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The Non-vanishing Cohomology of Orlik-Solomon Algebras

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Abstract. The cohomology of the complement of hyperplanes with coefficients in the rank one local system associated to a generic weight vanishes except in the highest dimension. In this paper, we construct matroids or arrangements admitting weights for which the Orlik-Solomon algebra has non-vanishing cohomology, using decomposable relations arising from Latin hypercubes.

1. Introduction

Let R be a commutative ring with 1. Write $[n] := \{1, 2, \ldots, n\}$. Let $E = E_R$ denote the graded exterior algebra over R generated by 1 and degree-one elements e_i for $i \in [n]$. Define an R-linear map $\partial : E^p \to E^{p-1}$ by $\partial 1 = 0$, $\partial e_i = 1$ for $i \in [n]$, and

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}$$

for $p \geq 2$ and $i_j \in [n]$. Let M be a loopless matroid on $[n]$ with rank $\ell + 1$.

DEFINITION 1.1. The Orlik-Solomon algebra of M is the quotient $A(M)$ of E by the ideal $\langle \partial M \rangle$ generated by $\partial(e_{i_1} \wedge \cdots \wedge e_{i_s})$ for every circuit $c = (i_1, \ldots, i_s)$ of M.

If 1 and 2 are parallel, that is, $\{1, 2\}$ is a circuit, then $e_1 = e_2$. So the Orlik-Solomon algebra of the simple matroid associated with M is equal to that of M. The ideal $\langle \partial M \rangle$ is homogeneous, so $A(M)$ inherits a natural grading from the exterior algebra E. The linear map ∂ on E induces the linear map ∂_M on $A(M)$. Let $e_\lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n \in E^1$. We call $\lambda = (\lambda_1, \ldots, \lambda_n)$ a weight of M. The left multiplication $e_\lambda \wedge : A^p(M) \to A^{p+1}(M)$ defines the complex $(A(M), e_\lambda)$. Let $H(A(M), e_\lambda)$ denote the cohomology of this complex. If $\lambda = 0$ then $H(A(M), e_\lambda)$ is just $A(M)$, otherwise we have $H^0(A(M), e_\lambda) = 0$. If $\sum_{j=1}^n \lambda_j \neq 0$ then we have $H^p(A(M), e_\lambda) = 0$ for all p (see [15]). If $\partial e_\lambda = \sum_{j=1}^n \lambda_j = 0$ then e_λ induces

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the complex $(\partial_M(A(M)), e_\lambda)$ and the cohomology $H(\partial_M(A(M)), e_\lambda)$, where $\partial_M(A(M))$ is the image of ∂_M. It is known that

$$H^{p+1}(A(M), e_\lambda) = H^{p+1}(\partial_M(A(M)), e_\lambda) \oplus H^p(\partial_M(A(M)), e_\lambda)\,.$$

For a generic weight λ, Yuzvinsky [15] proved the following vanishing theorem:

$$H^k(\partial_M(A(M)), e_\lambda) = 0 \quad \text{for } k \neq \ell\,.$$

Hence, we have

$$H^k(A(M), e_\lambda) = 0 \quad \text{for } k \neq \ell, \ell+1\,.$$

An arrangement $\mathcal{A}$ of hyperplanes in $\mathbf{P}^\ell$ has the rank $\ell+1$ matroid $M(\mathcal{A}) = M$ as underlying combinatorial structure. The cohomology of the complement of $\mathcal{A}$ is isomorphic to $\partial_M(A(M))$ (see [10] and [7]). If a weight $\lambda = (\lambda_i)_{i\in\mathcal{A}}$ satisfies a certain generic condition, then the cohomology of the complement of $\mathcal{A}$ with coefficients in the rank one local system associated to λ is isomorphic to $H(\partial_M(A(M)), e_\lambda)$ (see [5, 14]). The local system cohomology is an important subject in the multivariable theory of hypergeometric functions [2, 11]. By the vanishing theorem [15], for a generic weight λ, the local system cohomology vanishes in all but the top dimension. In this paper, our purpose is to construct matroids and arrangements with non-vanishing cohomology of Orlik-Solomon algebras, more precisely, with $H^{\ell-1}(A(M), e_\lambda) \neq 0$.

The case $\ell = 2$ has been studied in several papers, including [6, 9]. Falk [6] defined the *resonance variety* of the Orlik-Solomon algebra, as the space of weights with non-vanishing cohomology. The resonance variety is deeply related to the cohomology support loci [1] and the characteristic variety [8, 13] of the arrangement complement. Libgober and Yuzvinsky [9] showed that, under some condition, weights with non-vanishing first cohomology are parametrized by Latin squares.

In this paper, we prove that, in general, matroids associated to Latin hypercubes have weights with non-vanishing cohomology, by using decomposable relations arising from Latin hypercubes. This decomposable relation is a generalization of the relation discovered by Rybnikov (see [6]). Moreover, in the case $\ell = 2$, we give details of their matroids and derivations, using terms of Latin squares. In the last section, we shall give examples of realizations including the higher case. Some of them appear in classical projective geometry (see Figure 1, 2 and 3).

We shall use the following notation and terminology. A k-set is a set with cardinality k. Denote the family of all k-subset of a set S by $\binom{S}{k}$. Often, we regard a p-tuple $(i_1, \ldots, i_p)$ as a p-set $\{i_1, \ldots, i_p\}$. We refer to [12] for terminology of matroid theory.

2. Non-vanishing Theorem

A Latin hypercube of dimension ℓ and order m is an m^ℓ-array such that, if $\ell-1$ coordinates are fixed, the m positions so determined contain a permutation of m symbols. Let

$K = [k(i_1,\ldots,i_\ell)]_{1\le i_1,\ldots,i_\ell\le m}$ be a Latin ℓ-dimensional hypercube on $[m]$, that is, an m^ℓ-matrix satisfying the condition

$$\{k(i'_1,i_2,\ldots,i_\ell) : i'_1 \in [m]\} = \{k(i_1,i'_2,\ldots,i_\ell) : i'_2 \in [m]\} = \cdots$$

$$\cdots = \{k(i_1,i_2,\ldots,i'_\ell) : i'_\ell \in [m]\} = [m]\,,$$

for $1 \le i_1,\ldots,i_\ell \le m$. Define the family of $(\ell+1)$-subsets in $[n]$ associated to K by

$$\mathcal{C}[K] = [(i_1, m+i_2, 2m+i_3, \ldots, (\ell-1)m+i_\ell, \ell m + k(i_1,\ldots,i_\ell))]_{1\le i_1,\ldots,i_\ell\le m}\,.$$

On the other hand, a matroid is said to be *ℓ-generic* if it has no i-circuits for $i \le \ell$. Note that a 1-generic matroid is just a loopless matroid and a 2-generic matroid is just a simple matroid. The uniform matroid $U_{m,n}$ of rank m is m-generic. So we can state the main theorem as follows.

THEOREM 2.1. *Let $m \ge 2$, $\ell \ge 2$ and $n = (\ell+1)m$. Let K be a Latin ℓ-dimensional hypercube on $[m]$. Then there exists a unique ℓ-generic matroid $M[K]$ on $[n]$ with rank $\ell+1$, for which the family of all $(\ell+1)$-circuits is equal to $\mathcal{C}[K]$. Suppose that R is a field of characteristic zero (or the characteristic of the ring R does not divide m). This matroid has weights with non-vanishing cohomology; more precisely*

$$H^k(A(M[K]), e_\lambda) = 0 \quad \text{for } k \le \ell-2\,,$$
$$H^{\ell-1}(A(M[K]), e_\lambda) \ne 0\,,$$

for each non-zero weight

$$\lambda = (\underbrace{\lambda_1,\ldots,\lambda_1}_{m}, \underbrace{\lambda_2,\ldots,\lambda_2}_{m}, \cdots\cdots, \underbrace{\lambda_{\ell+1},\ldots,\lambda_{\ell+1}}_{m})\,; \quad \sum_{j=1}^{\ell+1}\lambda_j = 0\,.$$

We assume that R is a field of characteristic zero until the end of the paper.

In the rest of this section, we will prove this theorem. First of all, we prove some lemmas.

LEMMA 2.2. *A family $\mathcal{C}$ of $(\ell+1)$-subsets in $[n]$ satisfies the condition*

($\mathbf{C}_{\ell+1}$) *if $C_1, C_2 \in \mathcal{C}$ and $|C_1 \cup C_2| = \ell+2$ then every $(\ell+1)$-subset C_3 of $C_1 \cup C_2$ is a member of $\mathcal{C}$,*

if and only if, there exists an ℓ-generic matroid on $[n]$ for which the family of all $(\ell+1)$-circuits is equal to $\mathcal{C}$.

PROOF. It is clear when $n < \ell+1$. Assume that $n \ge \ell+1$. Let $\mathcal{C}$ be a family of $(\ell+1)$-subsets in $[n]$ satisfying ($\mathbf{C}_{\ell+1}$). Let I be an ℓ-subset of $[n]$. Define $X_I = I \cup \{e \in [n] : I \cup e \in \mathcal{C}\}$, $\binom{X_I}{\ell+1} = \{$all $(\ell+1)$-subsets of X_I.$\}$, and $\binom{X_I}{\ell+1}_s = \left\{S \in \binom{X_I}{\ell+1} : |S \setminus I| = s\right\}$. Note that $\binom{X_I}{\ell+1} = \bigcup_{s=1}^{\ell+1}\binom{X_I}{\ell+1}_s$. First of all, we show that $\binom{X_I}{\ell+1}_s$ is a subfamily of $\mathcal{C}$ by induction on s. For $s=1$, since $\binom{X_I}{\ell+1}_1 = \{I \cup e \in \mathcal{C}\}$, it is clear. Let assume that $\binom{X_I}{\ell+1}_s \subset \mathcal{C}$

for $s \geq 1$. Take a member S of $\binom{X_I}{\ell+1}_{s+1}$. Let $T := S \setminus I$ and $I' := S \cap I$. Note that $S = I' \cup T$, $I' \subset I$, $T \subset X_I \setminus I$, $|I'| = \ell - s$ and $|T| = s + 1$. Now we can choose $e \in I \setminus I'$ and $f_1, f_2 \in T$ with $f_1 \neq f_2$. By the inductive assumption, $C_1 := I' \cup e \cup (T \setminus \{f_1\})$ and $C_2 := I' \cup e \cup (T \setminus \{f_2\})$ are in $\binom{X_I}{\ell+1}_s \subset \mathcal{C}$. We can check C_1 and C_2 satisfy the condition in $(\mathbf{C}_{\ell+1})$, and S is a $(\ell + 1)$-subset of $C_1 \cup C_2$. So we have $S \in \mathcal{C}$. Therefore, we have $\binom{X_I}{\ell+1}_s \subset \mathcal{C}$ and hence $\binom{X_I}{\ell+1} \subset \mathcal{C}$.

Assume that $\mathcal{C}$ is not the family of all $(\ell + 1)$-subsets of $[n]$. We shall show that

$$\mathcal{I} = \{I \subset [n] : |I| \leq \ell + 1, I \notin \mathcal{C}\}$$

is a matroid complex (see [12]). Note that I have all i-subsets of $[n]$ for $i < \ell + 1$. Since $\emptyset \in \mathcal{I}$ and if $I' \subset I \in I$ then $I' \in \mathcal{I}$, we should prove the independence augmentation axiom for $\mathcal{I}$, that is, for $I_1, I_2 \in \mathcal{I}$ with $|I_2| = |I_1| + 1$, there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$. If $|I_1| < \ell$, it is clear. Let $|I_1| = \ell$. Suppose that $I_1 \cup \{e\} \notin \mathcal{I}$ for all $e \in I_2 \setminus I_1$. Then we have $I_2 \subset X_{I_1}$. By the above claim, we have $\binom{X_{I_1}}{\ell+1} \subset \mathcal{C}$ and hence we have $I_2 \in \mathcal{C}$, this is a contradiction. Therefore, $\mathcal{I}$ defines the matroid of rank $\ell + 1$. The converse is easy by the circuit elimination axiom of the matroids (see [12, 1.1.4]). □

REMARK 2.3. (1) When $\mathcal{C} = \emptyset$, the uniform matroid $U_{m,n}$ of rank m with $m \geq \ell+1$ is the matroid in the above lemma.

(2) If $\mathcal{C}$ consists of all $(\ell + 1)$-subsets of $[n]$, the uniform matroid $U_{\ell,n}$ of rank ℓ is the only ℓ-generic matroid which satisfies the condition of the above lemma. Otherwise, the rank of such a matroid is greater than ℓ, and there exists uniquely such an ℓ-generic matroid with rank $\ell + 1$.

LEMMA 2.4. *Let $n = (\ell+1)m$. Let $a_s = e_{(s-1)m+1} + \cdots + e_{sm}$ for $1 \leq s \leq \ell+1$. For a Latin ℓ-dimensional hypercube K on $[m]$, we obtain the following decomposable relation*

$$\begin{aligned}\partial(a_1 \wedge a_2 \wedge \cdots \wedge a_{\ell+1}) &= -(a_1 - a_2) \wedge (\partial(a_2 \wedge \cdots \wedge a_{\ell+1}))\\ &= (-1)^\ell m(a_1 - a_2) \wedge (a_2 - a_3) \wedge \cdots \wedge (a_\ell - a_{\ell+1})\\ &= m \sum_{S \in \mathcal{C}[K]} \partial(e_S)\,,\end{aligned}$$

where $e_S = e_{i_1} \wedge \cdots \wedge e_{i_p}$ for a p-tuple $(i_1, \ldots, i_p)$.

PROOF. The first and second equations are obtained by

$$\begin{aligned}\partial(a_1 \wedge a_2 \wedge \cdots \wedge a_{\ell+1}) &= \partial((a_1 - a_2) \wedge a_2 \wedge \cdots \wedge a_{\ell+1})\\ &= \partial(a_1 - a_2) \wedge a_2 \wedge \cdots \wedge a_{\ell+1} - (a_1 - a_2) \wedge (\partial(a_2 \wedge \cdots \wedge a_{\ell+1}))\\ &= -(a_1 - a_2) \wedge (\partial(a_2 \wedge \cdots \wedge a_{\ell+1})) = \cdots\\ &= (-1)^\ell (a_1 - a_2) \wedge (a_2 - a_3) \wedge \cdots \wedge (a_\ell - a_{\ell+1}) \wedge \partial(a_{\ell+1})\\ &= (-1)^\ell m(a_1 - a_2) \wedge (a_2 - a_3) \wedge \cdots \wedge (a_\ell - a_{\ell+1})\,.\end{aligned}$$

Let $E_s = \{(s-1)m+1, (s-1)m+2, \dots, sm\}$ for $1 \le s \le \ell+1$. Note that $E_1 \cup \cdots \cup E_{\ell+1} = [n]$. We regard K as a Latin hypercube $\tilde{K} = (\tilde{k}(i_1, \dots, i_\ell))$ with s-axis indexed by E_s for $1 \le s \le \ell$ and symbol set $E_{\ell+1}$. We note that $\tilde{k}(i_1, \dots, i_\ell) = \ell m + k(i_1, \dots, i_\ell) \in E_{\ell+1}$. Since $\partial(e_1 \wedge \cdots \wedge e_k \wedge e_{k+1}) = \partial(e_1 \wedge \cdots \wedge e_k) \wedge e_{k+1} + (-1)^k e_1 \wedge \cdots \wedge e_k$, we have

$$(-1)^k e_1 \wedge e_2 \wedge \cdots \wedge e_k = -\partial(e_1 \wedge \cdots \wedge e_k) \wedge e_{k+1} + \partial(e_1 \wedge \cdots \wedge e_k \wedge e_{k+1}) .$$

Hence, we can get

$$\begin{aligned}(-1)^\ell m \cdot a_1 \wedge \cdots \wedge a_\ell = m \sum_{i_1 \in E_1, \dots, i_\ell \in E_\ell} (-1)^\ell e_{i_1} \wedge \cdots \wedge e_{i_\ell} = m \times \\ \sum_{i_1 \in E_1, \dots, i_\ell \in E_\ell} \big\{ -\partial(e_{i_1} \wedge \cdots \wedge e_{i_\ell}) \wedge e_{\tilde{k}(i_1, \dots, i_\ell)} + \partial(e_{i_1} \wedge \cdots \wedge e_{i_\ell} \wedge e_{\tilde{k}(i_1, \dots, i_\ell)}) \big\} .\end{aligned}$$

The second term is

$$\sum_{i_1 \in E_1, \dots, i_\ell \in E_\ell} \partial(e_{i_1} \wedge \cdots \wedge e_{i_\ell} \wedge e_{\tilde{k}(i_1, \dots, i_\ell)}) = \sum_{S \in \mathcal{C}[K]} \partial(e_S) .$$

On the other hand, since K is a Latin hypercube, we have

$$\begin{aligned}&\sum_{i_1 \in E_1, \dots, i_\ell \in E_\ell} \partial(e_{i_1} \wedge \cdots \wedge e_{i_\ell}) \wedge e_{\tilde{k}(i_1, \dots, i_\ell)} \\ &= \sum_{i_1, \dots, i_\ell} \left(\sum_{p=1}^{\ell} (-1)^{p-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_\ell} \right) \wedge e_{\tilde{k}(i_1, \dots, i_\ell)} \\ &= \sum_{p=1}^{\ell} \sum_{i_1, \dots, i_\ell} \big((-1)^{p-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_\ell} \big) \wedge e_{\tilde{k}(i_1, \dots, i_\ell)} \\ &= \sum_{p=1}^{\ell} \sum_{i_1, \dots, \widehat{i_p}, \dots, i_\ell} \big((-1)^{p-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_\ell} \big) \wedge \sum_{i_p} e_{\tilde{k}(i_1, \dots, i_\ell)} \\ &= \sum_{p=1}^{\ell} \sum_{i_1, \dots, \widehat{i_p}, \dots, i_\ell} \big((-1)^{p-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_\ell} \big) \wedge a_{\ell+1} ,\end{aligned}$$

and

$$\begin{aligned}\partial(a_1 \wedge \cdots \wedge a_\ell) &= \partial(a_p) \sum_{p=1}^{\ell} (-1)^{p-1} a_1 \wedge \cdots \wedge \widehat{a_p} \wedge \cdots \wedge a_\ell \\ &= m \sum_{p=1}^{\ell} (-1)^{p-1} \sum_{i_1, \dots, \widehat{i_p}, \dots, i_\ell} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_\ell} .\end{aligned}$$

Therefore we obtain

$$(-1)^{\ell} m \cdot a_1 \wedge \cdots \wedge a_\ell = -\partial(a_1 \wedge \cdots \wedge a_\ell) \wedge a_{\ell+1} + m \sum_{S \in \mathcal{C}[K]} \partial(e_S)$$

and hence we have

$$\begin{aligned} \partial(a_1 \wedge \cdots \wedge a_\ell \wedge a_{\ell+1}) &= \partial(a_1 \wedge \cdots \wedge a_\ell) \wedge a_{\ell+1} + (-1)^{\ell} m \cdot a_1 \wedge \cdots \wedge a_\ell \\ &= m \sum_{S \in \mathcal{C}[K]} \partial(e_S)\,. \end{aligned}$$

□

PROOF OF THEOREM 2.1. Let K be a Latin ℓ-dimensional hypercube on $[m]$. By the construction of $\mathcal{C}[K]$, for $C_1, C_2 \in \mathcal{C}[K]$ with $C_1 \neq C_2$, we have $|C_1 \cap C_2| = \ell - 1$ and $|C_1 \cup C_2| = \ell + 3$. Hence, due to Lemma 2.2 and the remark following it, there exists a unique ℓ-generic matroid $M[K]$ with rank $\ell + 1$. In general, for an ℓ-generic matroid M and a non-zero weight λ of M, we have $H^k(M, e_\lambda) = 0$ for $k \leq \ell - 2$. Thus, we only need to prove $H^{\ell-1}(A(M[K]), e_\lambda) \neq 0$. Let λ be a weight given in the statement, and assume without loss of generality that $\lambda_1 \neq 0$. Since $\sum_{j=1}^{\ell+1} \lambda_j = 0$, we have

$$\begin{aligned} e_\lambda &= \lambda_1(e_1 + \cdots + e_m) + \cdots + \lambda_{\ell+1}(e_{\ell m+1} + \cdots + e_{(\ell+1)m}) \\ &= \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_{\ell+1} a_{\ell+1} \\ &= \lambda_1(a_1 - a_2) + (\lambda_1 + \lambda_2)(a_2 - a_3) + \cdots + (\lambda_1 + \cdots + \lambda_\ell)(a_\ell - a_{\ell+1})\,, \end{aligned}$$

where a_j is defined in Lemma 2.4. Define an $(\ell - 1)$-form

$$\begin{aligned} b &:= \partial(a_2 \wedge a_3 \wedge \cdots \wedge a_{\ell+1}) \\ &= (-1)^{\ell-1} m (a_2 - a_3) \wedge (a_3 - a_4) \wedge \cdots \wedge (a_\ell - a_{\ell+1})\,. \end{aligned}$$

By Lemma 2.4, we have

$$e_\lambda \wedge b = \lambda_1(a_1 - a_2) \wedge b \in \langle \partial M[K] \rangle\,,$$

that is, $e_\lambda \wedge b$ vanishes in the Orlik-Solomon algebra $A(M[K])$. Since $M[K]$ is ℓ-generic, the $(\ell - 1)$-form b is not in $\langle \partial M[K] \rangle$. Finally, we shall check that b is a non-vanishing cohomology class in $H^{\ell-1}(M[K], e_\lambda)$.

For a finite set $\{e_1, \ldots, e_n\}$, denote by $E(e_1, \ldots, e_n)$ the graded exterior algebra over R generated by 1 and degree-one elements $e_1, \ldots, e_n$. Note that $E(e_2, \ldots, e_n)$ is a subalgebra of $E(e_1, \ldots, e_n)$. Let $e_\lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n$ with $\lambda_i \in R$ and $\lambda_1 \neq 0$. Then we have $E(e_1, \ldots, e_n) = E(e_\lambda, e_2, \ldots, e_n)$. It is easy to see the following: if $\omega \in E(e_2, \ldots, e_n)$ with $\omega \neq 0$, then ω is not belong to the ideal of $E(e_1, \ldots, e_n)$ generated by e_λ.

By the above, since b is in $E(e_{m+1}, \ldots, e_n)$ and $\lambda_1 \neq 0$, b is not in the ideal of $E(e_1, \ldots, e_n)$ generated by e_λ, that is, there exists no $(\ell - 2)$-form η with $e_\lambda \wedge \eta = b$. This completes the proof. □

3. The case of $\ell = 2$

We refer to [4] for more background on Latin squares. A *Latin square* of order m is a Latin hypercube of dimension 2 and order m, that is, an $m \times m$ matrix with entries in an m-set (which we call the *symbol set*), such that each element occurs exactly once in each row and exactly once in each column. Two Latin squares K and K' are *isotopic* if K' is obtained by permutations of rows, permutations of columns, and a bijection from the symbol set of K. Let E_1, E_2 and E_3 be three m-sets and let K be a Latin square with rows indexed by E_1, columns by E_2, and symbols by E_3. Define $T(K) = \{\{x_1, x_2, x_3\} : x_i \in E_i (i = 1, 2, 3), k_{x_1,x_2} = x_3\}$. For any permutation σ of $\{1, 2, 3\}$, the *σ-conjugate* of L is the Latin square K_σ with rows indexed by $E_{\sigma 1}$, columns by $E_{\sigma 2}$, and symbols by $E_{\sigma 3}$, defined by $T(K) = T(K_\sigma)$. Two Latin squares K and K' are *main class isotopic* if K' is isotopic to some conjugate of K.

Let $K = (k_{i,j})$ be a a Latin square on $[m]$, that is, an $m \times m$-matrix satisfying the condition $\{k_{i,1}, k_{i,2}, \dots, k_{i,m}\} = \{k_{1,j}, k_{2,j}, \dots, k_{m,j}\} = [m]$ for $1 \le i, j \le m$. As in the previous section, we define $\mathcal{C}[K]$ by the family

$$\begin{bmatrix} (1, m+1, 2m+k_{1,1}) & (1, m+2, 2m+k_{1,2}) & \cdots & (1, 2m, 2m+k_{1,m}) \\ (2, m+1, 2m+k_{2,1}) & (2, m+2, 2m+k_{2,2}) & \cdots & (1, 2m, 2m+k_{2,m}) \\ \vdots & & & \vdots \\ (m, m+1, 2m+k_{m,1}) & (m, m+2, 2m+k_{m,2}) & \cdots & (1, 2m, 2m+k_{m,m}) \end{bmatrix}.$$

We can view K as a Latin square $\tilde{K}$ with rows indexed by $\{1, 2, \dots, m\}$, columns by $\{m+1, m+2, \dots, 2m\}$, and symbols by $\{2m+1, 2m+2, \dots, 3m\}$. So we can consider $\mathcal{C}[K] = T(\tilde{K})$. By Theorem 2.1, there exists a unique simple matroid $M[K]$ on $[n]$ with rank 3, for which the family of all 3-circuits is equal to $\mathcal{C}[K]$. The simple matroid $M[K]$ has weights with non-vanishing first cohomology.

PROPOSITION 3.1. *Let $m \ge 2$. If K_1 and K_2 are main class isotopic Latin squares then the matroids $M[K_1]$ and $M[K_2]$ are isomorphic. If a Latin square K_1 is not main class isotopic to K_2 then the matroid $M[K_1]$ is not isomorphic to $M[K_2]$.*

PROOF. This is clear by the definition of main class isotopic Latin squares. □

REMARK 3.2. The number of main class isotopic Latin squares of order $m \le 8$ is known (see [4]).

$m =$	1	2	3	4	5	6	7	8
main classes	1	1	1	2	2	12	147	283, 657

Two Latin squares $K = (k_{i,j})$ and $K' = (k'_{i,j})$ of same order are *orthogonal* if all pairs $(k_{i,j}, k'_{i,j})$ are distinct. A set of Latin squares of order m is *mutually orthogonal* if any two distinct squares are orthogonal.

THEOREM 3.3. *Let $m \ge 1$, $s \ge 1$ and $n = (s+2)m$. Let $K_1, \dots, K_s$ be mutually orthogonal Latin squares on $[m]$. Then there exists a simple matroid $M[K_1, \dots, K_s]$ on $[n]$*

satisfying

$$\dim H^1(A(M[K_1,\dots,K_s]),e_\lambda)=s$$

for each non-zero weight

$$\lambda=(\underbrace{\lambda_1,\dots,\lambda_1}_m,\underbrace{\lambda_2,\dots,\lambda_2}_m,\cdots\cdots,\underbrace{\lambda_{s+2},\dots,\lambda_{s+2}}_m)\,;\quad \sum_{j=1}^{s+2}\lambda_j=0\,.$$

PROOF. By Lemma 2.2 in the case of $\ell=2$, a family $\mathcal{C}$ of 3-subsets in $[n]$ satisfies the condition

($\mathbf{C_3}$) if $\{i,j,k\}$ and $\{i,j,l\}$ are members of $\mathcal{C}$ then $\{i,k,l\}$ and $\{j,k,l\}$ are members of $\mathcal{C}$,

if and only if, there exists a simple matroid on $[n]$ for which the family of all 3-circuits is equal to $\mathcal{C}$. Recall that the set of flats of a matroid is a geometric lattice. The closure of $C\in\mathcal{C}$ is the set $\cup\{C'\in\mathcal{C}:|C'\cap C|\geq 2\}$, that is a flat of rank 2. A 2-subset contained in no $C\in\mathcal{C}$ is a flat of rank 2.

Construction of $M[K_1,\dots,K_s]$: Let $K_1,\dots,K_s$ be mutually orthogonal Latin squares on $[m]$. A Latin square $\tilde{K}_p=(\tilde{k}^p_{i,j})$ associated to $K_p=(k^p_{i,j})$ is given by a Latin square with rows indexed by $\{1,2,\dots,m\}$, columns by $\{m+1,m+2,\dots,2m\}$, and symbols by $\{(p+1)m+1,(p+1)m+2,\dots,(p+2)m\}$, given by $\tilde{k}^p_{i,j}=(p+1)m+k^p_{i,j}$ for $1\leq i\leq m$ and $m+1\leq j\leq 2m$. We define

$$\mathcal{C}[K_1,\dots,K_s]:=T(\tilde{K}_1)\cup\cdots\cup T(\tilde{K}_s)\,,$$

$$X_{i,j}:=\{i,j,\tilde{k}^1_{i,j},\dots,\tilde{k}^s_{i,j}\}\quad\text{for } 1\leq i\leq m, m+1\leq j\leq 2m,\ \text{ and}$$

$$\mathcal{C}:=\mathcal{C}[K_1,\dots,K_s]\cup\left(\bigcup_{1\leq i\leq m,m+1\leq j\leq 2m}\binom{X_{i,j}}{3}\right).$$

By mutually orthogonality, we have $|C\cap X_{i,j}|=1$ for any $C\in\mathcal{C}[K_1,\dots,K_s]$ not contained in $X_{i,j}$, and $|X_{i,j}\cap X_{k,l}|=1$ for $(i,j)\neq(k,l)$. This implies that $\mathcal{C}$ satisfies ($\mathbf{C_3}$). If $m\geq 2$ then we obtain a simple matroid $M[K_1,\dots,K_s]$ on $[n]$ with rank 3 such that $\mathcal{C}$ is the family of all 3-circuits. If $m=1$ then $\mathcal{C}$ gives the uniform matroid $U_{2,n}$.

Non-vanishing: Let $a_i=e_{(i-1)m}+e_{(i-1)m+1}+\cdots+e_{(i-1)m}$ for $i=1,2,\dots,s+2$. By Lemma 2.4, we have

$$(a_1-a_i)\wedge(a_2-a_i)\in\langle\partial M[K_1,\dots,K_s]\rangle$$

for $3\leq i\leq s+2$. We take two one-forms

$$e_{\lambda^t}=\lambda^t_1a_1+\lambda^t_2a_2+\cdots+\lambda^t_{s+2}a_{s+2}$$

with $\sum_{j=1}^{s+2}\lambda^t_j=0$ for $t=1,2$. Since $e_{\lambda^1}=\lambda^1_2(a_2-a_1)+\cdots+\lambda^1_{s+2}(a_{s+2}-a_1)$ and $e_{\lambda^2}=\lambda^2_1(a_1-a_2)+\cdots+\lambda^2_{s+2}(a_{s+2}-a_2)$, we have $e_{\lambda^1}\wedge e_{\lambda^2}\in\langle\partial M[K_1,\dots,K_s]\rangle$. This implies $\dim H^1(A(M[K_1,\dots,K_s]),e_\lambda)=s$. □

REMARK 3.4. When $m = 1$, the matroid in this theorem is the uniform matroid $U_{2,n}$ with rank 2. When $m \geq 2$, the matroid $M[K_1, \dots, K_s]$ has rank 3.

REMARK 3.5. There exists a Latin square of order m for $m \geq 1$. Let $N(m)$ be the maximum number of mutually orthogonal Latin squares of order m. The following is known (see [4]).

- $N(0) = N(1) = \infty$ and $1 \leq N(m) \leq m - 1$ for every $m > 1$.
- If m is a prime power then $N(m) = m - 1$.
- If $m \not\equiv 2 \mod 4$, then $N(m) \geq 2$.
- $N(p \times q) \geq \min\{N(p), N(q)\}$.
- $N(2) = 1$, $N(3) = 2$, $N(4) = 3$, $N(5) = 4$, $N(6) = 1$, $N(7) = 6$, $N(8) = 7$.

REMARK 3.6. In the case of $s = 1$, we have $\dim H^1(A(M[K]), e_\lambda) = 1$ for non-zero one-form

$$e_\lambda = \lambda_1(e_1 + \cdots + e_m) + \lambda_2(e_{m+1} + \cdots + e_{2m}) + \lambda_3(e_{2m+1} + \cdots + e_{3m})$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

Let M and M' be loopless matroids M on $[n]$ of rank 3. We call M' a *degeneration* of M if the family of 3-circuits of M' contains that of M. Mostly, degenerations of $M[K_1, \dots, K_s]$ have weights with non-vanishing first cohomology. It is trivial that the uniform matroid $U_{2,n}$ of rank 2 is one of its degenerations. Next, we shall construct its non-trivial degeneration with non-vanishing first cohomology.

PROPOSITION 3.7. *Let $m \geq 2$, $s \geq 1$ and $n = (s + 2)m$. Let $K_1, \dots, K_s$ be mutually orthogonal Latin squares on $[m]$. Let M_i be a simple matroid on $I_i := \{(i-1)m + 1, (i-1)m + 2, \dots, im\}$ for $i = 1, 2, \dots, s + 2$. There exists a simple matroid $M[K_1, \dots, K_s : M_1, \dots, M_{s+2}]$ with rank 3 such that it is a degeneration of $M[K_1, \dots, K_s]$ and its restriction on I_i is M_i for $i = 1, 2, \dots, s + 2$. For this matroid, we have*

$$\dim H^1(A(M[K_1, \dots, K_s : M_1, \dots, M_{s+2}]), e_\lambda) = s$$

for a weight λ given in Theorem 3.3.

PROOF. Let $\mathcal{C}_3(M_1, \dots, M_{s+2})$ be the union of families of 3-circuits of $M_i; i = 1, \dots, s + 2$. For a 3-circuit C_i of M_i and $C \in \mathcal{C}[K_1, \dots, K_s]$, we have $C_i \cap C_j = \emptyset$ for $i \neq j$ and $|C_i \cap C| = 1$. Thus $\mathcal{C}[K_1, \dots, K_s] \cup \mathcal{C}_3(M_1, \dots, M_{s+2})$ satisfies ($\mathbf{C_3}$) and it yields a simple matroid $M[K_1, \dots, K_s : M_1, \dots, M_{s+2}]$ in this statement. By the same argument as that in the proof of Theorem 3.3, we can prove the proposition. □

REMARK 3.8. A realization of $M[K_1, \dots, K_s : M_1, \dots, M_{s+2}]$ is an $(s + 2, m)$-net in $\mathbf{P}^2$ defined in [17]. Therefore, there is no (k, m)-net for $k > N(m) + 2$. In particular, there is no $(k, 6)$-net for $k > 3$.

In a Latin square K, an $s \times s$-matrix obtained by s rows and s columns is called a Latin *s-subsquare* of K if it forms a Latin square of order s. Let K be a Latin square on $[m]$

and J be a subsquare of K. We treat $\tilde{J}$ as a subsquare of $\tilde{K}$. $\tilde{J}$ has row index set $I_1(J)$, column index set $I_2(J)$ and symbol set $I_3(J)$ where $I_1(J) \subset I_1$, $I_2(J) \subset I_2$, $I_3(J) \subset I_3$ and $|I_1(J)| = |I_2(J)| = |I_3(J)|$. We define $X(J) = I_1(J) \cup I_2(J) \cup I_3(J)$.

PROPOSITION 3.9. *Let J be a subsquare of a Latin square K on $[m]$. There exists a simple degeneration $M[K; J]$ of $M[K]$, whose restriction on $X(J)$ is the uniform matroid of rank 2. For this matroid, we have*

$$\dim H^1(A(M[K; J]), e_\lambda) = 1$$

for a weight λ given in Remark 3.6.

PROOF. Let $\mathcal{C} = \mathcal{C}[K] \cup \binom{X(J)}{3}$. Since J is a subsquare of K, for $C \in \mathcal{C}[K] \setminus \binom{X(J)}{3}$, we have $|C \cap X(J)| = 1$. This leads to $(\mathbf{C_3})$ for $\mathcal{C}$. The conclusion follows as in the proof of Proposition 3.7. □

REMARK 3.10. The following is known (see [4]).

- There exists a Latin square of order m with a proper k-subsquare if and only if $k \leq \left[\frac{m}{2}\right]$.
- There exists a Latin square of order m with no proper subsquares if $m \neq 2^a 3^b$ or if $m = 3,9,12,16,18,27,81$ or 243.

There are other degenerations of matroids associated to Latin squares with non-vanishing cohomology, for example, see Section 4.5.

4. Arrangements

For a matroid M, an arrangement over a field F with underlying matroid M is called an *F-realization* or *representation* of M. A matroid is said to be *realizable* or *representable* over F if M has an F-realization. We shall find realizations of matroids obtained in the previous section. In this section, we will see the following:

PROPOSITION 4.1. *If $1 \leq m \leq 4$ then the matroid $M[K]$ associated to a Latin square K on $[m]$ is realizable over the reals.*

In addition, these realizations are arrangements appearing in classical projective geometry (Figure 1, 2 and 3). Besides, we shall give many other examples including the higher dimensional case.

4.1. $m = 1$**.** Lemma 2.4 implies $(e_1 - e_3) \wedge (e_2 - e_3) = \partial(e_1 \wedge e_2 \wedge e_3)$. The matroid $M[K]$ is realized by the arrangement in $\mathbf{P}^2$ consisting of three lines through one point.

4.2. $m = 2$ (Falk [6])**.** We have only one main class isotopic Latin square $K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. The decomposable relation is $(e_1 + e_2 - e_5 - e_6) \wedge (e_3 + e_4 - e_5 - e_6) = \partial(e_1 \wedge e_3 \wedge e_5) + \partial(e_1 \wedge e_4 \wedge e_6) + \partial(e_2 \wedge e_3 \wedge e_6) + \partial(e_2 \wedge e_4 \wedge e_5)$. The matroid $M[K]$ is realized by the arrangement in $\mathbf{P}^2$ arising from the Ceva Theorem (the left side in Figure 1).

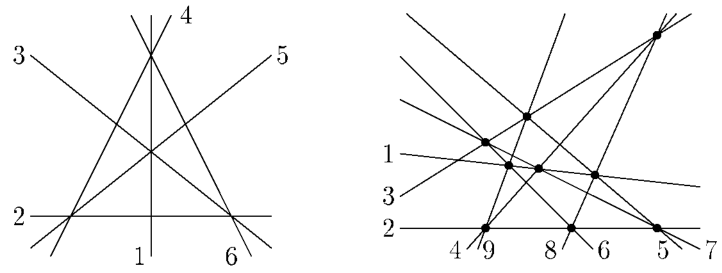

FIGURE 1. The Ceva Theorem and the Pappus Theorem.

4.3. $m = 3$**.** We have only one main class isotopic Latin square, which is given by

$$K = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mathcal{C}[K] = \begin{bmatrix} (1,4,7) & (1,5,8) & (1,6,9) \\ (2,4,9) & (2,5,7) & (2,6,8) \\ (3,4,8) & (3,5,9) & (3,6,7) \end{bmatrix}.$$

The realization is given by the arrangement of 9 lines in $\mathbf{P}^2$ arising from the Pappus Theorem (the right side in Figure 1).

4.4. $m = 4$**.** There are two main class isotopic Latin squares, that we can give by

$$K_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \mathcal{C}[K_1] = \begin{bmatrix} (1,5,9) & (1,6,10) & (1,7,11) & (1,8,12) \\ (2,5,12) & (2,6,9) & (2,7,10) & (2,8,11) \\ (3,5,11) & (3,6,12) & (3,7,9) & (3,8,10) \\ (4,5,10) & (4,6,11) & (4,7,12) & (4,8,9) \end{bmatrix},$$

$$K_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad \mathcal{C}[K_2] = \begin{bmatrix} (1,5,9) & (1,6,10) & (1,7,11) & (1,8,12) \\ (2,5,10) & (2,6,9) & (2,7,12) & (2,8,11) \\ (3,5,11) & (3,6,12) & (3,7,9) & (3,8,10) \\ (4,5,12) & (4,6,11) & (4,7,10) & (4,8,9) \end{bmatrix}.$$

The matroid $M[K_1]$ or $M[K_2]$ is realized by the arrangement of 12 lines in $\mathbf{P}^2$ defined by Figure 2 or 3, which is arising from the Kirkman Theorem or the Steiner Theorem, respectively (see [13, Chapter 16]).

4.5. Degenerations. Let K_1 and K_2 be in the preceding section. Let J be the subsquare of K_1 given by

$$J = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}.$$

By Proposition 3.9, we obtain $X(J) = \{1, 3, 6, 8, 10, 12\}$ and the matroid $M[K_1; J]$. Let M_1 be a simple matroid on $[4]$ for which the family of 3-circuits is $\{(1, 2, 4)\}$. By Proposition 3.7,

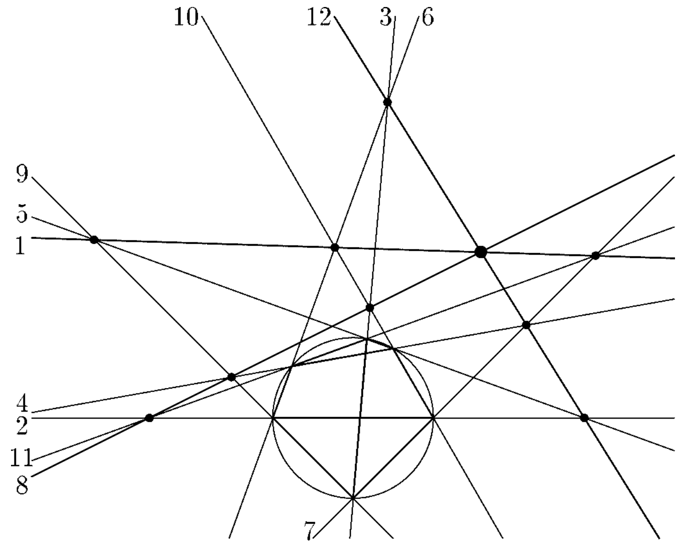

FIGURE 2. The Kirkman Theorem.

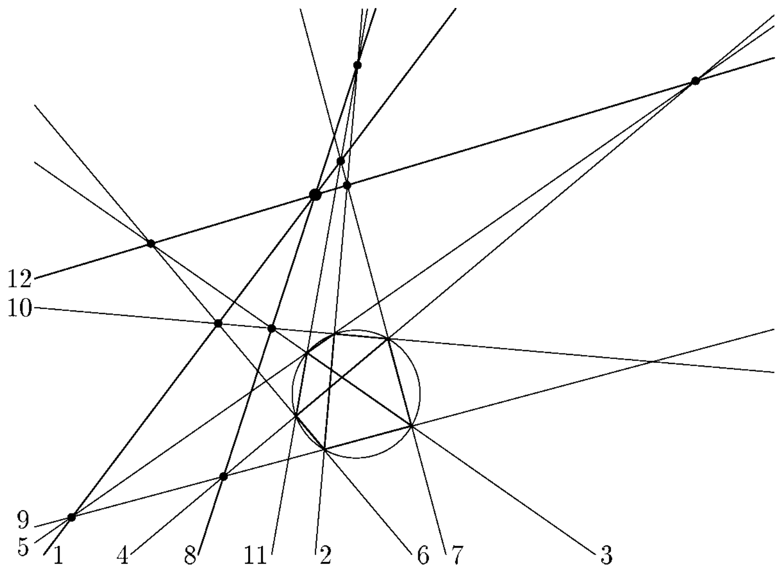

FIGURE 3. The Steiner Theorem.

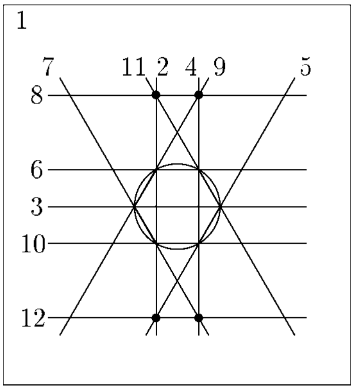

FIGURE 4. Degeneration of Kirkman's arrangement.

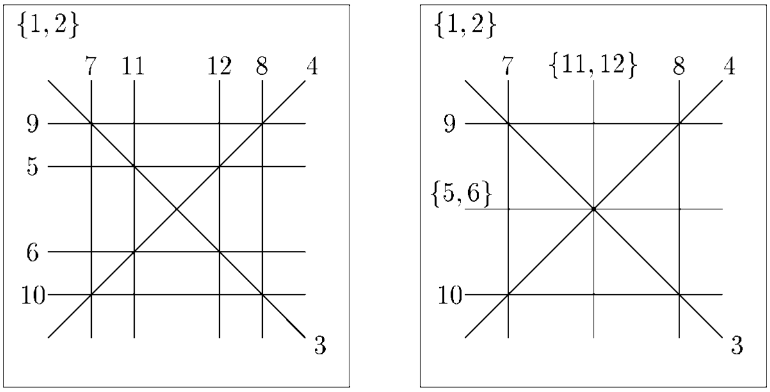

FIGURE 5. Degenerations of Steiner's arrangement.

we have the matroid $M[K_1; M_1]$. Furthermore, the family $\mathcal{C}[K_1] \cup \binom{X(J)}{3} \cup \mathcal{C}_3(M_1)$ satisfies ($\mathbf{C_3}$) and then yields the matroid $M[K_1 : M_1; J]$ with non-vanishing first cohomology. This matroid $M[K_1 : M_1; J]$ is realized by the arrangement of 11 lines in $\mathbf{C}^2$ with the line 1 at infinity in Figure 4.

The degeneration of $M[K_2]$ such that 1 and 2 are parallel, that is, $\{1, 2\}$ is a circuit, has a realization defined by the left one in Figure 5. Moreover, the degeneration of $M[K_2]$ such that $\{1, 2\}$, $\{5, 6\}$ and $\{11, 12\}$ are circuits, is realizable. This realization is the B_3-arrangement (the right one in Figure 5). Therefore, these two arrangements have weights with non-vanishing first cohomology in the same way as in Remark 3.6.

4.6. $m = 3$ **and** $s = 2$ (Libgober [8]). The two Latin squares

$$K_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \text{and} \qquad K_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

are mutually orthogonal. We have

$$\mathcal{C}[K_1] = \begin{bmatrix} (1,4,7) & (1,5,8) & (1,6,9) \\ (2,4,9) & (2,5,7) & (2,6,8) \\ (3,4,8) & (3,5,9) & (3,6,7) \end{bmatrix}, \ \mathcal{C}[K_2] = \begin{bmatrix} (1,4,10) & (1,5,11) & (1,6,12) \\ (2,4,11) & (2,5,12) & (2,6,10) \\ (3,4,12) & (3,5,10) & (3,6,11) \end{bmatrix}.$$

The matroid $M[K_1, K_2]$ is AG(2, 3) (see [12]) and realized as the Hessian configuration. The Hessian configuration is the arrangement of 12 projective lines passing through the nine inflection points of a nonsingular cubic in $\mathbf{P}^2(\mathbf{C})$ [10, Example 6.30], which we can define by lines

$$H_1 = \{x = 0\}, H_2 = \{y = 0\}, H_3 = \{z = 0\},$$

$$H_4 = \{x + y + z = 0\}, H_5 = \{x + \omega^2 y + \omega z = 0\}, H_6 = \{x + \omega y + \omega^2 z = 0\},$$

$$H_7 = \{x + \omega y + \omega z = 0\}, H_8 = \{x + y + \omega^2 z = 0\}, H_9 = \{x + \omega^2 y + z = 0\},$$

$$H_{10} = \{x + \omega^2 y + \omega^2 z = 0\}, H_{11} = \{x + \omega y + z = 0\}, H_{12} = \{x + y + \omega z = 0\},$$

where $\omega = e^{2\pi i/3}$. The underlying matroids of arrangements

$$\{H_1, \dots, H_6, H_7, H_8, H_9\} \quad \text{and} \quad \{H_1, \dots, H_6, H_{10}, H_{11}, H_{12}\}$$

are $M[K_1]$ and $M[K_2]$, respectively. The Hessian configuration $\{H_1, \dots, H_{12}\}$ has underlying matroid $M[K_1, K_2]$ and we have $\dim H^1(A(M[K_1, K_2]), e_\lambda) = 2$ for a non-zero one-form

$$e_\lambda = \lambda_1(e_1 + e_2 + e_3) + \lambda_2(e_4 + e_5 + e_6) + \lambda_3(e_7 + e_8 + e_9) + \lambda_4(e_{10} + e_{11} + e_{12})$$

with $\sum_{j=1}^4 \lambda_j = 0$.

4.7. Monomial arrangements (Cohen and Suciu [3]). Let K be the Latin square of order m defined by the addition table for $\mathbf{Z}_m \times \mathbf{Z}_m$ for $m \geq 2$. The monomial arrangement $\mathcal{A}_{m,m,3}$ in $\mathbf{C}^3$ is given by the defining polynomial

$$Q(\mathcal{A}_{m,m,3}) = (x_1^m - x_2^m)(x_1^m - x_3^m)(x_2^m - x_3^m).$$

Set $\zeta = \exp(2\pi i/m)$. Define

$$\mathcal{A}_{ij} = \{H_{i,j}^k = \operatorname{Ker}(x_i - \zeta^k x_j) : 1 \leq k \leq m\}$$

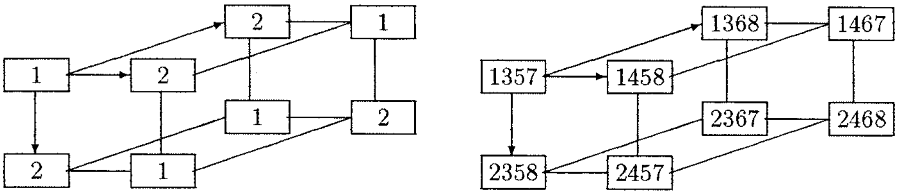

FIGURE 6. K and $\mathcal{C}[K]$.

for $1 \leq i < j \leq 3$. So we have $\mathcal{A}_{m,m,3} = \mathcal{A}_{12} \cup \mathcal{A}_{23} \cup \mathcal{A}_{13}$. Since $\cap_{k=1}^{m} H_{i,j}$ has rank two, the underlying matroid $M(\mathcal{A}_{ij})$ of $\mathcal{A}_{ij}$ is isomorphic to the uniform matroid $U_{2,m}$ of rank two. Other rank two intersections are $H_{1,2}^{p} \cap H_{2,3}^{q} \cap H_{1,3}^{r}$ for $p+q \equiv r \mod m$. Hence, K can be considered as the Latin square with rows indexed by $\mathcal{A}_{12}$, columns by $\mathcal{A}_{23}$, and symbols by $\mathcal{A}_{13}$. The underlying matroid of $\mathcal{A}_{m,m,3}$ is the matroid $M[K; M(\mathcal{A}_{12}), M(\mathcal{A}_{23}), M(\mathcal{A}_{13})]$. By Proposition 3.7, $\mathcal{A}_{m,m,3}$ has weights with non-vanishing first cohomology.

4.8. Higher dimensional case ($\ell = 3$). Let K be a Latin 3-dimensional hypercube on [2] defined by Figure 6. The matroid $M[K]$ is the matroid of type L_8 in [12, p.510]. Let $\mathcal{A}$ be an 4-arrangement defined by the polynomial

$$x_1x_2x_3x_4(x_1+x_2+x_3+x_4)(x_1+bcx_2+bx_3+cx_4)(x_1+cx_2+x_3+cx_4)(x_1+bx_2+bx_3+x_4)\,,$$

where $0, 1, b, c, bc$ are distinct from each other. By a simple computation, $\mathcal{A}$ is a realization of $M[K]$. Therefore, $\mathcal{A}$ has weights with non-vanishing second cohomology (cf. A. Libgober, arXiv: math/0404341, Example 7.4). Let $\mathcal{B}$ be an 4-arrangement defined by the defining polynomial

$$(x_1 - x_2)(x_1 + x_2)(x_2 - x_3)(x_2 + x_3)(x_3 - x_4)(x_3 + x_4)(x_4 - x_1)(x_4 + x_1)\,.$$

By a simple computation, we can check that $\mathcal{B}$ has no 3-circuits and the family of 4-circuits is

$$\mathcal{C}[K] \cup \{(1, 2, 3, 4), (1, 2, 7, 8), (3, 4, 5, 6), (5, 6, 7, 8)\}\,.$$

Therefore, $\mathcal{B}$ has weights with non-vanishing second cohomology.

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