# On the Diophantine Equation $2^aX^4 + 2^bY^4 = 2^cZ^4$

By Yasutaka SUZUKI

Department of Mathematics, Faculty of Education, Niigata University

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**1. Introduction.** In this paper, an integer means a rational integer. The greatest common divisor of the integers $a$ and $b$ is denoted by $(a, b)$. We shall prove the following main theorems.

**Theorem 1.** *Let $a$, $b$, $c$ be non-negative integers. If $X$, $Y$, $Z$ is a solution of the equation*

$$2^aX^4 + 2^bY^4 = 2^cZ^4$$

*in positive odd integers, then*

$$X = Y = Z \text{ and } a + 1 = b + 1 = c.$$

**Theorem 2.** *Let $m$ be a non-negative integer. Then the equation*

$$X^4 + 2^mY^2 = Z^4$$

*has no solutions in nonzero integers $X$, $Y$, $Z$.*

**2. Preliminaries.** We remind first the following three theorems which are all well-known (see [1], [2] or [3]).

**Theorem 3.** *Let $X$, $Y$, $Z$ be a solution of the equation*

$$X^2 + Y^2 = Z^2$$

*with positive integers $X$, $Y$, $Z$ such that $(X, Y) = 1$ and $X$ odd. Then there exist unique positive integers $u$ and $v$ of opposite parity with $(u, v) = 1$ and $u > v > 0$ such that*

$$X = u^2 - v^2,$$
$$Y = 2uv,$$
$$Z = u^2 + v^2.$$

**Theorem 4.** *The equation*

$$X^4 + Y^4 = Z^2$$

*has no solutions in nonzero integers $X$, $Y$, $Z$.*

**Theorem 5.** *The equation*

$$X^4 + Y^2 = Z^4$$

*has no solutions in nonzero integers $X$, $Y$, $Z$.*

**3. On the equation $X^4 + 2^mY^4 = Z^4$**

In this section, we shall give a simple proof of the following theorem which is slightly stronger than, and implies Fermat's last theorem for $n = 4$ (see [4]).

**Theorem 6.** *Let $m$ be a non-negative integer. Then the equation*

$$X^4 + 2^mY^4 = Z^4$$

*has no solutions in odd integers $X$, $Y$, $Z$.*

*Proof.* Suppose that $u$ is the least integer for which

$$x^4 + 2^my^4 = u^4$$

has a solution in positive odd integers $x$, $y$, $u$ for some non-negative integer $m$. The statement that $u$ is least immediately implies that three integers $x$, $y$, $u$ are pairwise relatively prime. Since the fourth power of an odd integer is congruent to 1 modulo 16, we have

$$2^my^4 = u^4 - x^4 \equiv 1 - 1 = 0 \pmod{16}.$$

Then $m > 3$. Since $u$ and $x$ are both odd and relatively prime, we have

$$u^2 + x^2 \equiv 2 \pmod 4$$

and

$$\begin{aligned}(u^2 + x^2, u + x) &= (u^2 + x^2, u - x)\\ &= (u + x, u - x) = 2.\end{aligned}$$

And since

$$2^my^4 = u^4 - x^4 = (u - x)(u + x)(u^2 + x^2),$$

there exist positive odd integers $a$, $b$, $c$ such that

$$u - x = 2a^4,\ u + x = 2^{m-2}b^4,\ u^2 + x^2 = 2c^4$$

or

$$u - x = 2^{m-2}b^4,\ u + x = 2a^4,\ u^2 + x^2 = 2c^4.$$

Hence

$$\begin{aligned}4c^4 = 2(u^2 + x^2) &= (u - x)^2 + (u + x)^2\\ &= 4a^8 + 2^{2m-4}b^8\end{aligned}$$

and so we obtain

$$(a^2)^4 + 2^{2m-6}(b^2)^4 = c^4$$

in positive odd integers $a$, $b$, $c$.

Moreover, since $0 < x < u$, we have $c^4 < 2c^4 = u^2 + x^2 < 2u^2 < u^4$ and so $0 < c < u$. Thus $u$ was not least after all and the theorem is proved.

**4. Proofs of the main theorems.**

**Lemma 7.** *Let $X$, $Y$, $Z$ be a solution of the equation*

$$X^4 + Y^4 = 2Z^2$$

*in non-negative integers. Then*

$$X^2 = Y^2 = Z.$$

*Proof.* Let $X$, $Y$, $Z$ be a solution of the equation $X^4 + Y^4 = 2Z^2$ in non-negative integers. If one of $X$, $Y$ and $Z$ is zero, then $X = Y = Z = 0$. Thus, we suppose that $X$, $Y$ and $Z$ are positive. Let $d$ be the greatest common divisor of $X$ and $Y$, then $d \mid X$, $d \mid Y$, and also $d^2 \mid Z$. We set $X = dx$, $Y = dy$ and $Z = d^2z$. Hence we have

$$x^4 + y^4 = 2z^2$$

with positive integers $x$, $y$, $z$ which are pairwise relatively prime. Furthermore, we note that $x$, $y$ and $z$ be all odd. Thus, we obtain

$$(2z^2)^2 = (x^4 + y^4)^2 = (x^4 - y^4)^2 + 4x^4y^4.$$

Since $x$ and $y$ are both odd, $\dfrac{x^4 - y^4}{2}$ is an integer. Thus

$$(xy)^4 + \left(\frac{x^4 - y^4}{2}\right)^2 = (z)^4$$

where $xy > 0$, $z > 0$ and $\dfrac{x^4 - y^4}{2}$ is an integer. By Theorem 5, we have $\dfrac{x^4 - y^4}{2} = 0$ and $xy = z$. Therefore $x^2 = y^2 = z$, and so $X^2 = Y^2 = Z$. This completes the proof.

**Corollary 8.** *Let $X$, $Y$, $Z$ be a solution of the equation*

$$X^4 + Y^4 = 2Z^4$$

*in non-negative integers. Then*

$$X = Y = Z.$$

**Proof of Theorem 1.** Let $a$, $b$ and $c$ be non-negative integers. Let $X$, $Y$, $Z$ be a solution of the equation

$$2^aX^4 + 2^bY^4 = 2^cZ^4$$

in positive odd integers $X$, $Y$, $Z$.

We shall first show that $a = b$. If $a \neq b$, then, without loss of generality, we may assume that $a < b$. Set $b = a + m$. Consequently we obtain that $c = a$ and

$$X^4 + 2^mY^4 = Z^4,$$

where $X$, $Y$ and $Z$ are positive odd integers, and $m$ is a positive integer. By Theorem 6, this equation is impossible. Thus $a = b$.

It follows from $a = b$ that $c = a + 1$ and

$$X^4 + Y^4 = 2Z^4$$

with positive odd integers $X$, $Y$, $Z$. Hence according to Corollary 8, we have $X = Y = Z$. This completes the proof of Theorem 1.

**Lemma 9.** *Let $m$ be a non-negative integer. If a set of three odd integers $X$, $Y$, $Z$ satisfies the equation*

$$X^4 + 2^mY^4 = Z^2,$$

*then $m \geqq 3$ and $m \equiv -1 \pmod 4$.*

*Proof.* Since the square of an odd integer is congruent to 1 modulo 8, we have

$$2^mY^4 = Z^2 - X^4 \equiv 1 - 1 = 0 \pmod 8.$$

This implies $m \geqq 3$.

We suppose that there is a set of four integers $X$, $Y$, $Z$, $m$ satisfying $X^4 + 2^mY^4 = Z^2$ with $X$, $Y$, $Z$ odd, $m > 3$ and $m \not\equiv -1 \pmod 4$, and we assume that the set of positive integers $x$, $y$, $z$, $m$ is such that $m$ is the least positive integer. Canceling by the greatest common divisor of $x^4$ and $y^4$, we may assume that $x$, $y$, $z$ are pairwise relatively prime. We have $2^my^4 = z^2 - x^4 = (z + x^2)(z - x^2)$, and since $z$, $x$ are both odd integers and relatively prime, we have $(z + x^2, z - x^2) = 2$. Hence there exist positive odd integers $a$, $b$ with $(a, b) = 1$ such that

$$\text{(I)} \quad z + x^2 = 2a^4,\ z - x^2 = 2^{m-1}b^4$$

or

$$\text{(II)} \quad z + x^2 = 2^{m-1}b^4,\ z - x^2 = 2a^4.$$

In the case of (I) $z + x^2 = 2a^4$, $z - x^2 = 2^{m-1}b^4$, we obtain $x^2 = a^4 - 2^{m-2}b^4$, $2^{m-2}b^4 = a^4 - x^2 = (a^2 + x)(a^2 - x)$, $m - 2 \geqq 3$, and so $m \geqq 5$. Also note that $a$ and $x$ are both odd integers and relatively prime and $(a^2 + x, a^2 - x) = 2$.

Hence there exist positive odd integers $A$, $B$ with $(A, B) = 1$ such that

$$a^2 + x = 2A^4,\ a^2 - x = 2^{m-3}B^4$$

or

$$a^2 + x = 2^{m-3}B^4,\ a^2 - x = 2A^4.$$

Thus, we obtain $a^2 = A^4 + 2^{m-4}B^4$, where $a$, $A$, $B$ are odd integers. Further $m - 4 < m$ and $m - 4 \equiv m \not\equiv -1 \pmod 4$. This contradicts the choice of $m$.

In the case of (II) $z + x^2 = 2^{m-1}b^4$, $z - x^2 = 2a^4$, we obtain $x^2 = 2^{m-2}b^4 - a^4$. Since $2^{m-2}b^4 = x^2 + a^4 \equiv 1 + 1 = 2 \pmod 4$, we have $m - 2 = 1$, so $m = 3$. This contradicts the choice of $m$. Hence the lemma is proved.

**Proof of Theorem 2.** By Theorem 5, we may assume $m \geqq 1$. So if $X$ or $Z$ is even, $Z$ or $X$ should also be even, so that we may assume $X$, $Y$, $Z$ odd. From $X^4 + 2^mY^2 = Z^4$ follows $(2^mY^2)^2 = (Z^4 - X^4)^2 = (Z^4 + X^4)^2 - 4X^4Z^4$. Since $X$, $Z$ are both odd integers, so is $\dfrac{X^4 + Z^4}{2}$, and we obtain

$$(XZ)^4 + 2^{2m-2}Y^4 = \left(\frac{X^4 + Z^4}{2}\right)^2$$

where $XZ$, $Y$, $\dfrac{X^4 + Z^4}{2}$ are odd integers and $2m - 2 \not\equiv -1 \pmod 4$. By Lemma 9, the last equation is impossible. Hence the proof of Theorem 2 is complete.

## References

[1] W. W. Adams and L. J. Goldstein: Introduction to Number Theory. Prentice-Hall, New Jersey (1976).

[2] E. Grosswald: Topics from the Theory of Numbers. 2nd ed., Birkhäuser, Boston (1984).

[3] G. H. Hardy and E. M. Wright: An Introduction to the Theory of Numbers. Fifth ed., Oxford, London (1979).

[4] Y. Suzuki: Simple Proof of Fermat's Last Theorem for $n = 4$. Proc. Japan Acad., **62A**, 209–210 (1986).