Stability of branching laws for spherical varieties and highest weight modules

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Abstract: If a locally finite rational representation V of a connected reductive algebraic group G has uniformly bounded multiplicities, the multiplicities may have good properties such as stability. Let X be a quasi-affine spherical G-variety, and M be a $(\mathbf{C}[X], G)$-module. In this paper, we show that the decomposition of M as a G-representation can be controlled by the decomposition of the fiber $M/\mathfrak{m}(x_0)M$ with respect to some reductive subgroup $L \subset G$ for sufficiently large parameters. As an application, we apply this result to branching laws for simple real Lie groups of Hermitian type. We show that the sufficient condition on multiplicity-freeness given by the theory of visible actions is also a necessary condition for holomorphic discrete series representations and symmetric pairs of holomorphic type. We also show that two branching laws of a holomorphic discrete series representation with respect to two symmetric pairs of holomorphic type coincide for sufficiently large parameters if two subgroups are in the same ϵ-family.

Key words: Spherical variety; multiplicity-free representation; branching rule; symmetric pair; highest weight module; semisimple Lie group.

1. Introduction. Our main concern in this paper is to describe a behavior of multiplicities of a completely reducible representation with uniformly bounded multiplicities. Note that this paper is a short version of [7].

Before we state the main theorem, we prepare some notations. Let G be a connected reductive algebraic group over $\mathbf{C}$. We will say that a representation V of G is a *locally finite rational representation* if $\mathrm{span}_{\mathbf{C}}\{gv : g \in G\}$ is a finite dimensional regular representation of G for any $v \in V$. Fix a Borel subgroup B of G. For a locally finite rational representation V of G, we denote by $m_V^G(\lambda)$ the multiplicity of the representation with highest weight λ with respect to B, and denote by $\Lambda^+(V) := \Lambda_G^+(V)$ the set of characters λ of B satisfying $m_V^G(\lambda) \neq 0$. We write the supremum of $m_V^G(\lambda)$ with respect to λ by $C_G(V)$. For a G-variety X, we write $\Lambda^+(X) := \Lambda^+(\mathbf{C}[X])$ for short. We will say that a $\mathbf{C}[X]$-module M is a $(\mathbf{C}[X], G)$-module if M is a locally finite rational representation of G and two actions of $\mathbf{C}[X]$ and G are compatible:

$$g(fm) = (gf) \cdot (gm)$$

for any $g \in G$, $f \in \mathbf{C}[X]$ and $m \in M$.

Let G be a connected reductive algebraic group over $\mathbf{C}$, and X be an irreducible G-variety. We assume the following two conditions:

(a) the quotient field of the regular function ring on X is equal to the function field on X,

(b) X is a spherical G-variety (i.e., a Borel subgroup B of G has an open dense orbit in X).

Usually, spherical varieties are defined to be normal. In this paper, however, we do not assume normality since we use only multiplicity-freeness and the Borel open orbit. The structure of spherical varieties such as their weight monoids are recently studied by F. Knop and I. Losev (see e.g., [12]).

We fix a point $x_0 \in X$ such that $Bx_0 (:= \{bx_0 : b \in B\})$ is open dense in X. Put

$$P := \{g \in G : gBx_0 \subset Bx_0\},$$
$$L := P_{x_0}. \tag{1.0.1}$$

Here, we denote by P_{x_0} the stabilizer at x_0 in P.

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Then, P is a parabolic subgroup of G. Using the theorem of M. Brion, D. Luna and T. Vust [2] for the spherical pair (G, G_{x_0}), we obtain that L is a reductive subgroup of G containing B_{x_0}. Note that irreducible representations of L are parametrized by a subset of characters of B_{x_0}, and this correspondence comes from taking a unique B_{x_0}-eigenvector in an irreducible representation of L. We use same notations $m_V^L(\lambda)$ and $C_L(V)$ for a locally finite rational representation V of L and a character λ of B_{x_0}.

Then, our main result is

Theorem 1.1. *Assume the above conditions (a) and (b). Let M be a finitely generated $(\mathbf{C}[X], G)$-module. Suppose $\mathbf{C}[X]$ has no zero divisors in M:*

$$\mathrm{Ann}_{\mathbf{C}[X]}(m)(:= \{f \in \mathbf{C}[X] : fm = 0\}) = 0$$

for any $m \in M \setminus \{0\}$. Then, there exists a $\lambda_0 \in \Lambda^+(X)$ such that

$$m_M^G(\lambda + \lambda_0) = m^L_{M/\mathfrak{m}(x_0)M}(\lambda|_{B_{x_0}})$$

for any $\lambda \in \Lambda^+(M)$. Here, $\mathfrak{m}(x_0)$ is the maximal ideal of $\mathbf{C}[X]$ corresponding to the point x_0 (i.e., $\mathfrak{m}(x_0) := \{f \in \mathbf{C}[X] : f(x_0) = 0\}$).

Remark 1.2. For any $\lambda_0 \in \Lambda^+(X)$, $\lambda_0|_{B_{x_0}} = 0$ holds. Then, we have $\lambda|_{B_{x_0}} = (\lambda + \lambda_0)|_{B_{x_0}}$.

This theorem asserts two things: the multiplicity function $m_V^G(\lambda)$ is periodic for sufficiently large parameter λ with respect to the translation by $\Lambda^+(X)$, and the multiplicities in sufficiently large parameters can be described by the decomposition of the 'fiber' $M/\mathfrak{m}(x_0)M$ with respect to L. The first property is called stability. If M can be realized as a set of global sections of an algebraic vector bundle over X, $M/\mathfrak{m}(x_0)M$ is actually equal to the fiber at x_0.

Stability was appeared in [8, Lemma 3.4] for example. F. Satō formulated and generalized stability for reductive spherical homogeneous spaces in [15]. Our theorem is a natural generalization of Satō's stability theorem for spherical varieties.

Retain the notation of Theorem 1.1. As a corollary of the theorem, the supremum of the multiplicities in M can be controlled by that of the fiber $M/\mathfrak{m}(x_0)M$.

Corollary 1.3. *Let M be a $(\mathbf{C}[X], G)$-module with no zero divisors. Then, we have*

$$C_G(M) = C_L(M/\mathfrak{m}(x_0)M).$$

Especially, M is multiplicity-free as a representation of G if and only if $M/\mathfrak{m}(x_0)M$ is multiplicity-free as a representation of L.

2. Examples. By applying Theorem 1.1 for some explicit varieties, we can obtain "stability theorems".

2.1. Quasi-affine spherical homogeneous spaces. Let G be a complex connected reductive algebraic group, and H be a Zariski-closed subgroup of G. We assume that (G, H) is a spherical pair and G/H is a quasi-affine variety. Note that the assumption "quasi-affine" is equivalent to the assumption (a) in Section 1 for homogeneous spaces (see [1]). Then, there exists a Borel subgroup B of G such that BH is open dense in G. Set $L := \{g \in H : gBH \subset BH\}$.

We apply Theorem 1.1 to $X = G/H$ and $M = \mathrm{Ind}_H^G(W) := (\mathbf{C}[G] \otimes W)^H$ for a finite dimensional rational representation W of H.

Theorem 2.1. *In the above settings, there exists a $\lambda_0 \in \Lambda^+(G/H)$ such that*

$$m^G_{\mathrm{Ind}_H^G(W)}(\lambda + \lambda_0) = m_W^L(\lambda|_{B_{x_0}})$$

for any $\lambda \in \Lambda^+(\mathrm{Ind}_H^G(W))$.

If H is semisimple, this theorem is equal to Satō's stability theorem [15].

2.2. Spherical projective varieties. Theorem 1.1 is not true for projective varieties. However, we can obtain a weaker result from the theorem.

Let G be a complex connected reductive algebraic group, P be a parabolic subgroup of G, and H be a connected reductive subgroup of G. We assume that G/P is a spherical H-variety. There exists a point $x_0 \in G$ such that Bx_0P is open dense in G for a Borel subgroup B of H. Set $L := \{g \in H : gx_0P = x_0P, gBx_0P \subset Bx_0P\}$. Then, we obtain the following theorem:

Theorem 2.2. *Let W be an irreducible rational representation of P. Then, there exists a character λ_0 of P such that*

$$C_H(\mathrm{Ind}_P^G(W \otimes \mathbf{C}_{\lambda_0+\lambda})) = C_L(W)$$

for any character λ of P satisfying $\mathrm{Ind}_P^G(\mathbf{C}_\lambda) \neq 0$. Here, we consider W as a representation of L via

$$g \cdot v = ({x_0}^{-1} g x_0) v$$

for $g \in L$ and $v \in W$.

In other words, if the parameter of W is sufficiently large in some sense, the supremum of the multiplicities in $\mathrm{Ind}_P^G(W)|_H$ is equal to that in $W|_L$.

Sketch of proof. Let $P = QN$ be a Levi decomposition of P. Then, $G/([Q,Q]N)$ is a quasi-affine spherical $H \times Q/[Q,Q]$-variety. Applying Theorem 1.1 to $X = G/([Q,Q]N)$ and $M = \mathrm{Ind}_{[Q,Q]N}^G(W)$, we obtain the theorem. □

2.3. Unitary highest weight modules. In Sections 2.3 and 2.4, we treat branching laws for infinite dimensional unitary representations of a simple real Lie group of Hermitian type. In this setting, G in Theorem 1.1 is $K_{\mathbf{C}}$ and X is the associated variety of a unitary representation. For a Lie group G, we write its Lie algebra by a German letter as $\mathfrak{g} := \mathrm{Lie}(G)$, and we write its complexification by a subscript $(\cdot)_{\mathbf{C}}$ as $\mathfrak{g}_{\mathbf{C}} := \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$.

Let G be a connected simple real Lie group of Hermitian type with finite center, and θ be a Cartan involution of G. Let K be the fixed point subgroup of θ in G. We fix a element Z of the center $Z(\mathfrak{k}_{\mathbf{C}})$ of $\mathfrak{k}$ such that $\mathrm{ad}(Z)$ has eigenvalues $\pm 1, 0$ in $\mathfrak{g}_{\mathbf{C}}$. We decompose $\mathfrak{g}_{\mathbf{C}}$ as

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_-$$

corresponding to the eigenvalues $1, 0, -1$, respectively.

We will say that an irreducible $(\mathfrak{g}, K)$-module $\mathcal{H}$ is a *highest weight module* if $\mathfrak{p}_+$-null part $\mathcal{H}^{\mathfrak{p}_+}$ is non-zero. If a highest weight module $\mathcal{H}$ is infinitesimally unitary, $\mathcal{H}$ is called a *unitary highest weight module*. Unitary highest weight modules are parametrized by highest weights of $\mathfrak{p}_+$-null part $\mathcal{H}^{\mathfrak{p}_+}$ with respect to $\mathfrak{t}_{\mathbf{C}}$. For $(\mathfrak{g}, K)$-module V and a unitary highest weight module V_λ with its highest weight λ, we write $m_V^G(\lambda) := \dim \mathrm{Hom}_{(\mathfrak{g},K)}(V_\lambda, V)$. Using this $m_V^G(\lambda)$, we redefine $\Lambda^+(V)$ and $C_G(V)$ in Section 1.

Since a unitary highest weight module $\mathcal{H}$ of G is a $(\mathfrak{g}_{\mathbf{C}}, K_{\mathbf{C}})$-module, $\mathcal{H}$ can be viewed as a $(\mathbf{C}[\mathfrak{p}_+], K_{\mathbf{C}})$-module via the isomorphism $\mathbf{C}[\mathfrak{p}_+] \simeq \mathcal{U}(\mathfrak{p}_-)$ determined by the Killing form of $\mathfrak{g}_{\mathbf{C}}$. We denote by $\mathcal{AV}(\mathcal{H}) \subset \mathfrak{p}_+$ the zero set of $\mathrm{Ann}_{\mathbf{C}[\mathfrak{p}_+]}(\mathcal{H})$, and we call $\mathcal{AV}(\mathcal{H})$ the *associated variety* of $\mathcal{H}$.

Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$, and a positive system Δ^+ of the root system $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ such that $\Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbf{C}}) \subset \Delta^+$. Since $\mathfrak{g}$ is Hermitian type Lie algebra, $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{g}$.

We take a maximal set of strongly orthogonal roots $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ as follows:

(a) γ_1 is the lowest root in $\Delta(\mathfrak{p}_+, \mathfrak{t}_{\mathbf{C}})$,

(b) γ_i is the lowest root in the roots that are strongly orthogonal to $\gamma_1, \gamma_2, \ldots, \gamma_{i-1}$,

and take root vectors $\{X_{\gamma_i}\}_{i=1}^r$. Note that r is equal to the real rank of $\mathfrak{g}$.

For $1 \leq m \leq r$, we put

$$\begin{aligned} X_m &:= X_{\gamma_1} + X_{\gamma_2} + \cdots + X_{\gamma_m}, \\ \mathfrak{a}_m &:= \bigoplus_{i=1}^m \mathbf{R}(X_{\gamma_i} + \overline{X_{\gamma_i}}), \\ \mathcal{O}_m &:= \mathrm{Ad}(K_{\mathbf{C}})X_m \text{ and} \\ L_m &:= Z_K(\mathfrak{a}_m). \end{aligned}$$

Here, $\overline{(\cdot)}$ is the complex conjugate of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{g}$. Then, we have the following theorem.

Theorem 2.3. *Let $\mathcal{H}$ be a unitary highest weight module of G with associated variety $\mathcal{AV}(\mathcal{H}) = \overline{\mathcal{O}_m}$. Then, there exists a $\lambda_0 \in \Lambda^+(\overline{\mathcal{O}_m})$ such that*

$$m_{\mathcal{H}}^K(\lambda + \lambda_0) = m_{\mathcal{H}/\mathfrak{m}(X_m)\mathcal{H}}^{L_m}(\lambda|_{T_{X_m}})$$

for any $\lambda \in \Lambda^+(\mathcal{H})$.

Remark 2.4. (1) By B. Kostant, L. K. Hua [6] and W. Schmid [16], the explicit form of $\Lambda^+(\overline{\mathcal{O}_m})$ was computed as:

$$\Lambda^+(\overline{\mathcal{O}_m}) = \left\{ -\sum_{i=1}^m c_i\gamma_i : c_1 \geq c_2 \geq \cdots \geq c_m \geq 0 \right\}.$$

(2) The representation $\mathcal{H}/\mathfrak{m}(X_m)\mathcal{H}$ is called an *isotropy representation* of $\mathcal{H}$. 'Isotropy representations' were introduced by D. Vogan ([17,18]) for general settings as a generalization of the multiplicity of associated cycles. H. Yamashita describe the isotropy representations of unitary highest weight modules by using Howe duality in [19].

Sketch of proof. Let us apply Theorem 1.1 to $X = \mathcal{AV}(\mathcal{H})$ and $M = \mathcal{H}$. The condition that $\mathbf{C}[\mathcal{AV}(\mathcal{H})]$ has no zero divisors in $\mathcal{H}$ is a direct consequence of A. Joseph's result:

Fact 2.5. *Let $\mathcal{H}$ be a unitary highest weight module of G. Then, the annihilator $\mathrm{Ann}_{S(\mathfrak{p}_-)}(\mathcal{H})$ is a prime ideal in $S(\mathfrak{p}_-)$, and $\mathrm{Ann}_{S(\mathfrak{p}_-)}(v) = \mathrm{Ann}_{S(\mathfrak{p}_-)}(\mathcal{H})$ for any $v \in \mathcal{H}$.*

Then, we obtain the theorem without the explicit form of L (defined in (1.0.1)).

$L = L_m$ comes from Moore's theorem (see e.g., [5, Proposition 4.8 in Chapter 5]) and some straightforward calculations. □

2.4. Holomorphic discrete series representations. Now, we will consider the restriction of holomorphic discrete series representations with respect to symmetric pairs of holomorphic type. Let G, K and θ be as in the previous section. Let τ be an involutive automorphism of G such that $\tau(Z) = Z$.

We put $H = (G^{\tau})_0$, the identity component of the fixed point group of τ. Such pair (G, H) is called a *symmetric pair of holomorphic type*. (This is because τ induces a holomorphic automorphism of G/K.) Note that $(H, H \cap K)$ is also a Hermitian symmetric pair.

For a unitary highest weight module $\mathcal{H}$ of G, if the completion of $\mathcal{H}$ with respect to its Hermitian inner product is a discrete series representation of G (i.e., any matrix coefficients of $\mathcal{H}$ is L^2-function on G), $\mathcal{H}$ is said to be a *holomorphic discrete series representation*.

We will reduce the branching law of $\mathcal{H}|_H$ to the maximal compact subgroup case (in Section 2.3). Here, we use the notation $\mathcal{H}|_H$ as the restriction of $\mathcal{H}$ with respect to $(\mathfrak{h}, H \cap K)$. To do this, we use the following fact (see e.g., [9,11]):

Fact 2.6. *Let $\mathcal{H}$ be a holomorphic discrete series representation of G, Suppose $S(\mathfrak{p}_-^{-\tau}) \otimes \mathcal{H}^{\mathfrak{p}_+}$ is decomposed as a $K \cap H$-representation as follows:*

$$S(\mathfrak{p}_-^{-\tau}) \otimes \mathcal{H}^{\mathfrak{p}_+} \simeq \bigoplus_{\pi \in \widehat{K \cap H}} m(\pi)\pi.$$

Then, $\mathcal{H}|_H$ is decomposed as

$$\mathcal{H}|_H \simeq \bigoplus_{\pi \in \widehat{K \cap H}} m(\pi)(N^{\mathfrak{h}}(\pi)).$$

Here, $\widehat{K \cap H}$ denotes the set of equivalent classes of finite dimensional representations of $K \cap H$, and $N^{\mathfrak{h}}(V)$ denotes the generalized Verma module:

$$\mathcal{U}(\mathfrak{h}_{\mathbf{C}}) \otimes_{\mathcal{U}((\mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_+) \cap \mathfrak{h}_{\mathbf{C}})} V$$

for a irreducible representation V of $K \cap H$. Moreover, each summand is also a holomorphic discrete series representation of H.

We take $\mathfrak{t}, \Delta^+$, $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ and $\mathfrak{a}_m$ as in Section 2.3, considering $\mathfrak{g}^{\theta\tau}$ as $\mathfrak{g}$. It is known that $\mathfrak{a}_r$ is a maximal abelian subspace of $\mathfrak{p}_+^{-\tau}$, and then we write $\mathfrak{a} := \mathfrak{a}_r$. We set $L := Z_{K \cap H}(\mathfrak{a})$. From Fact 2.6, we obtain the following theorem:

Theorem 2.7. *Let $\mathcal{H}$ be a holomorphic discrete series representation of G. Then, there exists a $\lambda_0 \in \Lambda^+_{K \cap H}(\mathfrak{p}_+^{\tau})$ such that*

$$m^H_{\mathcal{H}}(\lambda + \lambda_0) = m^L_{\mathcal{H}^{\mathfrak{p}_+}}(\lambda|_{T \cap L})$$

for any $\lambda \in \Lambda^+_H(\mathcal{H})$.

As a corollary of Theorem 2.7, we obtain a necessary and sufficient condition for multiplicity-freeness.

Corollary 2.8. *Let $\mathcal{H}$ be a holomorphic discrete series representation of G. Then, we have*

$$C_H(\mathcal{H}) = C_L(\mathcal{H}^{\mathfrak{p}_+}).$$

In particular, $\mathcal{H}|_H$ is multiplicity-free if and only if $\mathcal{H}^{\mathfrak{p}_+}|_L$ is multiplicity-free.

In [10, Theorems 18 and 38], T. Kobayashi showed 'uniformly boundedness' and 'If' part of multiplicity-freeness in this corollary.

3. Sketch of proof of Theorem 1.1. We will sketch the proof of Theorem 1.1. Let G, B, X and x_0 be as in Section 1. Suppose $B = TN$ is a Levi decomposition of B, where T is a maximal torus of G and N is the unipotent radical of B.

For the proof of Theorem 1.1, we use the following result. This property is called stability.

Proposition 3.1. *Let M be a finitely generated $(\mathbf{C}[X], G)$-module with no zero divisors. Then, there exists a $\lambda_0 \in \Lambda^+(X)$ such that*

$$m^G_M(\lambda + \lambda_0) = m^G_M(\lambda + \lambda_0 + \mu)$$

for any $\lambda \in \Lambda^+(M)$ and $\mu \in \Lambda^+(X)$.

Since $\mathbf{C}[X]$ has no zero divisors in M, the multiplication map $f\cdot : M \to M$ is injective for any $f \in \mathbf{C}[X]$. Especially, a B-eigenvector $f \in \mathbf{C}[X]^N(\mu)$ with weight $\mu \in \Lambda^+(X)$ induces an injection $f\cdot : M^N(\lambda) \to M^N(\lambda + \mu)$ for any $\lambda \in \Lambda^+(M)$. Here, we denote by $V(\lambda)$ the weight space with weight λ in a locally finite rational representation V of T. Since M is finitely generated and $\mathbf{C}[X]$ is multiplicity-free, then M has uniformly bounded multiplicities (see [11]). Proposition 3.1 is a direct consequence of the uniformly boundedness and the following proposition.

Proposition 3.2. *Let $\mathcal{A}$ be a Noetherian G-algebra, and M be a finitely generated $(\mathcal{A}, G)$-module. Then, M^N is a finitely generated $\mathcal{A}^N$-module.*

If $\mathcal{A}$ is finitely generated algebra, this proposition (for arbitrary characteristics) was appeared in [4].

Proof. $\mathcal{A}^N$ and M^N are isomorphic to $(\mathcal{A} \otimes \mathbf{C}[G/N])^G$ and $(M \otimes \mathbf{C}[G/N])^G$, respectively. Since $\mathbf{C}[G/N]$ is finitely generated (see [3]), $\mathcal{A} \otimes \mathbf{C}[G/N]$ is a Noetherian $\mathbf{C}$-algebra. By a similar proof as Hilbert's fourteenth problem for reductive groups, we can show that $(M \otimes \mathbf{C}[G/N])^G$ is finitely generated as an $(\mathcal{A} \otimes \mathbf{C}[G/N])^G$-module. $\square$

We take $\lambda_0 \in \Lambda^+(X)$ satisfying the condition of Proposition 3.1. We consider the evaluation map:

$$\mathrm{ev}_{x_0} : M \to M/\mathfrak{m}(x_0)M.$$

Put $M_{x_0} := M/\mathfrak{m}(x_0)M$. Recall that Bx_0 is open dense in X. To prove Theorem 1.1, it suffices to show that

$$\mathrm{ev}_{x_0} : M^N(\lambda + \lambda_0) \to M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}}) \tag{3.2.1}$$

is bijective for any $\lambda \in \Lambda^+(M)$.

We use the following two lemmas:

Lemma 3.3. *Let M be a $(\mathbf{C}[X], G)$-module with no zero divisors. Then, we have*

$$\bigcap_{b \in B} \mathfrak{m}(bx_0)M = 0.$$

Lemma 3.4. *The regular function ring on Bx_0 has the following explicit form:*

$$\mathbf{C}[Bx_0] = \mathbf{C}[X][1/f : f \in \mathbf{C}[X]^N(\lambda) \setminus \{0\}, \lambda \in \Lambda^+(X)].$$

From Lemma 3.3, the map (3.2.1) is injective. First, we prove the surjectivity under the assumption that there exists a finite dimensional representation W of G such that $\mathbf{C}[X] \otimes W \simeq M$. Fix $\lambda \in \Lambda^+(M)$. We define

$$\varphi(bx_0) = b^{-\lambda-\lambda_0}(bm)$$

for any $m \in W^{N_{x_0}}(\lambda|_{B_{x_0}})$ and $b \in B$. Here, we denote by $b^{-\lambda-\lambda_0}$ the value of the character $-\lambda - \lambda_0$ at $b \in B$. Then, φ is well-defined as an element of $\mathbf{C}[Bx_0] \otimes W$, and φ is a B-eigenvector with weight $\lambda + \lambda_0$. By Lemma 3.4, there exist a weight $\mu \in \Lambda^+(X)$ and $f \in \mathbf{C}[X]^N(\mu)$ satisfying $f\varphi \in \mathbf{C}[X] \otimes W$. From Proposition 3.1, the multiplication map

$$\begin{aligned} f\cdot :& (\mathbf{C}[X] \otimes W)^N(\lambda + \lambda_0) \\ &\to (\mathbf{C}[X] \otimes W)^N(\lambda + \lambda_0 + \mu) \end{aligned}$$

is bijective. Then, we have $\varphi \in (\mathbf{C}[X] \otimes W)^N(\lambda + \lambda_0)$. This implies that ev_{x_0} in (3.2.1) is surjective in this case.

Next, we consider general cases. We take a finite dimensional subrepresentation $W \subset M$ of G that generates M as a $\mathbf{C}[X]$-module. Then, we have the following commutative diagram:

$$\begin{tikzcd} \mathbf{C}[X] \otimes W \arrow[r, "\mathrm{ev}_{x_0}"] \arrow[d, "\times"] & W \arrow[d] \\ M \arrow[r, "\mathrm{ev}_{x_0}"] & M_{x_0}, \end{tikzcd}$$

and all arrows are surjective. Take $\lambda_0' \in \Lambda^+(X)$ described in Proposition 3.1 for $M = \mathbf{C}[X] \otimes W$. By restricting the above diagram to the subspace of B-eigenvectors of weight $\lambda + \lambda_0'$, we have the following commutative diagram.

$$\begin{tikzcd} (\mathbf{C}[X] \otimes W)^N(\lambda + \lambda_0') \arrow[r, "\mathrm{ev}_{x_0}"] \arrow[d] & W^{N_{x_0}}(\lambda|_{B_{x_0}}) \arrow[d] \\ M^N(\lambda + \lambda_0') \arrow[r, "\mathrm{ev}_{x_0}"] & M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}}). \end{tikzcd}$$

Since G and L are reductive, the vertical arrows are surjective. From the free module case, the above horizontal arrow is surjective. Then, $\mathrm{ev}_{x_0} : M^N(\lambda + \lambda_0') \to M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}})$ is also surjective.

Since $\dim(M^N(\lambda + \lambda_0)) \geq \dim(M^N(\lambda + \lambda_0'))$ by the result of Proposition 3.1, $\mathrm{ev}_{x_0} : M^N(\lambda + \lambda_0) \to M_{x_0}^{N_{x_0}}(\lambda|_{B_{x_0}})$ is also surjective. This completes the proof.

4. Branching laws and ϵ-family. In this section, we treat the relation between branching laws and ϵ-family. Let G be a connected simple Lie group of Hermitian type with finite center, and θ be a Cartan involution of G. Let K be the fixed point subgroup of θ in G. Suppose τ is an involutive automorphism of G commuting with θ, and $(\mathfrak{g}, \mathfrak{g}^\tau)$ is of holomorphic type (see Section 2.4). Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}^{-\tau}$.

We introduce ϵ-family of symmetric pairs. The following definitions are due to T. Ōshima and J. Sekiguchi [13]. We denote by $\Sigma(\mathfrak{a}) := \Sigma(\mathfrak{g}, \mathfrak{a})$ the set of restricted roots with respect to $\mathfrak{a}$. Rossmann (see [14]) showed that $\Sigma(\mathfrak{a})$ is a root system.

We will say a map $\epsilon : \Sigma(\mathfrak{a}) \cup \{0\} \to \{1, -1\}$ is a *signature* of $\Sigma(\mathfrak{a})$ if $\epsilon(\alpha + \beta) = \epsilon(\alpha)\epsilon(\beta)$ for any $\alpha, \beta \in \Sigma(\mathfrak{a}) \cup \{0\}$. For a signature ϵ, we define an involutive automorphism τ_ϵ of $\mathfrak{g}$ as follows:

$$\tau_\epsilon(X) = \epsilon(\alpha)\tau(X) \text{ for } X \in \mathfrak{g}(\mathfrak{a}; \alpha), \alpha \in \Sigma(\mathfrak{a}) \cup \{0\}.$$

Here, we put

$$\mathfrak{g}(\mathfrak{a}; \alpha) := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for any } H \in \mathfrak{a}\}.$$

We define

$$F((\mathfrak{g}, \mathfrak{g}^\tau)) := \{(\mathfrak{g}, \mathfrak{g}^{\tau_\epsilon}) : \epsilon \text{ is a signature of } \Sigma(\mathfrak{a})\},$$

and call it an *ϵ-family of symmetric pairs.* If $\tau = \theta$, we call $F((\mathfrak{g}, \mathfrak{k}))$ a *$\mathfrak{k}_\epsilon$-family of symmetric pairs.*

We fix a signature ϵ of $\Sigma(\mathfrak{a})$. We assume that $(\mathfrak{g}, \mathfrak{g}^{\tau_\epsilon})$ is of holomorphic type. Suppose H and H'

are analytic subgroups with Lie algebra $\mathfrak{g}^\tau$ and $\mathfrak{g}^{\tau_\epsilon}$. As an application of Corollary 2.8, we have the following theorem:

Theorem 4.1. *Let $\mathcal{H}$ be a holomorphic discrete series representation of G. Then, we have $C_H(\mathcal{H}) = C_{H'}(\mathcal{H})$.*

Sketch of proof. By the definition of ϵ-family ($\epsilon(0) = 1$), we have $Z_{H\cap K}(\mathfrak{a}) = Z_{H'\cap K}(\mathfrak{a})$. This shows the theorem. □

More precisely, two branching laws of $\mathcal{H}|_H$ and $\mathcal{H}|_{H'}$ coincide for sufficiently large parameters. We fix a Cartan subalgebra $\mathfrak{t}^\tau \subset \mathfrak{k}^{\tau,\tau_\epsilon}$. $\mathfrak{t}^\tau$ is also a Cartan subalgebra of $\mathfrak{g}^\tau$ and $\mathfrak{g}^{\tau_\epsilon}$. The following theorem is proved by using Weyl's character formula.

Theorem 4.2. *Let $\mathcal{H}$ be a holomorphic discrete series representation of G. Suppose $(\mathcal{H}^{\mathfrak{p}_+})^*$ has the following formal character with respect to $\mathfrak{t}^\tau$:*

$$\mathrm{ch}((\mathcal{H}^{\mathfrak{p}_+})^*) = \bigoplus_{\nu\in\sqrt{-1}(\mathfrak{t}^\tau)^*} m(\nu)e^\nu.$$

We put $\mathcal{V} := \{\nu \in \sqrt{-1}(\mathfrak{t}^\tau)^* : m(\nu) \neq 0\}$. Then, there exists a total order on $\sqrt{-1}(\mathfrak{t}^\tau)^*$ such that

$$m^H_{\mathcal{H}}(\lambda) = m^{H'}_{\mathcal{H}}(\lambda),$$

for any $\lambda \in \sqrt{-1}(\mathfrak{t}^\tau)^*$ satisfying $(\lambda+\nu,\alpha) \geq 0$ for any $\alpha \in \Delta^+(\mathfrak{k}_{\mathbf{C}}^{-\tau\tau_\epsilon}, \mathfrak{t}^\tau_{\mathbf{C}})$ and $\nu \in \mathcal{V}$. Here, we take positive systems of $\mathfrak{g}^\tau$ and $\mathfrak{g}^{\tau_\epsilon}$ by the ordering on $\sqrt{-1}(\mathfrak{t}^\tau)^*$.

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