On a naturality of Chern-Mather classes

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Introduction. The Chern-Schwartz-MacPherson class (or more precisely natural transformation) is the unique natural transformation from the covariant constructible function functor to the covariant integral homology functor satisfying the normalization that the value of the characteristic function of a nonsingular compact complex analytic variety is equal to the Poincaré dual of the total Chern cohomology class of the tangent bundle. The existence of such a transformation was conjectured by Deligne and Grothendieck and was proved by MacPherson [10]. The novelity of MacPherson's proof is introducing the notion of local Euler obstruction (which was independently introduced by Kashiwara [7] also) and assigning the Chern-Mather class to this local Euler obstruction, not to the characteristic function. Although the Chern-Mather class is a very geometrically simple homology class, the assignment of the Chern-Mather class to the characteristic function does not give such a natural transformation.

It is often said that few "functorial" properties are know for the Chern-Mather class (e.g., see [6, *Note*, page 377, right after Example 19.1.7]), although the assignment of the Chern-Mather class to the local Euler obstruction is perfectly "natural", which is the main part of MacPherson's proof. In this paper, using this fine naturality of this assingment, we interpret the Chern-Mather class in the same way as the Chern-Schwartz-MacPherson class. Furthermore, by introducing the notion of a "q-deformed" local Euler obstruction which unifies the characteristic function and the local Euler obstruction, we give a "q-deformed" Chern-Schwartz-MacPherson class natural transformation, which specializes to the Chern-Mather class natural transformation for $q = 0$ and the Chern-Schwartz-MacPherson class natural transformation for $q = 1$.

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1. Constructible functions and Chern-Schwartz-MacPherson classes. Let $\mathcal{F}(X)$ denote the abelian group of constructible functions on X. The correspondence $\mathcal{F}$ assigning to each variety X the abelian group $\mathcal{F}(X)$ becomes a covariant functor when we consider the following "geometrically defined" pushforward:

$$\big(f_*\mathbb{1}_W\big)(y) := \chi\big(f^{-1}(y) \cap W\big),$$

which is linearly extended with respect to the generators $\mathbb{1}_W$ (see [10, 12]).

For the algebraic category Deligne and Grothendieck conjectured and MacPherson proved:

Theorem (1.1) (MacPherson's theorem [10]). *For the covariant functors $\mathcal{F}$ and H_* there exists a unique natural transformation*

$$C_* : \mathcal{F} \to H_*$$

satisfying (*normalization condition*) *that for X nonsingular*

$$C_*(\mathbb{1}_X) = c(TX) \cap [X],$$

where $c(TX)$ is the total Chern cohomology class of the tangent bundle TX and $[X]$ is the fundamental homology class of X.

MacPherson first observed that the abelian group of constructible functions are freely generated by local Euler obstructions of the closed subvarieties and via the graph construction method he proved that the association of the Chern-Mather class $C^M(W)$ to the local Euler obstruction Eu_W;

$$C_* : \mathrm{Eu}_W \longmapsto C^M(W),$$

not to the characteristic function $\mathbb{1}_W$, is natural, i.e.,

$$f_*C_*(\mathrm{Eu}_W) = C_*(f_*\,\mathrm{Eu}_W). \tag{1.2}$$

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In particular, we have

$$f_* C_*(\mathbb{1}_W) = C_*(f_* \mathbb{1}_W). \tag{1.3}$$

This is the so-called "naturality" of the Chern-Schwartz-MacPherson class, since $C_*(\mathbb{1}_W) = C_*(W)$ is the Chern-Schwartz-MacPherson class and we have that $\int_X C_*(X) = \chi(X)$, the topological Euler-Poincaré characteristic of the variety X.

In the analytic category MacPherson's proof works in parallell, except for the analyticity of the graph construction. However it was solved affirmatively by M. Kwieciński in his thesis [8]. So now the above conjecture is true in both the algebraic and analytic categories. It turns out that the so-called Schwartz characteristic cohomology classes, which M.-H. Schwartz [11] had constructed before the above conjecture was made, are in fact isomorphic to the homology class $C_*(\mathbb{1}_X)$ via the Alexander duality isomorphism (see [2]). Thus the total homology class $C_*(\mathbb{1}_X)$ is nowadays called the *Chern-Schwartz-MacPherson class of* X. We call the above natural transformation C_* the *Chern-Schwartz-MacPherson class natural transformation*, emphasizing that it is a natural transformation, since the "naturality" is the main object in this paper.

2. A "q-deformed" local Euler obstruction. For the original definition of MacPherson's local Euler obstruction defined via the obstruction theory, see his paper [10]. Since for a smooth point of the subvariety W $\mathrm{Eu}_W(p) = 1$, the local Euler obstruction function Eu_W is almost like the characteristic function $\mathbb{1}_W$; they differ only along the singular set of W. The local Euler obstruction Eu_W is a constructible function (e.g., see [2, 4, 10]).

Definition (2.1). For a subvariety $W \subset X$ the "q-deformed" local Euler obstruction $\mathrm{Eu}_W^{(q)}$ of W is defined by

$$\mathrm{Eu}_W^{(q)} := \sum_S n_S q^{\dim W - \dim S} \mathbb{1}_S,$$

provided that

$$\mathrm{Eu}_W = \sum_S n_S \mathbb{1}_S,$$

where $S's$ are closed subvarieties of W.

Note that for any q

$$\mathrm{Eu}_W^{(q)} = \mathbb{1}_W + \sum_{S \subset \mathrm{Sing}(W)} n_S q^{\dim W - \dim S} \mathbb{1}_S,$$

therefore we have

(1) if W is nonsingular, then $\mathrm{Eu}_W^{(q)} = \mathrm{Eu}_W = \mathbb{1}_W$, and
(2) $\mathrm{Eu}_W^{(0)} = \mathbb{1}_W$,
(3) $\mathrm{Eu}_W^{(1)} = \mathrm{Eu}_W$.

In this sense the "q-deformed" local Euler obstruction $\mathrm{Eu}_W^{(q)}$ is a "deformation" of both the characteristic function and the local Euler obstruction.

Proposition (2.2).
(1) *The homomorphism*

$$\mathcal{E}^{(q)} : \mathcal{Z}(X) \otimes_{\mathbf{Z}} \mathbf{Z}[q] \to \mathcal{F}(X) \otimes_{\mathbf{Z}} \mathbf{Z}[q]$$

defined by

$$\mathcal{E}^{(q)} \left(\sum p_W [W] \right) := \sum p_W \, \mathrm{Eu}_W^{(q)}$$

is an isomorphism, which shall be called a q-Euler isomorphism. Here $p_W \in \mathbf{Z}[q]$.
(ii) *Furthermore* $\{\mathrm{Eu}_W^{(q)} | W\}$ *are the free generators of* $\mathcal{F}(X) \otimes_{\mathbf{Z}} \mathbf{Z}[q]$.

The proof of the lemma is done by the induction on dimension, in a way similar to that in the original MacPherson's proof. So it is left for the reader as an exercise.

3. The "q-deformed" Chern-Schwartz-MacPherson class. One of the basic requirements for a right notion of Chern homology class $C_*(X)$ of a possibly singular variety X is the very geometric one ("Euler-Poincaré characteristic condition") that the degree $\int_X C_*(X)$ of the 0-dimensional component of $C_*(X)$ must be equal to the topological Euler-Poincaré characteristic $\chi(X)$ of X. Hence, even if we have (1.2), which is a perfect and fine "naturality" of the Chern-Mather class, we cannot adopt the Chern-Mather class as a right singular Chern homology class, simply because $\int_X C^M(X) \neq \chi(X)$ for X singular. Thus it is often said that the "functoriality" of the Chern-Mather class is not known. However, we still want to consider (1.2) as the "naturality" of the Chern-Mather class. So, to avoid ambiguity, we want to express the naturality (1.2) in the same form as (1.3);

$$f_\bigstar C_\bigstar(\mathbb{1}_W) = C_\bigstar(f_\bigstar \mathbb{1}_W).$$

Certainly, the association $C_\bigstar$ is

$$C_\bigstar : \mathbb{1}_W \longmapsto C^M(W),$$

and the pushforward $f_\bigstar$ is

$$f_\bigstar \mathbb{1}_W = \sum n_S \mathbb{1}_S$$

provided that $f_* \operatorname{Eu}_W = \sum n_S \operatorname{Eu}_S$. It is easy to see that with the above pushforward $C_\bigstar : \mathcal{F} \to H_*$ is a unique natural transformation satisfying that for X nonsingular $C_\bigstar(\mathbb{1}_X) = c(TX) \cap [X]$. This $C_\bigstar$ shall be called *the Chern-Mather class natural transformation*. Indeed, with the following canonical automorphism

$$\Phi_X : \mathcal{F}(X) \to \mathcal{F}(X) \text{ defined by } \Phi_X(\mathbb{1}_W) = \operatorname{Eu}_W,$$

we have $C_\bigstar = C_* \circ \Phi$ and for a morphism $f : X \to Y$

$$f_\bigstar = \Phi_Y^{-1} \circ f_* \circ \Phi_X.$$

Thus the uniqueness of $C_\bigstar$ follows from the uniqueness of C_* via the isomorphism Φ.

We generalize $C_\bigstar$, using the "q-deformed" local Euler obstructions. We first define the following pushforward $f_\bigstar^{(q)}$.

Definition (3.1). For a morphism $f : X \to Y$ the pushforward

$$f_\bigstar^{(q)} : \mathcal{F}(X) \otimes_{\mathbf{Z}} \mathbf{Z}[q] \to \mathcal{F}(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[q]$$

is defined by, for each generator $\operatorname{Eu}_W^{(q)}$,

$$f_\bigstar^{(q)} \operatorname{Eu}_W^{(q)} := \sum n_S \operatorname{Eu}_S^{(q)},$$

provided that

$$f_* \operatorname{Eu}_W = \sum n_S \operatorname{Eu}_S .$$

It is obvious that $f_\bigstar^{(0)} = f_\bigstar$ and $f_\bigstar^{(1)} = f_*$. Certainly, the pushforward $f_\bigstar^{(q)}$ is well-defined and functorial. Indeed, there is a canonical automorphism

$$\Phi_X^{(q)} : \mathcal{F}(X) \otimes_{\mathbf{Z}} \mathbf{Z}[q] \to \mathcal{F}(X) \otimes_{\mathbf{Z}} \mathbf{Z}[q]$$

defined by

$$\Phi_X^{(q)} \left(\sum_W n_W \operatorname{Eu}_W^{(q)} \right) = \sum_W n_W \operatorname{Eu}_W .$$

Let

$$f_*^{[q]} : \mathcal{F}(X) \otimes_{\mathbf{Z}} \mathbf{Z}[q] \to \mathcal{F}(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[q]$$

be the linear extension of the original pushforward $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ with respect to the polynomial ring $\mathbf{Z}[q]$. Then the pushforward $f_\bigstar^{(q)}$ is described by

$$f_\bigstar^{(q)} = (\Phi_Y^{(q)})^{-1} \circ f_*^{[q]} \circ \Phi_X^{(q)}.$$

In other words the automorphism $\Phi^{(q)}$ is an equivalence of the two functors $\mathcal{F}^{(q)} := \mathcal{F} \otimes_{\mathbf{Z}} \mathbf{Z}[q]$ with the pushforward $f_\bigstar^{(q)}$ and $\mathcal{F}^{[q]} := \mathcal{F} \otimes_{\mathbf{Z}} \mathbf{Z}[q]$ with the pushforward $f_*^{[q]}$. And let $C_*^{[q]} : \mathcal{F} \to H_* \otimes_{\mathbf{Z}} \mathbf{Z}[q]$ be the linear extension of C_* with respect to the polynomial ring $\mathbf{Z}[q]$. Then we have the natural transformation

$$C_\bigstar^{(q)} = C_*^{[q]} \circ \Phi^{(q)} : \mathcal{F}^{(q)} \to H_* \otimes_{\mathbf{Z}} \mathbf{Z}[q].$$

This $C_\bigstar^{(q)}$ shall be called a *"q-deformed" Chern-Schwartz-MacPherson class natural transformation* and the total homology class $C_\bigstar^{(q)}(X) := C_\bigstar^{(q)}(\mathbb{1}_X)$ iscalled the *"q-deformed" Chern-Schwartz-MacPherson class of X* .

It should be noted that the above pushforward $f_\bigstar^{(q)}$ looks easy, but actually it requires representing or expressing constructible functions in terms of the "q-deformed" local Euler obstructions, therefore that it is a complicated pushforward.

We can paraphrase the transformation $C_\bigstar^{(q)}$ as follows:

Theorem (3.2). *Let the Chern-Mather correspondence*

$$C^M : \mathcal{Z}(X) \otimes_{\mathbf{Z}} \mathbf{Z}[q] \to H_*(X; \mathbf{Z}[q])$$

be defined by

$$C^M \left(\sum p_W[W] \right) := \sum p_W C^M(W) \ \textit{with} \ p_W \in \mathbf{Z}[q].$$

Then the correspondence

$$C_\bigstar^{(q)} := C^M \circ (\mathcal{E}^{(q)})^{-1} : \mathcal{F} \otimes_{\mathbf{Z}} \mathbf{Z}[q] \to H_* \otimes_{\mathbf{Z}} \mathbf{Z}[q]$$

is a unique natural transformation satisfying that for X nonsingular

$$C_\bigstar^{(q)}(\mathbb{1}_X) = c(TX) \cap [X].$$

In particular, $C_\bigstar^{(0)} = C_\bigstar : \mathcal{F} \to H_*$ is the Chern-Mather class natural transformation and $C_\bigstar^{(1)} = C_* : \mathcal{F} \to H_*$ is the Chern-Schwartz-MacPherson class natural transformation.

Certainly the theorem can be proved directly in the same way as in the original MacPherson's proof [10].

As to the "q-deformed" Chern-Schwartz-MacPherson class of a variety X we need the following:

Theorem (3.3).

$$\mathbb{1}_X = \operatorname{Eu}_X^{(q)} + \sum_{\substack{S \in \mathcal{S}_X \\ S \subset \operatorname{Sing}(X)}} \theta(S, X) q^{\dim X - \dim S} \operatorname{Eu}_{\overline{S}}^{(q)},$$

where $\mathcal{S}_X$ is a Whitney stratification of X and $\theta(S, X)$ is the Dubson-Kashiwara index.

Proof . First we observe the following result:

Lemma (3.3.1). *Let $A = (a_{ij})$ be an $n \times n$ lower triangular matrix such that each diagonal entry $a_{ii} = 1$. Then the (i,j)-entry b_{ij} $(i > j)$ of the inverse matrix A^{-1} is given by:*

$$b_{ij} = \sum_{i=i_0>i_1>\cdots>i_k=j} (-1)^k a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1}j}.$$

The proof of this is straightforward by induction on the size of the matrix and left for the leader.

Let us put a partial order on the set $\mathcal{S}_X$ of Whitney stratification of X by $S' < S$ if and only if $S' \subset \overline{S} \setminus S$. Then, following [3, 4], for an ordered pair (S', S), where $S' < S$, the Dubson-Kashiwara index $\theta(S', S)$ [3, 4, 7] is defined by

$$\theta(S', S) := 1 - V^{1+\dim S'}(S', S).$$

Here $V^{1+\dim S'}(S', S) := \chi(\overline{S} \cap H^{1+\dim S'}_{x,\varepsilon} \cap B_\delta(x))$, where $x \in S'$, $H^{1+\dim S'}_{x,\varepsilon}$ is a plane of codimension $1 + \dim S'$, which does not go through the point x and is close to x within the distance ε, $B_\delta(x)$ is a δ-ball centered at x and $0 < \varepsilon << \delta$. Next, using this Dubson-Kashiwara index we consider a big lower triangular θ-matrix Θ with the diagonal entry being 1 (for this matrix see [3, 4]):

$$\Theta := (\widetilde{\theta}(S', S))_{(S',S)},$$

where

$$\widetilde{\theta}(S', S) = \begin{cases} 1, & S' = S, \\ \theta(S', S), & S' < S, \\ 0, & \text{otherwise} \end{cases}$$

We consider the following "vector" of distinguished constructible functions:

$$(\mathbb{1}_{\overline{S}}), \qquad (\mathrm{Eu}_{\overline{S}}).$$

Dubson's theorem [4] relates these two distinguished vectors by the above θ-matrix Θ as follows:

$$(\mathbb{1}_{\overline{S}}) = (\mathrm{Eu}_{\overline{S}})\Theta.$$

In particular we have the following

$$\mathbb{1}_X = \mathrm{Eu}_X + \sum_S \theta(S, X)\, \mathrm{Eu}_{\overline{S}}.$$

As a corollary of Lemma (3.3.1) we have:

$$\Theta^{-1} = \left(\sum_{S=S_{i_0}>\cdots>S_{i_k}=S'} \widehat{\theta}(S', S) \right)_{(S',S)},$$

where $\widehat{\theta}(S', S) := (-1)^k \theta(S', S_{i_{k-1}}) \cdots \theta(S_{i_1}, S)$ for each $S = S_{i_0} > \cdots > S_{i_k} = S'$. For any lower triangular matrix $M = \big(m(S', S)\big)_{(S',S)}$, where $m(S', S)$ is a real number, we define the "q-deformed" matrix $M^{(q)}$ as follows:

$$M^{(q)} := \Big(q^{\dim S - \dim S'} m(S', S) \Big)_{(S',S)}.$$

Then the "q-deformed" θ-matrix $\Theta^{(q)}$ is:

$$\Theta^{(q)} = \Big(q^{\dim S - \dim S'} \widetilde{\theta}(S', S) \Big)_{(S',S)}.$$

Then again by Lemma (3.3.1) it is easy to see that

$$\big(\Theta^{(q)}\big)^{-1} = \big(\Theta^{-1}\big)^{(q)}.$$

Now we are ready to prove Theorem (3.3). Since $(\mathbb{1}_{\overline{S}}) = (\mathrm{Eu}_{\overline{S}})\Theta$, we have

$$(\mathrm{Eu}_{\overline{S}}) = (\mathbb{1}_{\overline{S}})\Theta^{-1}.$$

Therefore by the definition of the "q-deformed" local Euler obstruction we have

$$\big(\mathrm{Eu}^{(q)}_{\overline{S}}\big) = (\mathbb{1}_{\overline{S}})\big(\Theta^{-1}\big)^{(q)} = (\mathbb{1}_{\overline{S}})\big(\Theta^{(q)}\big)^{-1}.$$

Hence we have

$$(\mathbb{1}_{\overline{S}}) = \big(\mathrm{Eu}^{(q)}_{\overline{S}}\big)\Theta^{(q)}.$$

Therefore in particular we get that

$$\mathbb{1}_X = \mathrm{Eu}^{(q)}_X + \sum_{S \subset \mathrm{Sing}(X)} \theta(S, X) q^{\dim X - \dim S}\, \mathrm{Eu}^{(q)}_{\overline{S}}.$$

□

Corollary (3.4).

$$C^{(q)}_\star(X) = C_*(X) + \sum_{S \subset \mathrm{Sing}(X)} \theta(S, X)(q^{\dim X - \dim S} - 1) C^M(\overline{S})$$
$$= C_*(X) + \sum_{S_{i_0}>\cdots>S_{i_k}=S} \widehat{\widehat{\theta}}(S, X)(q^{\dim X - \dim S_{i_0}} - 1) C_*(\overline{S}),$$

where $\widehat{\widehat{\theta}}(S, X) := (-1)^k \theta(S, S_{i_{k-1}}) \cdots \theta(S_{i_0}, X)$ for each $S_{i_0} > \cdots > S_{i_k} = S$. In particular, $C^{(0)}_\star = C^M(X)$ the Chern-Mather class of X and $C^{(1)}_\star = C_*(X)$ the Chern-Schwartz-MacPherson class of X.

For the last equality use $(\mathrm{Eu}_{\overline{S}}) = (\mathbb{1}_{\overline{S}})\Theta^{-1}$.

Definition (3.5) (A "q-deformed" Euler-Poincaré characteristic)**.**

$$\chi^{(q)}(X) := \int_X C^{(q)}_\star(X) = \chi(X) + \sum_{S_{i_0}>\cdots>S_{i_k}=S} \widehat{\widehat{\theta}}(S, X)(q^{\dim X - \dim S_{i_0}} - 1)\chi(\overline{S}).$$

As an example for the "q-deformed" Euler-Poincaré characteristic, let us consider an n-dimensional singular variety X with isolated singularities $x_1, x_2, \dots, x_r$. In this case we have

$$C^{(q)}_{\bigstar}(X) = C_*(X) + \sum (\mathrm{Eu}_X(x_i) - 1)(q^n - 1)[x_i].$$

In particular for an n-dimensional local complete intersection X with isolated singularities $x_1,\ x_2,\ \dots,\ x_r$ we have

$$C^{(q)}_{\bigstar}(X) = C_*(X) + (-1)^n(q^n - 1)\sum \mu_i[x_i],$$

where μ_i is the Milnor number of X at the singularity x_i. This is due to the following formula due to L. Ernström [5]: for an isolated singularity of an n-dimensional local complete intersection X

$$\mathrm{Eu}_X(x) = 1 + (-1)^{n-1}\mu(X, x),$$

where $\mu(X, x)$ denotes the Milnor number of X at the isolated singularity x. (The hypersurface version of this formula was proved by Kashiwara [7].) Hence we have

$$\chi^{(q)}(X) = \chi(X) + (-1)^n(q^n - 1)\sum \mu_i$$

The following is a generalization of the product formula of the Chern-Schwartz-MacPherson class [8] (cf. [9]).

Theorem (3.6) (A product formula). *Let $\alpha \in \mathcal{F}^{(q)}(X)$ and $\beta \in \mathcal{F}^{(q)}(Y)$ and the exterior product $\alpha \otimes \beta \in \mathcal{F}^{(q)}(X \times Y)$ is defined to be $(\alpha \otimes \beta)(x, y) := \alpha(x)\beta(y)$. Then we have*

$$C^{(q)}_{\bigstar}(\alpha \otimes \beta) = C^{(q)}_{\bigstar}(\alpha) \times C^{(q)}_{\bigstar}(\beta),$$

where $\times$ is the homology cross product. In particular, we have

$$C^{(q)}_{\bigstar}(X \times Y) = C^{(q)}_{\bigstar}(X) \times C^{(q)}_{\bigstar}(Y),$$

hence

$$\chi^{(q)}(X \times Y) = \chi^{(q)}(X)\chi^{(q)}(Y).$$

Proof. Since $\mathrm{Eu}^{(q)}_W$ are the free generators, it suffices to show the following product formula:

$$C^{(q)}_{\bigstar}(\mathrm{Eu}^{(q)}_W \otimes \mathrm{Eu}^{(q)}_T) = C^{(q)}_{\bigstar}(\mathrm{Eu}^{(q)}_W) \times C^{(q)}_{\bigstar}(\mathrm{Eu}^{(q)}_T).$$

By the definition we have $C^{(q)}_{\bigstar}(\mathrm{Eu}^{(q)}_A) = C_*(\mathrm{Eu}_A) = C^M(A)$, thus it suffices to show the following equality:

$$\mathrm{Eu}^{(q)}_W \otimes \mathrm{Eu}^{(q)}_T = \mathrm{Eu}^{(q)}_{W \times T}.$$

Let

$$\mathrm{Eu}_W = \sum n_S \mathbb{1}_S \quad \text{and} \quad \mathrm{Eu}_T = \sum n_Q \mathbb{1}_Q.$$

Then by the basic property of the local Euler obstruction [7, 10] we have

$$\mathrm{Eu}_{W \times T} = \mathrm{Eu}_W \otimes \mathrm{Eu}_T = \sum n_S n_Q \mathbb{1}_{S \times Q}.$$

Hence by the definition we have

$$\begin{aligned}
\mathrm{Eu}^{(q)}_W \otimes \mathrm{Eu}^{(q)}_T &= \left(\sum n_S q^{\dim W - \dim S}\mathbb{1}_S\right) \otimes \\
&\qquad \left(\sum n_Q q^{\dim T - \dim Q}\mathbb{1}_Q\right) \\
&= \sum n_S n_Q q^{\dim W - \dim S} \cdot q^{\dim T - \dim Q}\mathbb{1}_S \otimes \mathbb{1}_Q \\
&= \sum n_S n_Q q^{\dim W + \dim T - \dim S - \dim Q}\mathbb{1}_{S \times Q} \\
&= \mathrm{Eu}^{(q)}_{W \times T}.
\end{aligned}$$

$\square$

Remark (3.7). The pushforward $f^{(q)}_{\bigstar}$ cannot be directly defined by using the "q-deformed" Euler-Poincaré characateristic like in the original definition of the pushforward f_*, i.e., in general we have

$$\left(f^{(q)}_{\bigstar}\mathbb{1}_W\right)(y) \neq \chi^{(q)}(f^{-1}(y) \cap W).$$

For example, let C be a nonsingular plane curve of degree $d(> 1)$, let $X(\subset \mathbf{P}^2)$ be the projective cone over C with v the cone point, and let $f : \widetilde{X} \to X$ be the blow-up of X at the cone point v (cf. [13, Examples (3.11) and (3.12)]). Then we have

$$\left(f^{(q)}_{\bigstar}\mathbb{1}_{\widetilde{X}}\right)(v) = 1 + (2d - d^2 - 1)q^2 + d,$$

$$\chi^{(q)}\left(f^{-1}(v) \cap \widetilde{X}\right) = \chi(C) = 3d - d^2.$$

Hence

$$\left(f^{(q)}_{\bigstar}\mathbb{1}_{\widetilde{X}}\right)(v) \neq \chi^{(q)}\left(f^{-1}(v) \cap \widetilde{X}\right).$$

Details are left for the reader as an exercise.

A crucial reason for why in the Chern-Schwartz-MacPherson class case we have the equality

$$(f_*\mathbb{1}_W)(y) = \chi\left(f^{-1}(y) \cap W\right)$$

is that the Euler-Poincaré characteristic χ satisfies the property ("strong" multiplicativity) that $\chi(E) = \chi(B)\chi(F)$ for a fiber bundle $E \to B$ with fiber F, which is much stronger than the multiplicativity $\chi(X \times Y) = \chi(X)\chi(Y)$. Hence we can see that $\chi^{(q)}$ is not "strong" multiplicative. So it remains to see whether one can define a "strong" multiplicative characteristic $\widetilde{\chi^{(q)}}$ of a variety so that the following holds:

$$\left(f^{(q)}_{\bigstar}\mathbb{1}_W\right)(y) = \widetilde{\chi^{(q)}}\left(f^{-1}(y) \cap W\right).$$

We do not know an answer even in the case when $q = 0$, i.e., in the Chern-Mather class case. This supposedly means "few functorial properties are known for the Chern-Mather class".

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