), \quad \delta_{11} = \delta(k).$$

$$\theta_1 = \theta(a, b, c), \quad \theta_2 = \theta(d, c, h), \quad \theta_3 = \theta(b, g, d), \quad \theta_4 = \theta(f, g, i),$$
$$\theta_5 = \theta(e, k, f), \quad \theta_6 = \theta(b, e, i), \quad \theta_7 = \theta(b, f, j), \quad \theta_8 = \theta(i, j, d),$$
$$\theta_9 = \theta(a, g, h), \quad \theta_{10} = \theta(b, g, d), \quad \theta_{11} = \theta(f, h, i), \quad \theta_{12} = \theta(b, h, k),$$
$$\theta_{13} = \theta(b, e, i), \quad \theta_{14} = \theta(e, d, f), \quad \theta_{15} = \theta(i, j, k), \quad \theta_{16} = \theta(b, f, j),$$
$$\theta_{17} = \theta(b, k, h), \quad \theta_{18} = \theta(a, d, k).$$

4.4. A presentation matrix of the linear map $Z_{W(n)}$. In this subsection, we consider the linear map $Z_{W(n)}$ which is obtained by the cobordism $W_{W(n)} := (M_{W(n)}, P_{W(n)})$, where $W_{W(n)}$ is defined by gluing $n-2$ copies of the cobordism $W_{W(3)}$, for detail see [11]. We note that the manifold $M_{W(n)}$ is homeomorphic to $\left(S^2 - \coprod_{i=1}^{n} \mathrm{Int}(D_i^2)\right) \times S^1$. Since we consider a presentation matrix of the linear map $Z_{W(n)} : V(\emptyset) \to V(\coprod_{i=1}^{n} \theta_i)$, we give an order to a basis of the vector space $V(\coprod_{i=1}^{n} \theta_i)$. The set $\prod_{i=1}^{n} \mathrm{Adm}^{(r)}(\theta_i) := \underbrace{\mathrm{Adm}^{(r)}(\theta_1) \times \mathrm{Adm}^{(r)}(\theta_2) \times \cdots \times \mathrm{Adm}^{(r)}(\theta_n)}_{n}$ is a basis of the vector space $V(\coprod_{i=1}^{n} \theta)$. We give an order to $\prod_{i=1}^{n} \mathrm{Adm}^{(r)}(\theta_i)$ by the following.

Step 1. We consider ordered n integers $(i_1, i_2, \ldots, i_n)$, where $i_k \in \left\{1, 2, \ldots, \frac{(r-1)r(r+1)}{6}\right\}$ and denote by N the set of all such elements.

Step 2. We give the dictionary-order to the set N and denote by $\mu_j := (j_1, j_2, \ldots, j_n)$ the j-th element of N.

Note that $\frac{(r-1)r(r+1)}{6}$ is the number of r-admissible colorings of the theta-curve θ.

DEFINITION 4.7. An element $(\tau_{j_1}, \tau_{j_2}, \ldots, \tau_{j_n})$ is the j-th element of $\prod_{i=1}^{n} \mathrm{Adm}^{(r)}(\theta_i)$.

By the element (τ_1, τ_2, τ_3), we define that the integers a_i, b_i, c_i $(i = 1, 2, 3)$ are assigned to the faces of the DS-diagram $\Delta_{W(3)}$ shown in Figure 11. Then, we consider the presentation matrix of the linear map $Z_{W(n)} : V(\emptyset) \to V(\coprod_{i=1}^{n} \theta_i)$ with respect to the ordered basis $\prod_{i=1}^{n} \mathrm{Adm}^{(r)}(\theta_i)$. It can be regarded as a vector in $V(\coprod_{i=1}^{n} \theta_i)$ since $V(\emptyset) = \mathbf{C}$. We denote it by $v_{W(n)}$. The i-th element $W(n)_i = W(n)_{(i_1, i_2, \ldots, i_n)}$ of the vector $v_{W(n)}$ is given by the following lemma.

LEMMA 4.8. *The element $W(n)_{(i_1, i_2, \ldots, i_n)}$ of the presentation matrix $v_{W(n)}$ of the linear map $Z_{W(n)} : \mathbf{C} \to V(\coprod_{i=1}^{n} \theta_i)$ is given by the following.*

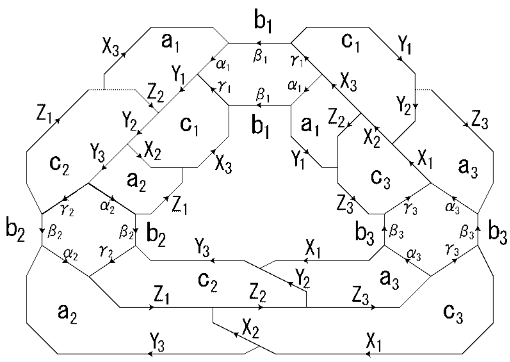


FIGURE 11. DS-diagram $\Delta_{W(3)}$ colored by (τ_1, τ_2, τ_3).

1. *In case* $n = 3$
   (a) *In case that* $b_{i_1} = b_{i_2} = b_{i_3}$ *and triple integers* $(a_{i_1}, c_{i_2}, c_{i_3})$, $(a_{i_2}, c_{i_3}, c_{i_1})$, $(a_{i_3}, c_{i_1}, c_{i_2})$ *and* (b, a_{i_j}, c_{i_j}) $(j = 1, 2, 3)$ *are* r*-admissible.*
   We put $b := b_{i_1} = b_{i_2} = b_{i_3}$. *Then,*

$$\begin{aligned} W(3)_{(i_1,i_2,i_3)} &= Z^{(r)}(P_{W(3)}, \tau_{i_1}(\theta) \sqcup \tau_{i_2}(\theta) \sqcup \tau_{i_3}(\theta)) \\ &= \frac{1}{\sqrt{\delta(b)}} \left(\prod_{i=1}^{3} Tet_i \right) \left(\prod_{i=1}^{6} \delta_i \right) \left(\prod_{i=1}^{6} \theta_i \right)^{-1}, \end{aligned}$$

where

$$Tet_1 = Tet \begin{bmatrix} b & a_{i_2} & c_{i_3} \\ c_{i_1} & a_{i_3} & c_{i_2} \end{bmatrix}, \qquad Tet_2 = Tet \begin{bmatrix} b & a_{i_3} & c_{i_1} \\ c_{i_2} & a_{i_1} & c_{i_3} \end{bmatrix},$$

$$Tet_3 = Tet \begin{bmatrix} b & a_{i_1} & c_{i_2} \\ c_{i_3} & a_{i_2} & c_{i_1} \end{bmatrix}.$$

$$\delta_1 = \sqrt{\delta(a_1)}, \quad \delta_2 = \sqrt{\delta(a_2)}, \quad \delta_3 = \sqrt{\delta(a_3)}, \quad \delta_4 = \sqrt{\delta(c_1)},$$
$$\delta_5 = \sqrt{\delta(c_2)}, \quad \delta_6 = \sqrt{\delta(c_3)}.$$

$$\theta_1 = \theta(b, a_{i_3}, c_{i_3}), \quad \theta_2 = \theta(b, a_{i_1}, c_{i_1}), \quad \theta_3 = \theta(b, a_{i_2}, c_{i_2}), \quad \theta_4 = \theta(a_{i_3}, c_{i_1}, c_{i_2}),$$
$$\theta_5 = \theta(a_{i_1}, c_{i_2}, c_{i_3}), \quad \theta_6 = \theta(a_{i_2}, c_{i_3}, c_{i_1}).$$

   (b) *Otherwise*

$$W(3)_{(i_1,i_2,i_3)} = 0.$$

2. *In case* $n > 3$

$$W(n+1)_{(i_1,i_2,\dots,i_{n+1})} = \sum_{j=0}^{\frac{(r-1)r(r+1)}{6}} W(n)_{(i_1,i_2,\dots,i_{n-1},j)} \times \begin{cases} W(3)_{(j,i_n,i_{n+1})} & (n \text{ is even}), \\ \overline{W(3)_{(j,i_n,i_{n+1})}} & (n \text{ is odd}), \end{cases}$$

where $\bar{c}$ *is the conjugation of a complex number* c.

5. The Turaev-Viro invariants for all orientable closed Seifert fibered manifolds

In this section, we give a formula of the Turaev-Viro invariants for all orientable closed Seifert fibered manifolds. Our formula is obtained by applying the "gluing lemma" shown in Section 5.1 to special DS-spines yielding a special spine of any orientable closed Seifert fibered manifold.

5.1. Topological quantum field theory. As mentioned in the previous section, any orientable closed Seifert fibered manifold and its special spine can be obtained by gluing the cobordisms $W_\phi = (V, P_\phi)$, $W_L = (U, P_L)$, $W_R = (U, P_R)$, $W_J = (J, P_J)$ and $W_{W(3)} = (W(3), P_{W(3)})$. We review the definition of gluing maps between these cobordisms [11]. Each connected boundary component of the compact 3-manifolds V, U, J and $W(3)$ is a torus T^2. So, the gluing map is an orientation preserving homeomorphism $f : T^2 \to T^2$. For $Q, R \in \{\phi, L^{(0)}, L^{(1)}, R^{(0)}, R^{(1)}, J, W(3)\}$, let $\Gamma_Q := (T^2, \theta, \varphi_Q)$ and $\Gamma_R := (T^2, \theta, \varphi_R)$ be two objects obtained from boundary components of these cobordisms. By definition, the gluing map $f : T^2 \to T^2$ satisfies $\varphi_R \cdot \mathrm{id} = f \cdot \varphi_Q$. Thus, the gluing map f is an element of $\mathrm{Hom}_2(\Gamma_Q, \Gamma_R)$. So, we apply Definition 3.7 to f, we have a $\mathbf{C}$-linear map $f_* : V(\theta) \to V(\theta)$.

When we restrict ourselves to the cobordisms W_ϕ, W_L, W_R, W_J and $W_{W(3)}$ and to these gluing maps, we have the following five properties on the assignments $\Gamma \mapsto V(\Gamma)$, $W \mapsto Z_W$ and $f \mapsto f_*$. They are called an axiom of (2 + 1)-dimensional topological quantum field theory (TQFT) as posed by Atiyah [1].

1. (a) Suppose that two cobordisms $W_1 = (M_1, P_1; \Gamma_1, \Gamma_2)$ and $W_2 = (M_2, P_2; \Gamma_2, \Gamma_3)$ are obtained from a cobordism $W = (M, P; \Gamma_1, \Gamma_3)$ by cutting along a closed surface Σ_2 such that $M_1 \cup_{\mathrm{id}_{\Sigma_2}} M_2 = M$ and $M_1 \cap M_2 = \Sigma_2$. Then, we have $Z_W = Z_{W_2} \cdot Z_{W_1}$.
   (b) $Z_{\mathrm{id}_\Gamma} = \mathrm{id}_{V(\Gamma)}$.
2. (a) For three objects $\Gamma_i = (\Sigma_i, G_i, \varphi_i)$, $i = 1, 2, 3$ such that $\Gamma_1 \overset{f}{\sim} \Gamma_2$ and $\Gamma_2 \overset{g}{\sim} \Gamma_3$, the equation $(g \cdot f)_* = g_* \cdot f_*$ holds.
   (b) $(\mathrm{id}_\Gamma)_* = \mathrm{id}_{V(\Gamma)}$
3. Let $W = (M, P\,;\, \Gamma_1, \Gamma_2)$ and $W' = (M', P'\,;\, \Gamma_1', \Gamma_2')$ be two cobordisms. Suppose that there exists an orientation preserving homeomorphism $f : M \to M'$ such that $\Gamma_i \overset{f_i}{\sim} \Gamma_i'$, $i = 1, 2$, where $f_1 := -f|_{\Sigma_1} : \Sigma_1 \to \Sigma_1'$ and $f_2 := f|_{\Sigma_2} : \Sigma_2 \to \Sigma_2'$. Then, the following diagram is commutative.

$$\begin{array}{ccc} V(\Gamma_1) & \xrightarrow{f_{1*}} & V(\Gamma_1') \\ {\scriptstyle Z_W}\big\downarrow & & \big\downarrow{\scriptstyle Z_{W'}} \\ V(\Gamma_2) & \xrightarrow[f_{2*}]{} & V(\Gamma_2') \end{array}$$

4. For two cobordisms $W_1 = (M, P_M; \Gamma_1, \Gamma_2)$ and $W_2 = (N, P_N; \Gamma_3, \Gamma_4)$ and a homeomorphism $f \in \mathrm{Hom}_2(\Gamma_2, \Gamma_3)$, the equation $Z_W = Z_{W_2} \cdot f_* \cdot Z_{W_1}$ holds, where W is the cobordism $(M \cup_f N, P_M \cup_f P_N\,;\, \Gamma_1, \Gamma_4)$.
5. There exists natural isomorphisms. (a) $V(\Gamma_1 \sqcup \Gamma_2) \cong V(\Gamma_1) \otimes V(\Gamma_2)$. (b) $V(\emptyset) \cong \mathbf{C}$. (c) $V(-\Gamma) \cong V(\Gamma)^*$, where $-\Gamma := (-\Sigma, G, \varphi)$, and $-\Sigma$ means Σ with the opposite orientation, and $V(\Gamma)^*$ is the dual vector space of $V(\Gamma)$.

By the axiom, we have the following lemma to calculate invariants called "gluing lemma" [1].

LEMMA 5.1. *Let Z be a $(2+1)$-dimensional TQFT. If a closed 3-manifold M is obtained by gluing two compact 3-manifolds M_1 and M_2 by an orientation preserving homeomorphism $f : \partial M_1 \to \partial M_2$, then we have*

$$Z(M) = \langle Z(f) \cdot Z(M_1)\,,\ Z(M_2)\rangle\,,$$

where $Z(M_1) = Z_{W_1}(1)$ and W_1 is a cobordism from the empty surface $\emptyset$ to ∂M_1 and $Z(M_2) = Z_{W_2}(1)$ and W_2 is a cobordism from ∂M_2 to the empty surface $\emptyset$, and the notation $\langle\ ,\ \rangle$ is the pairing between the vector space $Z(\partial M_1)$ and its dual space $Z(\partial M_2)^*$.

By using Lemma 5.1 and the presentation matrices v_ϕ, v_L, v_R, v_J and $v_{W(n)}$ of the linear maps Z_ϕ, Z_L, Z_R, Z_J and $Z_{W(n)}$, we get a formula of the Turaev-Viro invariants for all orientable closed Seifert fibered manifolds.

5.2. The Turaev-Viro invariants for lens spaces. A level $r \geq 3$ is fixed. The vector $v_\phi^{(r)}$ and the two matrices $M_L^{(r)}$ and $M_R^{(r)}$ are given in Lemma 4.4 and Lemma 4.5. For simplicity, we use the notations v_ϕ, M_L and M_R instead of $v_\phi^{(r)}$, $M_L^{(r)}$ and $M_R^{(r)}$ respectively.

THEOREM 5.2. *For two coprime natural numbers p and q such that $0 < q < p$, the Turaev-Viro invariant of lens space $L(p,q)$ at the level r is obtained by the Hermitian product of the two vectors $v_{p,q}^{(r)} = (v_i)_{i=1}^n$ and $u^{(r)} = (u_i)_{i=1}^n$, that is, $TV^{(r)}(L(p,q)) = \langle\, v_{p,q}^{(r)}\,,\ u^{(r)}\rangle := \sum_{i=1}^n v_i\overline{u_i}$, where the two vectors u and v are defined by the following.*

$$u^{(r)} := (M_L)^{-1} \cdot v_\phi\,,$$
$$v_{p,q}^{(r)} := \begin{cases} (M_L)^{a_n} \cdots (M_L)^{a_3} \cdot (M_R)^{a_2} \cdot (M_L)^{a_1}\, v_\phi \cdot & (n \text{ is odd})\,, \\ (M_R)^{a_n} \cdots (M_L)^{a_3} \cdot (M_R)^{a_2} \cdot (M_L)^{a_1}\, v_\phi \cdot & (n \text{ is even})\,, \end{cases}$$

where the natural numbers $\{a_i\}$ are determined by an expansion into continued fraction $q/p = [a_1, a_2, \ldots, a_n, 1]$, where we use the following notation.

$$[k_1,\, k_2,\, \ldots\,,\, k_{n-1},\, k_n] := \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{\ddots + \cfrac{1}{k_n}}}}\,.$$

PROOF. In [11], we show that a special spine of the lens space $L(p,q)$ can be obtained by gluing two cobordisms U and V, where

$$U := W_\phi \cup W_{\bar{L}}, \quad V = V(p,q) := \begin{cases} W_\phi \cup W_L^{a_1} \cup W_R^{a_2} \cup W_L^{a_3} \cup \cdots \cup W_L^{a_n} & (n \text{ is odd})\,, \\ W_\phi \cup W_L^{a_1} \cup W_R^{a_2} \cup W_L^{a_3} \cup \cdots \cup W_R^{a_n} & (n \text{ is even})\,. \end{cases}$$

The cobordism U is obtained by gluing W_ϕ and $W_{\bar{L}}$ by $f : \partial W_\phi \to \partial W_{\bar{L}}$. So, we get $Z_U = Z_{\bar{L}} \cdot f_* \cdot Z_\phi : \mathbf{C} \to V(\theta)$ by the axiom of TQFT shown in Section 5.1. Since f

identifies edges of the theta-curve θ assigned with the same label, f_* is the identity map on $V(\theta)$. So, we have $Z_U = Z_{\bar{L}} \cdot Z_\phi$. Similarly, for the cobordism V we see that Z_V is given by

$$Z_V = \begin{cases} (Z_L)^{a_n} \cdots (Z_L)^{a_3} \cdot (Z_R)^{a_2} \cdot (Z_L)^{a_1} \cdot Z_\phi & (n \text{ is odd}), \\ (Z_R)^{a_n} \cdots (Z_L)^{a_3} \cdot (Z_R)^{a_2} \cdot (Z_L)^{a_1} \cdot Z_\phi & (n \text{ is even}). \end{cases}$$

Thus, the presentation matrices $u^{(r)}$ and $v^{(r)}(p,q)$ of the linear maps $Z_U^{(r)}$ and $Z_V^{(r)}$ with respect to the basis $\{\mu_i\} = \mathrm{Adm}^{(r)}(\theta)$ are given by

$$u^{(r)} := M_{\bar{L}} \cdot v_\phi ,$$

$$v^{(r)}(p,q) := \begin{cases} (M_L)^{a_n} \cdots (M_L)^{a_3} \cdot (M_R)^{a_2} \cdot (M_L)^{a_1} \cdot v_\phi & (n \text{ is odd}), \\ (M_R)^{a_n} \cdots (M_L)^{a_3} \cdot (M_R)^{a_2} \cdot (M_L)^{a_1} \cdot v_\phi & (n \text{ is even}). \end{cases}$$

By definition of $\bar{L}$-diagram [11], we have $M_{\bar{L}} = (M_L)^{-1}$. So, we get $u^{(r)} = (M_L)^{-1} \cdot v_\phi$. The gluing map $f : \partial U \to \partial V$ induces the identity map on $V(\theta)$, and oriented lens space $L(p,q)$ is obtained if one of the orientation of U or V is reversed. Thus, we have $L(p,q) \cong V \cup_f -U$. So, we have

$$TV^{(r)}(L(p,q)) = \langle f_* \cdot Z_V \ , \ Z_U \rangle = \langle Z_V \ , \ Z_U \rangle .$$

Thus, the Turaev-Viro invariant of the lens space $L(p,q)$ at the level r is obtained the Hermitian product of the two vectors $v_{p,q}^{(r)}$ and $u^{(r)}$. ∎

REMARK 5.3. By calculation, we know that all elements of the vector v_ϕ and the matrix M_L are real number at the level $r = 3, 4, 5$. So, all elements of the vector u and v in Theorem 5.2 are real number at the level $r = 3, 4, 5$.

5.3. The Turaev-Viro invariants of all orientable closed Seifert fibered manifolds. Let $g \geq 0$ and b be integers, and let p_i and q_i be coprime natural numbers such that $q_i < p_i$ $(i = 1, 2, \ldots, n)$. In [11], we define the closed 3-manifold $M(F_g, (1,b), (p_1,q_1), \ldots, (p_n,q_n))$ by gluing cobordisms $\coprod_{i=1}^{g} W_J$, $W(b) := W_\phi \cup W_L \cup (W_R)^b \cup W_{\bar{L}}$, $\coprod_{i=1}^{n} V(p_i,q_i)$ and $W(n+g+1)$, where $W_{\bar{L}} := (D^2 \times S^1, P_L\ ;\ \Gamma_L^{(1)}, \Gamma_L^{(0)})$. It is an orientable closed Seifert fibered manifold on an orientable closed surface F_g with genus g which has n singular fibers with indices (α_i, β_i), where $\alpha_i = p_i$ and $\beta_i q_i \equiv 1 \pmod{p_i}$. Oppositely, any orientable closed Seifert fibered manifold has such presentation. (for detail see [11]).

In this section, we give a formula of the Turaev-Viro invariants of $M := M(F_g, (1,b), (p_1,q_1), \ldots, (p_n,q_n))$. Our formula shown in Theorem 5.4 is directly obtained by applying Lemma 5.1 and the axiom of TQFT to these cobordisms.

A level $r \geq 3$ is fixed. The vectors and matrices $v_\phi^{(r)}$, $v_L^{(r)}$, $v_R^{(r)}$, $v_J^{(r)}$ and $v_{W(n)}^{(r)}$ are given in Lemma 4.4, Lemma 4.5, Lemma 4.6 and Lemma 4.8. For simplicity, we use the notations v_ϕ, v_L, v_R, v_J and $v_{W(n)}$ instead of $v_\phi^{(r)}$, $v_L^{(r)}$, $v_R^{(r)}$, $v_J^{(r)}$ and $v_{W(n)}^{(r)}$ respectively.

THEOREM 5.4. *For pairs of coprime natural numbers* (p_i, q_i) *such that* $0 < q_i < p_i$ $(i = 1, 2, \ldots, n)$, *an integer* b *and a natural number* g, *the Turaev-Viro invariant of the orientable closed Seifert fibered manifold* $M := M(F_g, (1, b), (p_1, q_1), \ldots, (p_n, q_n))$ *at the level* r *is obtained by the Hermite product of the two vectors* $v_{W(n+g+1)}$ *and* $v_J{}^{\otimes g} \otimes v_b \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_n$, *that is,*

$$TV^{(r)}(M) = \langle v_{W(n+g+1)}\ ,\ v_J{}^{\otimes g} \otimes v_b \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_n \rangle\ ,$$

where the vector v_b *is defined by* $v_b := (M_L)^{-1} M_R^b M_L v_\phi$, *and the vector* $v_i := v^{(r)}(p_i, q_i)$ *is defined in Theorem* 5.2.

5.4. Corollaries of main theorems. In this subsection, we give a sufficient condition that the values of the Turaev-Viro invariant of two orientable closed Seifert fibered manifolds coincide when a level r is fixed.

At first, we prepare some notations. Let (p, q) and (p', q') be pairs of coprime natural numbers such that $0 < q < p$ and $0 < q' < p'$ and $n \leq n'$, where n and n' are natural numbers defined by $q/p = [a_1, a_2, a_3, \ldots, a_n, 1]$ and $q'/p' = [a'_1, a'_2, a'_3, \ldots, a'_{n'}, 1]$. Then, for a natural number $k \in \mathbf{N}$, we define that *(p', q') is obtained from (p, q) by a k-move* if either (1) or (2) holds.

(1) $n' = n$ and $a_l \equiv a'_l \pmod{k}$ for all l $(1 \leq l \leq n)$.

(2) $n' = n + 2$ and there exists an element a_l $(1 \leq l \leq n)$ such that

$$a'_i = a_i\ ,\quad a'_l = a_l - m\ ,\quad a'_{l+1} = k\ ,\quad a'_{l+2} = m\ ,\quad a'_{j+2} = a_j\ ,$$

where m is an arbitrary natural number such that $a_l - m > 0$ and $i = 1, 2, \ldots, l-1$ and $j = l+1, l+2, \ldots, n$.

Then, we call *(p', q') and (p, q) are k-equivalent*, denoted by $(p, q) \overset{k}{\sim} (p', q')$ if one of (p', q') or (p, q) is obtained from the other by a finite sequence of k-moves.

COROLLARY 5.5. *For any level* r *and natural number* k *such that* $(M_L^{(r)})^k = (M_R^{(r)})^k = E$ *where* E *is the unit matrix, the values of the Turaev-Viro invariant of two lens spaces* $L(p, q)$ *and* $L(p', q')$ *at the level* r, *where* $0 < q < p$ *and* $0 < q' < p'$, *are coincident if* (p, q) *and* (p', q') *are* k*-equivalent.*

PROOF. For simplicity, we denote M_L and M_R instead of $M_L^{(r)}$ and $M_R^{(r)}$. By Theorem 5.2, we have $TV^{(r)}(L(p, q)) = \langle u\ ,\ v \rangle$.

For any $i \in \{1, 2, \ldots, n\}$ where n is the length of the expansion into continued fraction of q/p, we get the following equation about the vector v.

$$\begin{aligned} v &= v_{p,q}^{(r)} \\ &= M_{X_n}{}^{a_n} \cdots M_L{}^{a_3} \cdot M_R{}^{a_2} \cdot M_L{}^{a_1} \cdot v_\phi \\ &= M_{X_n}{}^{a_n} \cdots M_{X_i}{}^{a_i - m} E M_{X_i}{}^{m} \cdots M_L{}^{a_3} \cdot M_R{}^{a_2} \cdot M_L{}^{a_1} \cdot v_\phi \end{aligned}$$

$$= {M_{X_n}}^{a_n} \cdots {M_{X_i}}^{a_i - m} {M_Y}^k {M_{X_i}}^m \cdots {M_L}^{a_3} \cdot {M_R}^{a_2} \cdot {M_L}^{a_1} \cdot v_\phi \,,$$

where $X_i, Y \in \{L, R\}$. If $X_i = Y$, we have $v^{(r)}_{p,q} = v^{(r)}_{p',q'}$, where $q'/p' = [a_1, a_2, \dots, a_{i-1}, a_i + k, a_{i+1}, \dots, a_n, 1]$. If $X_i \neq Y$, we have $v^{(r)}_{p,q} = v^{(r)}_{p',q'}$, where $q'/p' = [a_1, a_2, \dots, a_{i-1}, m, k, a_i - m, a_{i+1}, \dots, a_n, 1]$. Thus, when (p', q') is obtained by a 4-move from (p, q) we have $TV^{(r)}(L(p, q)) = \langle u^{(r)}, v^{(r)}_{p,q} \rangle = \langle u^{(r)}, v^{(r)}_{p',q'} \rangle = TV^{(r)}(L(p', q'))$. So, $TV^{(r)}(L(p, q))$ and $TV^{(r)}(L(p', q'))$ are equal if (p, q) and (p', q') are k-equivalent. ∎

In case $r = 3, 4, 5$, we set $k = 4, 16, 64$ respectively. Then, we have $(M^{(r)}_L)^k = (M^{(r)}_R)^k = E$. In case $r > 5$, the natural number k will be satisfied the following equation.

CONJECTURE 5.6. For any level $r \geq 3$, we set $k = 4^{r-2}$. Then, we have $(M^{(r)}_L)^k = (M^{(r)}_R)^k = E$.

We show an example of coincidence of the values of the Turaev-Viro invariant of lens spaces.

EXAMPLE 5.7 ($r = 3$).

$$\frac{1}{5} = [4, 1] \overset{4-move}{\longrightarrow} [8, 1] = \frac{1}{9}\,, \quad \frac{1}{5} = [4, 1] \overset{4-move}{\longrightarrow} [2, 4, 2, 1] = \frac{13}{29}\,.$$

So, we have $(4, 1) \overset{4}{\sim} (9, 1)$ and $(4, 1) \overset{4}{\sim} (29, 13)$. Thus, we get

$$TV^{(3)}\,(L(4, 1)) = TV^{(3)}\,(L(9, 1)) = TV^{(3)}\,(L(29, 13))\,.$$

We get a sufficient condition of coincidence of the values of the Turaev-Viro invariant of orientable closed Seifert fibered manifolds.

COROLLARY 5.8. *For any level r and natural number k such that $(M^{(r)}_L)^k = (M^{(r)}_R)^k = E$ where E is the unit matrix, the value of the Turaev-Viro invariant of two orientable closed Seifert fibered manifolds $M := M\big(F_g, (1, b), (p_1, q_1), \dots, (p_n, q_n)\big)$ and $M' := M\big(F_g, (1, b'), (p'_1, q'_1), \dots, (p'_n, q'_n)\big)$ at the level r, where $0 < q_i < p_i$ and $0 < q'_i < p'_i$ for all $i = 1, 2, \dots, n$, are coincident if two conditions* (1) *and* (2) *are hold.*

$$(1)\ \ b \equiv b' \ (mod\ k)\,, \quad (2)\ \ (p_i, q_i) \overset{k}{\sim} (p'_i, q'_i)\ for\ any\ i = 1, 2, \dots, n\,.$$

PROOF. The proof is similar to Corollary 5.5. By Theorem 5.4, the Turaev-Viro invariant of M is given by the inner product of the two vectors $v_{W(n+g+1)}$ and ${v_J}^{\otimes g} \otimes v_b \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_n$. In the proof of Corollary 5.5, we show $v_{p_i,q_i} = v_{p'_i,q'_i}$ if $(p_i, q_i) \overset{k}{\sim} (p'_i, q'_i)$. By the same reason, we have $v_b = v_{b'}$ if $b \equiv b'$ (mod k) because the vector v_b is defined by $v_b := (M_L)^{-1}\,(M_R)^b\,M_L v_\phi$. Thus, the conditions (1) and (2) hold, and we have $TV^{(r)}(M) = TV^{(r)}(M')$. ∎

At last, we show two examples of calculation of the Turaev-Viro invariant at the level $r = 3$.

1. Quaternionic space $Q = M(F_0, (2,1), (2,1), (2,1))$. By definition, we have $v_1^{(3)} = v_2^{(3)} = v_3^{(3)} = v^{(3)}(2,1) = M_L^{(3)} v_\phi^{(3)} = (1,0,1,0)$ and

$$v_{W(3)}^{(3)} = (1, 0, \dots, 0, 1_{(11)}, 0, \dots, 0, 1_{(22)}, 0, \dots, 0, 1_{(32)}, 0,\\ 0, \dots, 0, 1_{(35)}, 0, \dots, 0, 1_{(41)}, 0, \dots, 0, 1_{(56)}, 0, \dots, 0, 1_{(62)}, 0, 0),$$

where $1_{(i)}$ means that the i-th element of the vector $v_{W(3)}^{(3)}$ is 1. So, we get

$$TV^{(3)}(Q) = \langle v_{W(3)}^{(3)}, v_1^{(3)} \otimes v_2^{(3)} \otimes v_3^{(3)} \rangle = 4.$$

2. Brieskorn manifold $M = \Sigma(2,3,5) = M(F_0, (2,1), (3,1), (5,1))$
We have the following equations.

$$\begin{aligned} v_1^{(3)} &= v^{(3)}(2,1) = M_L^{(3)} v_\phi = (1,0,1,0), \\ v_2^{(3)} &= v^{(3)}(3,1) = (M_L^{(3)})^2 v_\phi = (1,1,0,0), \\ v_3^{(3)} &= v^{(3)}(5,1) = (M_L^{(3)})^4 v_\phi = (1,-1,0,0). \end{aligned}$$

Thus, we get $TV^{(3)}(M) = \langle v_{W(3)}^{(3)}, v_1^{(3)} \otimes v_2^{(3)} \otimes v_3^{(3)} \rangle = 1$.

6. Appendix

6.1. The vector v_ϕ and the matrices M_L and M_R

6.1.1. $r = 3$

$$v_\phi = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \qquad M_L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

6.1.2. $r = 4$

$$v_\phi = (1, -\sqrt{2}, 1, 0, 0, 0, 0, 0, 0, 0)^T,$$

where T means the transposition.

$$
M_L = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & -\frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{\sqrt{2}} & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & -\frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix},
$$

$$
M_R = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{pmatrix}.
$$

where $\cdot$ means 0.

6.1.3. $r = 5$

We put $a = \dfrac{5-\sqrt{5}}{5+\sqrt{5}}$.

$$
v_\phi = (1, -1-a^{1/2}, 1+a^{1/2}, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T ,
$$

where T means the transposition.

$$M_L = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & -a^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a^{1/4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a^{1/4} & \cdot & \cdot \\
\cdot & -a^{1/4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a^{1/4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a^{1/2} & \cdot & \cdot \\
\cdot & \cdot & a^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a^{1/4} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a^{1/4} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & a^{1/4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a^{1/4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a^{1/2} & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}$$

where $\cdot$ means 0.

$$
M_R = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -a^{1/2} & -a^{1/4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a^{1/2} & a^{1/4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -a^{1/4} & a^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a^{1/4} & -a^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a^{1/2} & a^{1/4} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a^{1/2} & a^{1/4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a^{1/4} & -a^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a^{1/4} & -a^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot
\end{pmatrix}.
$$

6.2. The functions δ, θ and Tet. For each level $r \geq 3$, the functions δ, θ and Tet are defined as follows [4].

- The function $\delta : \mathcal{C}^{(r)} \to \mathbf{C}$, where $\mathcal{C}^{(r)} := \{0, 1, 2, \ldots, r-2\}$.

$$\begin{aligned} \delta_{n+1} &= d\delta_n - \delta_{n-1}\,, & & \\ \delta_0 &:= 1\,, & d &:= -A^2 - A^{-2}\,, \\ \delta_{-1} &:= 0\,, & A &:= e^{i\pi/2r}\,. \end{aligned}$$

- The function $\theta : \{(a,b,c) \mid a,b,c \in \mathcal{C}^{(r)},\ (a,b,c) \text{ is } r\text{-admissible triple}\} \to \mathbf{C}$.

$$\begin{aligned} \theta(a,b,c) &:= \frac{(-1)^{m+n+p}[m+n+p+1]![n]![m]![p]!}{[m+n]![n+p]![p+m]!}\,, \\ m &:= (a+b-c)/2\,, \\ n &:= (-a+b+c)/2\,, \\ p &:= (a-b+c)/2\,, \\ [n] &:= (-1)^{n-1}\delta_{n-1}\,, \\ [n]! &:= [n][n-1]\cdots[2][1]\,, \\ [0]! &:= 1 \end{aligned}$$

- The function Tet : $\left\{ (a,b,c,d,e,f) \;\middle|\; \begin{array}{l} a,b,c,d,e,f \in \mathcal{C}^{(r)}, \\ (b,c,e), (a,b,f), (c,d,f), (a,d,e) \\ \text{are } r\text{-admissible triples} \end{array} \right\} \to \mathbf{C}$.

$$\mathrm{Tet}\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix} := \frac{\tau!}{\varepsilon!} \sum_{m \leq s \leq M} \frac{(-1)^s [s+1]!}{\prod_i [s-a_i]! \prod_j [b_j - s]!}\,.$$

$$\begin{aligned} \tau! &:= \textstyle\prod_{i,j}[b_j - a_i]!\,, & b_1 &:= (a+b+c+d)/2\,, \\ \varepsilon! &:= [a]![b]![c]![d]![e]![f]!\,, & b_2 &:= (a+c+e+f)/2\,, \\ a_1 &:= (a+b+e)/2\,, & b_3 &:= (b+e+d+f)/2\,, \\ a_2 &:= (a+d+f)/2\,, & m &:= \max\{a_i\}\,, \\ a_3 &:= (b+c+d)/2\,, & M &:= \min\{b_i\}\,, \\ a_4 &:= (c+d+e)/2\,. & & \end{aligned}$$

ACKNOWLEDGEMENTS. The author would like to express his gratitude to Professor I. Ishii and Professor M. Wakui for their advices.

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