

The Gibbs phenomenon for Stromberg wavelets

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Abstract: The Gibbs phenomenon for Stromberg wavelets is studied. It is proved that the Gibbs phenomenon for partial sums of Fourier-Stromberg series occurs for almost all points of \mathbf{R} .

Key words: Gibbs phenomenon; wavelets; Stromberg systems.

The Gibbs phenomenon, discovered by Henry Wilbraham in 1848 and rediscovered by Josiah Willard Gibbs in 1899, is the peculiar manner in which the Fourier series of some function behaves at a jump discontinuity. The n -th partial sum of the Fourier series has large oscillations near the jump, which might increase the maximum of the partial sum above that of the function itself. The overshoot does not die out as n increases, but approaches a finite limit.

Modifying the Franklin system, Strömberg [11] obtained a system, which is an unconditional basis in $H^p(\mathbf{R})$, for $p > \frac{1}{2}$, where the space $H^p(\mathbf{R})$ is the boundary values on the real axis of the real parts of those analytic functions F in the upper half-plane which satisfy

$$\sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|^p dx < \infty.$$

Let $m \geq 0$, $A_0 = \mathbf{N} \cup \{0\} \cup \frac{1}{2}(-\mathbf{N})$ and $A_1 = A_0 \cup \left\{ \frac{1}{2} \right\}$, where \mathbf{N} is the set of natural numbers.

The points of the set A_0 split \mathbf{R} into intervals $\{I_\sigma\}_{\sigma \in A_0}$, where σ is the left endpoint of the interval I_σ . Let S_0^m be the subspace of functions $f \in L^2(\mathbf{R})$, such that $f \in L^2(\mathbf{R}) \cap C^m(\mathbf{R})$ and f is a real polynomial of degree not greater than $m+1$ on each I_σ , $\sigma \in A_0$. Let S_1^m be the corresponding subspace of $L^2(\mathbf{R})$ with the set A_0 replaced by A_1 . It is clear that $S_0^m \subset S_1^m$ and if $f(x) \in S_0^m$, then $f(x-1) \in S_1^m$. In general, the functions of S_1^m differ from the functions of S_0^m only by the following: a function from S_0^m is a polynomial on $[0, 1]$, while a

function from S_1^m can be polynomial only at each of the intervals $\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$. It follows that S_0^m had codimension 1 in S_1^m . Therefore there exists a function $f^{(m)} \in L^2(\mathbf{R})$, that is uniquely defined (up to the sign) by the following relations:

1. $f^{(m)} \in S_1^m$,
2. $f^{(m)} \perp S_0^m$, that is $\int_{\mathbf{R}} f^{(m)}(x)f(x)dx = 0$ for all $f \in S_0^m$,
3. $\|f^{(m)}\|_2 = 1$.

For all pairs $(i, j) \in \mathbf{Z}^2$ denote

$$(1) \quad f_{i,j}^{(m)}(x) = 2^{\frac{i}{2}} f^{(m)}(2^i x - j).$$

The system $\{f_{i,j}^{(m)}(x)\}_{i,j \in \mathbf{Z}}$ was introduced by Strömberg in [11]. He also proved that $\{f_{i,j}^{(m)}(x)\}_{i,j \in \mathbf{Z}}$ is a complete orthonormal system in $L^2(\mathbf{R})$ and it is an unconditional basis in $H^p(\mathbf{R})$ for all $p > \frac{1}{m+2}$.

Let t_0 be a discontinuity point of the first kind of a function $q \in L(\mathbf{R})$, such that $|q(t_0+) - q(t_0-)| = 2d > 0$, and let $\{q_{i,j}(t)\}_{i,j \in \mathbf{Z}}$ be any sequence of functions converging to $q(t)$ at every point of some neighborhood of t_0 , when $i, j \rightarrow +\infty$. Then the value of the Gibbs function for sequence $\{q_{i,j}(t)\}_{i,j \in \mathbf{Z}}$ at t_0 is defined by

$$\begin{aligned} G(t_0, q, \{q_{i,j}\}_{i,j \in \mathbf{Z}}) &= G(t_0) \\ &= \overline{\lim}_{\substack{t \rightarrow t_0 \\ i,j \rightarrow +\infty}} \frac{1}{d} \left| q_{i,j}(t) - \frac{q(t_0+) + q(t_0-)}{2} \right|. \end{aligned}$$

If $G(t_0) > 1$, we say that the sequence $\{q_{i,j}(t)\}_{i,j \in \mathbf{Z}}$ exhibits the Gibbs phenomenon at t_0 .

The Gibbs phenomenon has been studied for Fourier series with respect to the trigonometric system (see [2], pp. 123–126). In this case the value $G(t_0)$ is independent of t_0 and it is equal to the

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Gibbs constant:

$$G(t_0) = \frac{2}{\pi} \int_0^\pi \frac{\sin t}{t} dt \approx 1.17.$$

The Gibbs phenomenon for Fourier series with respect to the classical Franklin system has been studied in [10]. It is proved that the Gibbs phenomenon occurs everywhere and the function G is a constant almost everywhere. The case of the general Franklin system has been studied in [8]. It has been proved that the Gibbs phenomenon occurs almost everywhere.

The Gibbs phenomenon for Fourier series with respect to the Walsh system has been studied in the papers [12] and [1]. The existence of the Gibbs phenomenon for Fourier series with respect to Walsh system was established in [12]. In this case G is not a constant. In [1] exact upper and lower bounds for this function are found.

Such problems are also studied in [4,6].

Let

$$S_{i_0, j_0}^{(m)}(f, x) = \sum_{i=-\infty}^{i_0-1} \sum_{j=-\infty}^{+\infty} a_{i,j} f_{i,j}^{(m)}(x) + \sum_{j=-\infty}^{j_0} a_{i_0,j} f_{i_0,j}^{(m)}(x), \quad i_0, j_0 \in \mathbf{Z}$$

be the partial sum of Fourier-Stromberg series and let

$$G(t_0, m) = G(t_0, f, \{S_{i_0, j_0}^{(m)}(f, \cdot)\}_{i_0, j_0=-\infty}^{+\infty})$$

be the Gibbs function. Also denote

$$K_{i_0, j_0}^{(m)}(x, t) = \sum_{i=-\infty}^{i_0-1} \sum_{j=-\infty}^{+\infty} f_{i,j}^{(m)}(x) f_{i,j}^{(m)}(t) + \sum_{j=-\infty}^{j_0} f_{i_0,j}^{(m)}(x) f_{i_0,j}^{(m)}(t), \quad i_0, j_0 \in \mathbf{Z}.$$

It is easy to see that for all bounded functions $f : \mathbf{R} \rightarrow \mathbf{R}$

$$(2) \quad S_{i_0, j_0}^{(m)}(f, x) = \int_{-\infty}^{+\infty} K_{i_0, j_0}^{(m)}(x, t) f(t) dt.$$

In the case $m = 0$ the Gibbs phenomenon has been studied in [9], where the following theorem was proved:

Theorem 1. *Let $m = 0$. If t_0 is a discontinuity point of the first kind of a function $f \in L^2(\mathbf{R})$,*

then the Gibbs phenomenon occurs everywhere in \mathbf{R} , and for all $t_0 \in \mathbf{R}$

$$1 + \frac{48 - 28\sqrt{3} + 8\sqrt{2}(2 - \sqrt{3})}{27} \leq G(t_0, 0) \leq \frac{1 + 2\sqrt{3}}{3},$$

with $G(t_0, 0) = \frac{1 + 2\sqrt{3}}{3}$ almost everywhere.

The main result of the present paper is the following theorem:

Theorem 2. *Let $m \geq 0$. If t_0 is a discontinuity point of the first kind of a function $f \in L^2(\mathbf{R})$, then the Gibbs phenomenon occurs almost everywhere in \mathbf{R} , i.e. $G(t_0, m) > 1$ for almost all $t_0 \in \mathbf{R}$.*

Let's prove the following lemma:

Lemma 1. *There exists a natural number k , such that*

$$\int_{-\infty}^k K_{0,0}^{(m)}(k, t) dt > 1.$$

Proof. Since from (2) we have

$$\int_{-\infty}^{+\infty} K_{0,0}^{(m)}(k, t) dt = 1,$$

it suffices to prove that there exists a natural number k , such that

$$\int_k^{+\infty} K_{0,0}^{(m)}(k, t) dt < 0.$$

Recall, that B-spline with real knots $x_0 \leq x_1 \leq \dots \leq x_r$ is the following function

$$M(x) = r[x_0, x_1, \dots, x_r](\cdot - x)_+^{r-1}, \quad x \in \mathbf{R},$$

where square brackets mean divided differences (see [3], ch. 5, p. 2). For every $n \in A_1$ denote by $N_n(t)$ B-spline with $m + 3$ consecutive knots from A_1 , with starting knot n .

We have $K_{0,0}^{(m)}(1, t) = \sum_{n \in A_1} c_n N_n(t)$ and from (2)

follows that

$$\int_{-\infty}^{+\infty} K_{0,0}^{(m)}(1, t) N_i(t) dt = N_i(1) = 0,$$

for all $i \in \mathbf{N}$. Therefore

$$\sum_{n \in A_1} c_n \int_{-\infty}^{+\infty} N_n(t) N_i(t) dt = 0,$$

for all $i \in \mathbf{N}$. Since $\int_{-\infty}^{+\infty} N_n(t) N_i(t) dt = 0$, for $n + m + 2 \leq i$ and for $n \geq i + m + 2$, we have

$$(3) \quad \sum_{n=i-m-1}^{i+m+1} c_n \int_{-\infty}^{+\infty} N_n(t) N_i(t) dt = 0,$$

for all $i \in \mathbf{N}$. We regard (3) as a recurrence relation with respect to c_i . The characteristic polynomial of (3) has order $2m + 2$. It is clear that 0 is not a root of that polynomial and if x_0 is its root, then $\frac{1}{x_0}$ is also its root. Since $\lim_{i \rightarrow +\infty} c_i = 0$ (see [11]), for all $i \in \mathbf{N}$

$$c_i = \sum_{j=1}^t \alpha_j^i (a_{j,1} + a_{j,2}i + \dots + a_{j,n_j-1}i^{n_j-1}),$$

where $\alpha_1, \dots, \alpha_t$ are those roots of the characteristic polynomial which by modulo are less than one and n_j is the multiplicity of the root α_j and the coefficients $a_{j,h}$ are real (see [5], Section 3.3). Denote

$$d_j = \int_1^{j+2} N_1(t) dt,$$

for all $j \in \{0, 1, \dots, m\}$.

Let $|\alpha_1| = \max\{|\alpha_1|, |\alpha_2|, \dots, |\alpha_t|\}$. It is clear that if $\alpha_1 \in \mathbf{R}$, then

$$c_i = a_{1,n_1-1} i^{n_1-1} \alpha_1^i + o(i^{n_1-1} |\alpha_1|^i),$$

and if $\alpha_1 \in \mathbf{C} \setminus \mathbf{R}$, then

$$c_i = 2 \operatorname{Re}(a_{1,n_1-1} i^{n_1-1} \alpha_1^i) + o(i^{n_1-1} |\alpha_1|^i).$$

If $k > m + 3$ and $k \in \mathbf{N}$, then

$$\begin{aligned} \int_k^{+\infty} K_{0,0}^{(m)}(1, t) dt &= \sum_{n \in A_1} c_n \int_k^{+\infty} N_n(t) dt \\ &= \sum_{n=k-m-1}^{+\infty} c_n \int_k^{+\infty} N_n(t) dt \\ &= \sum_{n=k-m-1}^{k-1} c_n \int_k^{+\infty} N_n(t) dt + \sum_{n=k}^{+\infty} c_n \end{aligned}$$

$$= \sum_{l=0}^m c_{l+k-m-1} d_l + \sum_{n=k}^{+\infty} c_n.$$

Hence the sign of the number $\int_k^{+\infty} K_{0,0}^{(m)}(1, t) dt$ coincides with the sign of the following number

$$\begin{aligned} (4) \quad \operatorname{Re} &\left(\sum_{l=0}^m a_{1,n_1-1} \alpha_1^{l+k-m-1} (l+k-m-1)^{n_1-1} d_l \right. \\ &\left. + \sum_{n=k}^{+\infty} a_{1,n_1-1} \alpha_1^n n^{n_1-1} \right) + o(k^{n_1-1} |\alpha_1|^k) \\ &= \operatorname{Re}(\alpha_1^{k-m-1} (k-m-1)^{n_1-1} a_{1,n_1-1} x_k \\ &\quad + a_{1,n_1-1} \alpha_1^k k^{n_1-1} z_k) + o(k^{n_1-1} |\alpha_1|^k) \\ &= \operatorname{Re}(\alpha_1^{k-m-1} (k-m-1)^{n_1-1} a_{1,n_1-1} x_k) \\ &\quad + o(k^{n_1-1} |\alpha_1|^k), \end{aligned}$$

where

$$\begin{aligned} x_k &= \sum_{l=0}^m \alpha_1^l \left(1 + \frac{l}{k-m-1} \right)^{n_1-1} d_l \\ &\quad + \frac{\alpha_1^{m+1}}{1-\alpha_1} \left(1 + \frac{m+1}{k-m-1} \right)^{n_1-1} \end{aligned}$$

and

$$z_k = \sum_{n=0}^{+\infty} \alpha_1^n \left(1 + \frac{n}{k} \right)^{n_1-1} - \frac{1}{1-\alpha_1}.$$

For obtaining the formula (4) we used that $\lim_{k \rightarrow +\infty} z_k = 0$. Suppose there exists $\beta \in \{0, 1, \dots, n_1 - 1\}$, for which there exists a finite nonzero limit $\lim_{k \rightarrow +\infty} (k-m-1)^\beta x_k$. Denote $y_k = (k-m-1)^\beta x_k$, $y = \lim_{k \rightarrow +\infty} y_k$, $\psi_k = \arg y_k$, $\psi_0 = \arg y$, $\varphi = \arg \alpha_1$, $\varphi_0 = \arg a_{1,n_1-1}$. We can assume that $\varphi \in (0, \pi]$. If $\varphi = \pi$, then $\alpha_1 < 0$; therefore

$$\begin{aligned} \operatorname{Re}(\alpha_1^{k-m-1} (k-m-1)^{n_1-1} a_{1,n_1-1} x_k) \\ = \alpha_1^{k-m-1} (k-m-1)^{n_1-1-\beta} a_{1,n_1-1} y_k. \end{aligned}$$

Since $y \neq 0$, the expression above has different signs

for k and $k+1$. Therefore $\int_k^{+\infty} K_{0,0}^{(m)}(1, t) dt < 0$ or

$$\int_{k+1}^{+\infty} K_{0,0}^{(m)}(1, t) dt < 0.$$

Now let $\varphi \in (0, \pi)$. Note that

$$\begin{aligned} & \operatorname{Re}(\alpha_1^{k-m-1}(k-m-1)^{n_1-1-\beta} a_{1,n_1-1} y_k) \\ &= |\alpha_1|^{k-m-1}(k-m-1)^{n_1-1-\beta} |a_{1,n_1-1}| |y_k| \\ & \quad \cdot \cos((k-m-1)\varphi + \varphi_0 + \psi_k). \end{aligned}$$

We choose $\varepsilon > 0$ such that $\frac{\pi - 2\varepsilon}{\varphi} > 1$. Then for all $n \in \mathbf{N}$ the length of the segment

$$\left[\frac{\frac{\pi}{2} + 2\pi n - \varphi_0 - \psi_0 + \varepsilon}{\varphi}, \frac{\frac{3\pi}{2} + 2\pi n - \varphi_0 - \psi_0 - \varepsilon}{\varphi} \right]$$

will be greater than one. Since $\lim_{k \rightarrow +\infty} \psi_k = \psi_0$, there exists $k_0 \in \mathbf{N}$, such that $\psi_k \in (\psi_0 - \varepsilon, \psi_0 + \varepsilon)$ for all $k \geq k_0$. It follows that there are infinitely many k , such that

$$\cos((k-m-1)\varphi + \varphi_0 + \psi_k) < 0.$$

Now suppose that for no $\beta \in \{0, 1, \dots, n-1\}$, there exists a finite nonzero limit $\lim_{k \rightarrow +\infty} (k-m-1)^\beta x_k$. It is clear that in this case $x_k = 0$, for $k > m+3$. Hence taking into account (4) it suffices to prove that the number

$$\operatorname{Re}(a_{1,n_1-1} \alpha_1^k k^{n_1-1} z_k) + o(k^{n_1-1} |\alpha_1|^k),$$

has a negative sign. Suppose $n_1 \neq 1$. It is clear that in this case there exists a finite nonzero limit $\lim_{k \rightarrow +\infty} k z_k$. Hence, in the same way, the desired result can be obtained.

Now suppose $n_1 = 1$. Then we will change the form c_i in the following way

$$c_i = a_{1,0} \alpha_1^i + a_{1,n_2-1} i^{n_2-1} \alpha_2^i + o(i^{n_2-1} |\alpha_2|^i),$$

where $|\alpha_2| = \max\{|\alpha_2|, |\alpha_3|, \dots, |\alpha_t|\}$. Similarly, we find that it suffices to investigate the case $n_2 = 1$. Doing the same steps, we get that we can assume $n_1 = n_2 = \dots = n_t = 1$. We got that $x_k = 0$, if $k > m+3$, for α_1 ; in particular $\lim_{k \rightarrow \infty} x_k = 0$. Consequently, the following identity holds

$$\sum_{l=0}^m \alpha_1^l d_l + \frac{\alpha_1^{m+1}}{1 - \alpha_1} = 0.$$

This is equivalent to that α_1 is a root of the polynomial

$$\begin{aligned} P(x) &= d_0 + (d_1 - d_0)x + (d_2 - d_1)x^2 + \dots \\ & \quad + (d_m - d_{m-1})x^m + (1 - d_m)x^{m+1}. \end{aligned}$$

In the same way we get that $P(\alpha_i) = 0$, for all $i \in$

$\{1, 2, \dots, m+1\}$. From definitions of d_0, d_1, \dots, d_m it follows that $P\left(\frac{1}{\alpha_i}\right) = 0$, for all $i \in 1, 2, \dots, m+1$. Since $|\alpha_i| < 1$, for $i \in 1, 2, \dots, m+1$, the polynomial P will have $2m+2$ different zeros, which contradicts the fact that P has order $m+1$. \square

Lemma 2. *If $\alpha, \beta (\alpha < \beta)$ are real numbers and if $A_{i,j} = (\frac{\alpha+i}{2^i}, \frac{\beta+j}{2^j})$, $i, j \in \mathbf{Z}$, then*

$$\mu\left(\mathbf{R} \setminus \bigcap_{l=1}^{+\infty} \bigcup_{i=l}^{+\infty} \bigcup_{j \in \mathbf{Z}} A_{i,j}\right) = 0,$$

where μ is a Lebesgue measure.

Proof. Let $l \in \mathbf{N}$. Since the sequence $\{2^n x\}$, $n \in \mathbf{N}$, $n \geq l$ is everywhere dense in $[0, 1]$ for almost all $x \in [0, 1]$ (see [7], problems 4.3, 1.6), for almost all $x \in \mathbf{R}$ there exists $i \geq l$ such that $x \in A_{i, [2^i x]}$. It follows that

$$\mu\left(\mathbf{R} \setminus \bigcup_{i=l}^{+\infty} \bigcup_{j \in \mathbf{Z}} A_{i,j}\right) = 0,$$

for all $l \in \mathbf{N}$, which ends the proof. \square

Proof of theorem 2. From (2) and lemma 1 follows that there exist constants $\varepsilon, \delta > 0$ and $k_0 \in \mathbf{N}$ such that $S_{0,0}^{(m)}(\varphi_x, k_0) > 1 + \delta$ for all $x \in (k_0 - \varepsilon, k_0 + \varepsilon)$, where

$$\varphi_x(t) = \begin{cases} 1, & t \leq x \\ 0, & t > x. \end{cases}$$

Denote $I_{i,j} = (\frac{k_0 - \varepsilon + j}{2^i}, \frac{k_0 + \varepsilon + j}{2^i})$, $i, j \in \mathbf{Z}$. From (1) we obtain

$$K_{i,j}(k_0, t) = 2^i K_{0,0}(2^i k_0 - j, 2^i t - j);$$

thus

$$S_{i,j}^{(m)}\left(\varphi_x, \frac{k_0 + j}{2^i}\right) > 1 + \delta, \quad x \in I_{i,j}.$$

Therefore from lemma 2 follows that for almost all $x \in \mathbf{R}$ there exist sequences i_n, j_n such that $\lim_{n \rightarrow \infty} \frac{k_0 + j_n}{2^{i_n}} = x$ and

$$S_{i_n, j_n}^{(m)}\left(\varphi_x, \frac{k_0 + j_n}{2^{i_n}}\right) > 1 + \delta, \quad x \in I_{i,j}.$$

It follows that

$$(5) \quad \overline{\lim}_{\substack{t \rightarrow x \\ i, j \rightarrow +\infty}} S_{i,j}^{(m)}(\varphi_x, t) > 1 + \delta.$$

To complete the proof it remains to do the following steps as in [9]:

- 1) Prove that if g is an integrable function on \mathbf{R} which is continuous at x , then

$$\lim_{\substack{t \rightarrow x \\ i, j \rightarrow +\infty}} S_{i,j}^{(m)}(g, t) = g(x).$$

The main part of proof of this point, is that for all $x \in \mathbf{R}$ and $\delta > 0$

$$\lim_{\substack{t \rightarrow x \\ i, j \rightarrow +\infty}} \int_{|s-x| \geq \delta} |K_{i,j}^{(m)}(t, s)| ds = 0.$$

- 2) Prove that $G(t, m) > 1 + \delta$ for almost all $t \in \mathbf{R}$. Proof of this fact for characteristic functions follows from (5). \square

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References

- [1] L. A. Balashov and V. A. Skvortsov, Gibbs constants for partial sums of Fourier-Walsh series and their $(C, 1)$ -means, *Trudy Mat. Inst. Steklov.* **164** (1983), 37–48.
- [2] N. K. Bari, *Trigonometric series*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961. (in Russian).
- [3] R. A. DeVore and G. G. Lorentz, *Constructive approximation*, Grundlehren der Mathematischen Wissenschaften, 303, Springer-Verlag, Berlin, 1993.
- [4] C. Karanikas, Gibbs phenomenon in wavelet analysis, *Results Math.* **34** (1998), no. 3–4, 330–341.
- [5] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, 2nd ed., Academic Press, New York, 2001.
- [6] S. E. Kelly, Gibbs phenomenon for wavelets, *Appl. Comput. Harmon. Anal.* **3** (1996), no. 1, 72–81.
- [7] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience, New York, 1974.
- [8] V. G. Mikaelyan, The Gibbs phenomenon for general Franklin systems, *J. Contemp. Math. Anal.* **52** (2017), no. 4, 198–210; translated from *Izv. Nats. Akad. Nauk Armenii Mat.* **52** (2017), no. 4, 51–71.
- [9] V. G. Mikaelyan, The Gibbs phenomenon for Stromberg’s piecewise linear wavelet, *J. Contemp. Math. Anal.* **54** (2019), no. 2, 112–123; translated from *Izv. Nats. Akad. Nauk Armenii Mat.* **54** (2019), no. 2, 65–81.
- [10] O. G. Sargsyan, On the convergence and Gibbs phenomenon of Franklin series, *J. Contemp. Math. Anal.* **31** (1996), no. 1, 51–72; translated from *Izv. Nats. Akad. Nauk Armenii Mat.* **31** (1996), no. 1, 61–84.
- [11] J.-O. Strömberg, A modified Franklin system and higher-order spline systems on \mathbf{R}^n as unconditional bases for Hardy spaces, in *Conference on harmonic analysis in honor of Antoni Zygmund, (Chicago, 1981)*, 475–494, Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983.
- [12] A. M. Zubakin, The Gibbs phenomenon for multiplicative systems of Walsh or Vilenkin-Dzhofarli type, *Siberian Math. J.*, **12** (1971), no. 1, 147–157.