28. *On Total Risk Aversion and Differential Games for Controlled Parabolic Equations*

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1. Recently E. N. Barron and R. Jensen [1] investigated a connection between the theory of risk for controlled finite dimensional state systems and the theory of differential games. In the present note we will discuss the same problems for infinite dimensional state systems on a finite time interval $[0, T]$ governed by parabolic equations.

Let W be a standard d-dimensional Wiener process on a probability space $(\Omega, \mathcal{F}, P)$ and denote by $\mathcal{F}_t$ the σ-field generated by $\{W(s) ; s \leq t\}$. Let Γ be a compact subset of $\boldsymbol{R}^q$ and $\mathcal{A}$ be the space of all Borel measurable function $u : \boldsymbol{R}^n \to \Gamma$ endowed with the $L^2(\boldsymbol{R}^n \to \boldsymbol{R}^q ; e^{-|x|}dx)$-topology, called a control region. A map $U : [0, T] \times \Omega \to \mathcal{A}$ is called an admissible control, if it is $\mathcal{F}_t$-progressively measurable.

Putting $H = L^2(\boldsymbol{R}^n)$ and

$$A\zeta = \sum_{ij=1}^n \frac{\partial}{\partial x_i}(a^{ij}(x)\frac{\partial}{\partial x_j}\zeta) + \sum_{i=1}^n r^i(x)\frac{\partial}{\partial x_i}\zeta - c(x)\zeta,$$

we consider the controlled system ξ governed by the parabolic equation in a random environment:

(1) $\frac{\partial \xi}{\partial t}(t, x) = A\xi(t, x) + b(x, \xi(t, x), y + W(t), U(t, x))$, for $t \in (0, T)$, $x \in \boldsymbol{R}^n$ with initial condition $\xi(0, \ ) = \eta (\in H)$.

Let us assume the following conditions (A1) ~(A5).

(A1) a^{ij} and r^i are in $C^3(\boldsymbol{R}^n)$, with finite C^3-norm,

(A2) the matrix $(a^{ij}(x))$ is uniformly positive difinite, say $(a^{ij}(x)) \geq \hat{\mu} I$ where $I = n \times n$ identity matrix and $\hat{\mu} > 0$,

(A3) c is bounded, continuous and non-negative,

(A4) $b ; \boldsymbol{R}^n \times \boldsymbol{R}^1 \times \boldsymbol{R}^d \times \Gamma \to \boldsymbol{R}^1$ is bounded and Lipschitz continuous,

(A5) there is $\hat{b} \in H$, such that $\hat{b}(x)$ is decreasing, as $|x| \to \infty$, and $|b(x, a, y, u)| \leq \hat{b}(x)$ for any a, y and u.

By (A2), $-A$ is coercive, namely there is a non-negative constant μ, such that

(A2)′ $\langle -A\zeta, \zeta\rangle + \mu\|\zeta\|^2 \geq 0$ for $\zeta \in H^1$,

where $\langle\ ,\ \rangle$ = duality pairing between H^{-1} and H^1.

Define $\beta ; H \times \boldsymbol{R}^d \times \mathcal{A} \to H$ by $\beta(\zeta, y, u)(x) = b(x, \zeta(x), y, u(x))$. Let us put $m(a)^2 = \int \min(|a(x)|^2, \hat{b}(x)^2)dx$. Then (A4) implies

(2) $\|\beta(\zeta, y, u) - \beta(\zeta', y', u')\| \leq k(\|\zeta - \zeta'\| + m(y - y') + m(u - u'))$

with a constant k. In view of the fact that A generates a continuous semigroup e^{tA} on H, ξ is called *a mild solution* of (1), if

$$\xi(t) = e^{tA}\eta + \int_0^t e^{(t-s)A}\beta(\xi(s), y + W(s), U(s))ds \quad \text{for } t \in [0, T].$$

with probability 1. Using (2), we can easily prove the following:

Theorem 1. *There is a unique mild solution* $\xi = \xi(\ \ , \eta, y, U)$ *of* (1). *Moreover*

(3) $\|\xi(t, \eta, y, U) - \xi(t, \eta', y', U)\| \leq \hat{k}(\|\eta - \eta'\| + m(y - y'))$

holds with a constant $\hat{k}$ *independent of* U, t *and* ω.

2. Let h ; $H \to \boldsymbol{R}^1$ be bounded and Lipschitz continuous. Moreover we assume

(A6) there is F ; $[0, \infty) \to \boldsymbol{R}^1$, which is decreasing to 0 and $|h(\eta) - h(\eta')| \leq F(R)\ (\|\eta\| + \|\eta'\|)$ whenever $\eta = \eta'$ on $\{x \in \boldsymbol{R}^n, |x| \leq R\}$. For example, $h(\zeta) = \psi(\langle\zeta, e_1\rangle, \cdots \langle\zeta, e_p\rangle)$ with $e_i \in H$ and smooth function ψ on $\boldsymbol{R}^p$, where $\langle\ ,\ \rangle$ = inner product.

In this note, we deal with a running cost given by h and (risk averse) utility given by exponential function $Q_c(a) = -\exp(-a/c)$, with $c > 0$. So our problem is to maximize the expected value, $J_c(t, \eta, y, U) = EQ_c\left(\int_0^t h(\xi(s, \eta, y, U))ds\right)$, over the class of admissible controls. According to the theory of risk, $1/c$ is index of risk aversion, and the *value function* v_c and the *certainty equivalent value function* V_c are defined by

$v_c(t, \eta, y) = \sup_U J_c(t, \eta, y, U)$ and $V_c(t, \eta, y) = Q_c^{-1}(v_c(t, \eta, y))$

respectively. Therefore we see that

(4) $$|V_c(t, \eta, y)| \leq t|h|_\infty.$$

If V_c is smooth and its Fréchet derivative $\partial V_c(t,\ \ ,y)$ belongs to H^2, then V_c satisfies the so-called Bellman equation:

(5) $$\frac{\partial V}{\partial t} - \langle A^*\partial V, \eta\rangle - \frac{1}{2}\Delta_y V + \frac{1}{2c}|\nabla_y V|^2 - \sup_{u\in\mathscr{A}}\langle\partial V, \beta(\eta, y, u)\rangle$$

$- h(\eta) = 0$, where A^* = dual operator of A, ∇_y = gradient *w. r.* to y and Δ_y = Laplacian *w. r.* to y.

Although V_c is generally non-smooth, V_c still satisfies (5) in the viscosity sense; see the next section, Theorem 3.

3. Recalling the definition of viscosity solutions due to Crandall and Lions [3] and modifying it slightly, we will define the viscosity solutions of the equation (6) below, that is more general than (5). $\Phi \in C^{112}((0, T) \times H \times \boldsymbol{R}^d)$ is called a *test function,* if

(i) Φ and $\left(\frac{\partial\Phi}{\partial y_i}\right)^2$, $i = 1 \cdots d$, are weakly lower semi-continuous in $(0, T) \times H \times \boldsymbol{R}^d$ and bounded from below.

(ii) $\partial\Phi(t,\ \ ,y) \in H^2$ and $A^*\partial\Phi$ is continuous and locally bounded,

(iii) $\frac{\partial\Phi}{\partial t}$, $\frac{\partial\Phi}{\partial y_i}$, $\frac{\partial^2\Phi}{\partial y_i \partial y_j}$, $\partial\Phi$ and Φ are locally bounded.

g ; $H \to [0, \infty)$ is called *radial,* if $g(\zeta) = \hat{g}(\|\zeta\|)$ with $\hat{g} \in C^2(0, \infty)$ increasing from 0 to ∞.

We consider the following non-linear equation:

(6) $$\frac{\partial v}{\partial t} - \langle A^*\partial v, \eta\rangle + F\left(t, \eta, y, \partial v, \left(\frac{\partial v}{\partial y_i}\right), \left(\frac{\partial^2 v}{\partial y_i \partial v_j}\right)\right) = 0,$$

where $F \in C([0, T] \times H \times \boldsymbol{R}^d \times H \times \boldsymbol{R}^d \times S^d)$, S^d = set of $d \times d$ symmetric matrices and F satisfies the ellipticity condition: $F(t, \eta, y, \xi, z, B) \leq F(t, \eta, y, \xi, z, D)$ whenever $B \geq D$ and $(t, \eta, y, \zeta, z) \in [0, T] \times H \times \boldsymbol{R}^d \times H \times \boldsymbol{R}^d$.

Definition. Let $v \in C([0, T] \times H \times \boldsymbol{R}^d)$ be bounded and weakly continuous. v is called a *subsolution* (resp. *supersolution*) of (6), if the following condition (i) (resp. (ii)) holds for any test function Φ and any radial function g:

(i) If $v - \Phi - g$ has a global maximum at $\hat{z} = (\hat{t}, \hat{\eta}, \hat{y})$ with $\hat{t} \in (0, T)$, then

$$\frac{\partial \Phi}{\partial t}(\hat{z}) - \langle A^*\partial\Phi(\hat{z}), \hat{\eta}\rangle + F\left(\hat{z}, \partial\Phi(\hat{z}) + \partial g(\hat{z}), \left(\frac{\partial \Phi}{\partial y_i}(\hat{z})\right), \left(\frac{\partial^2 \Phi}{\partial y_i \partial y_j}(\hat{z})\right)\right)$$
$$\leq \mu g'(\|\hat{\eta}\|)\,\|\hat{\eta}\|,$$

(ii) If $v + \Phi + g$ has a global minimum at $\hat{z} = (\hat{t}, \hat{\eta}, \hat{y})$ with $\hat{t} \in (0, T)$, then

$$-\frac{\partial \Phi}{\partial t}(\hat{z}) + \langle A^*\Phi(\hat{z}), \hat{\eta}\rangle + F\left(\hat{z}, -\partial\Phi(\hat{z}) - \partial g(\hat{z}), \left(-\frac{\partial \Phi}{\partial y_i}(\hat{z})\right), \left(-\frac{\partial^2 \Phi}{\partial y_i \partial y_j}(\hat{z})\right)\right)$$
$$\geq -\mu g'(\|\hat{\eta}\|)\,\|\hat{\eta}\|,$$

where μ is the constant of (A2)$'$.

v is called a *viscosity solution*, if it is both a subsolution and a supersolution.

Remark. In the defenition above, we can replace "global" by "strictly global".

Let ρ ; $[0, \infty) \to [0, 1]$ be smooth and decreasing and $\rho = 1$ on $[0, 1]$ and $\rho = 0$ on $[2, \infty)$. For any fixed $R > 0$, we put $\Gamma_R(x) = \rho(|x|/R)$. According to [6], we will evaluate the modulus of continuity of V_c, which is used later. Hereafter k_i denotes a positive constant independent of t, η, y, ω, c and R.

Theorem 2. *There is a positive constant K independent of c and R such that*

(7) $$|V_c(t, \eta, y) - V_c(t', \eta', y')| \leq |t - t'|\,|h|_\infty + K\{F(R)(\|\eta\| + \|\eta'\| + \|\hat{b}\|) + \|\Gamma_R(\eta - \eta')\|_{-1} + \frac{1}{\sqrt{R}}\|\eta - \eta'\| + m(y - y')\}.$$

Outline of proof. Set $B = (I - \Delta)^{-1}$. Then we can easily show the structural condition:

$\langle -A^*B\zeta, \zeta\rangle \geq \lambda\|\zeta\|^2 + \theta\langle B\zeta, \zeta\rangle$ for any $\zeta \in H$ with $\lambda > 0$ and $\theta \in \boldsymbol{R}^1$.

Putting $\zeta(t) = \xi(t, \eta, y, U) - \xi(t, \eta', y, U)$ and $\Gamma = \Gamma_R$ for simplicity, we evaluate the dynamics of $\|\Gamma\zeta(t)\|_{-1}^2$ $(= \langle B\Gamma\zeta(t), \Gamma\zeta(t)\rangle)$ and use the structural condition to obtain

$$\frac{d}{dt}\|\Gamma\zeta(t)\|_{-1}^2 \leq -k_1\|\Gamma\zeta(t)\|^2 + k_2(\|\Gamma\zeta(t)\|_{-1}^2 + \frac{1}{R}\|\eta - \eta'\|^2).$$

Now, putting $\xi = \xi(\ \ , \eta, y, U)$, $\xi' = \xi(\ \ , \eta', y', U)$ and $\zeta = \xi - \xi'$, and using (3), we have

$$(8)\quad \int_0^t \| \Gamma\zeta(s) \|^2 ds \le k_3(\| \Gamma(\eta - \eta') \|_{-1}^2 + \frac{1}{R} \| \eta - \eta' \|^2 + m(y - y')^2).$$

Next we evaluate J_c, using [5].

$$\int_0^t h(\xi(s))ds = \int_0^t h(\Gamma\xi(s))ds + \int_0^t h(\xi(s)) - h(\Gamma\xi(s))ds$$

$$\le \int_0^t h(\Gamma\xi'(s))ds + \{k_4(\| \Gamma(\eta - \eta') \|_{-1} + \frac{1}{\sqrt{R}} \| \eta - \eta' \| + m(y - y')$$
$$+ F(R)(\| \eta \| + \| \eta' \| + \| \hat{b} \|)) + | t - t' | | h |_\infty\}$$

by virture of (8). Put $G =$ inside of $\{\ \ \}$. Then we have

$$J_c(t, \eta, y, U) \le J_c(t', \eta', y', U) \exp(-G/c).$$

In the same way, we have

$$J_c(t, \eta, y, U) \ge J_c(t', \eta', y', U) \exp(G/c).$$

So, v_c satisfies the same inequalities. Applying Q_c^{-1} to both sides of the inequality above, we complete the proof.

Since (7) implies the weak continuity of V_c, we use Itô's formula to prove that (4) implies the following theorem.

Theorem 3. *V_c is a viscosity solution of the Bellman equation* (5).

Outline of proof. For simplicity, we delete the suffix c. Let $\hat{z} = (\hat{t}, \hat{\eta}, \hat{y}) (\in (0, T) \times H \times \boldsymbol{R}^d)$ be a strictly global maximum point of $V - \Phi - g$. Since Q is increasing, we get

$$v(z) < Q(\Phi(z) - \Phi(\hat{z}) + g(\eta) - g(\hat{\eta}) + V(\hat{z})) \quad \text{for } z \ne \hat{z}.$$

Putting $\boldsymbol{I}(z) = Q(\Phi(z) - \Phi(\hat{z}) + g(\eta) - g(\hat{\eta}) + V(z)) - Q(g(\eta) - g(\hat{\eta}) + V(\hat{z}))$ and $\boldsymbol{J}(\eta) = Q(g(\eta) - g(\hat{\eta}) + V(\hat{z})) - Q(V(\hat{z}))$, we have $v(z) - v(\hat{z}) < \boldsymbol{I}(z) + \boldsymbol{J}(\eta)$. Since v satisfies the dynammic programming principle, we obtain

$$0 \le \sup_U E\Big[\boldsymbol{I}(\hat{t} - \theta, \xi(\theta), \hat{y} + W(\theta)) + \boldsymbol{J}(\xi(\theta))$$
$$+ v(\hat{t} - \theta, \xi(\theta), \hat{y} + W(\theta))\Big(\exp\Big(-\frac{1}{c}\int_0^\theta h(\xi(s))ds\Big) - 1\Big)\Big].$$

Using Itô's formula, we evaluate each term to conclude that v is a subsolution. In the same way we can prove that v is a supersolution. This completes the proof.

4. Let us set

$\boldsymbol{Y} = \{Y ; [0, T] \to \boldsymbol{R}^d$, image set $Y[0, T]$ is bounded$\}$
$\boldsymbol{Z} = \{Z ; [0, T] \to \mathscr{A}$, measurable$\}$
$M = \{\alpha ; \boldsymbol{Z} \to \boldsymbol{Y}$, non-anticipative$\}$
$\boldsymbol{N} = \{\beta ; \boldsymbol{Y} \to \boldsymbol{Z}$, non-anticipative$\}$

where α is said to be non-anticipative if $\alpha(Y)(\cdot) = \alpha(Y')(\cdot)$ on $[0, t]$, whenever $Y = Y'$ on $[0, t]$. Y(resp. Z) is called a control for nature (resp. player) and α(resp. β) a strategy for nature (resp. player). For a partition $\pi = \{0 = t_0 < \cdots < t_p = T\}$, Y is called π-admissible, if $Y(t) = Y(t_j)$ for $t \in [t_j, t_{j+1})$, and $\boldsymbol{Y}_\pi =$ the set of π-admissible controls. α is called π-admissible, if $\alpha ; \boldsymbol{Z} \to \boldsymbol{Y}_\pi$ and $\alpha(Z)(s)$ is independent of Z, for $s \in [0, t_1)$. $\boldsymbol{M}_\pi$ denotes the set of π-admissible strategies and $\boldsymbol{M}^* = \cup_\pi \boldsymbol{M}_\pi$. $\boldsymbol{Z}_\pi$,

$\boldsymbol{N}_\pi$ and $\boldsymbol{N}^*$ are defined in the same way.

For $Y \in \boldsymbol{Y}$ and $Z \in \boldsymbol{Z}$, we consider the system X in H, governed by (9).

$$\frac{dX}{dt}(t) = AX(t) + \beta(X(t), Y(t), Z(t)) \tag{9}$$

with initial condition $X(0) = \eta$.
(9) has a unique mild solution $X(\ \ , \eta, Y, Z)$ and a criterion $\mathscr{J}$ is given by $\mathscr{J}(t, \eta, Y, Z) = \int_0^t h(X(s, \eta, Y, Z))ds$. The *upper value* of this game ν; $[0, T] \times H \to \boldsymbol{R}^1$, is defined by

$$\nu(t, \eta) = \inf_{\alpha \in \boldsymbol{M}^*} \sup_{Z \in \boldsymbol{Z}} \mathscr{J}(t, \eta, \alpha(Z), Z).$$

Theorem 4. $\nu(t, \eta) \leq V_c(t, \eta, y)$ *for any* $y \in \boldsymbol{R}^d$ *and* $c > 0$.

Proof. Suppose that π_n is finer than π_{n-1} and the meth of $\pi_n \to 0$, as $n \to \infty$. Then, putting $\boldsymbol{M}_n = \boldsymbol{M}_{\pi_n}$, we see that $\inf_{\alpha \in \boldsymbol{M}_n} \sup_{z \in \boldsymbol{Z}} \mathscr{J}(t, \eta, \alpha(Z), Z)$ is decreasing to the upper value ν, as $n \to \infty$, and

$$\inf_{a \in \boldsymbol{M}_\pi} \sup_{z \in \boldsymbol{Z}} \mathscr{J}(t, \eta, \alpha(Z), Z) = \sup_{\beta \in N} \inf_{Y \in \boldsymbol{Y}_\pi} \mathscr{J}(t, \eta, Y, \beta(Y)) \tag{10}$$

holds [4].

On the other hand, if $\boldsymbol{Y} \supset C([0, T] \to \boldsymbol{R}^d)$, then

$$\sup_{\beta \in N} \inf_{Y \in \boldsymbol{Y}_\pi} Q_c(\mathscr{J}(t, \eta, Y, \beta(Y)) \leq v_c(t, \eta, y) + \varepsilon(\pi, y) \tag{11}$$

where $\varepsilon(\pi, y) \to 0$, as the meth of $\pi \to 0$. Now apply Q_c^{-1} to (11) to complete the proof.

5. Suppose that c_n converges to 0, as $n \to \infty$. Put $V_n = V_{c_n}$.

Lemma 1. *There exists a subsequence* n_j *such that* V_{n_j} *converges uniformly in any bounded set of* $[0, T] \times H \times \boldsymbol{R}^d$. *Moreover, its limit function* V *also satisfies* (4) *and* (7).

Lemma 2. *The limit function* V *is independent of* y.

Outline of proof. We can employ the same arguments as in Appendix of [1], with a slight modification. Let $\hat{z} = (\hat{t}, \hat{\eta}, \hat{y}) (\in (0, T) \times H \times \boldsymbol{R}^d)$ be a strictly global maximum point of $V - \Phi - g$. Then (4) and Lemma 1 imply that any global maximum point $\hat{z}_j$ of $V_j - \Phi - g$ converges to $\hat{z}$ weakly, as $j \to \infty$.

On the other hand, the condition (iii) on test functions implies that $|\nabla_y \Phi(z_j)|^2 \leq c_j k$, $j = 1, 2 \cdots$, with a constant $k = k(\Phi, g)$. So, we see that $|\nabla_y \Phi(\hat{z})|^2 \leq 0$ holds by the condition (i). Namely, V is a subsolution of the equation

$$|\nabla_y V|^2 = 0. \tag{12}$$

Since V is a supersolution, V is a viscosity solution of (12).

Fix $(\hat{t}, \hat{\eta}) \in (0, T) \times H$ arbitrarily and put $v^* = V(\hat{t}, \hat{\eta},\ \ )$. Let $\psi \in C^2(R^d)$ be bounded below and assume that $v^* - \psi$ has a strictly global maximum at $\hat{y}$. Let us define Φ_ε by

$$\Phi_\varepsilon(t, \eta, y) = \frac{1}{\varepsilon}(|t - \hat{t}|^2 + \langle B(\eta - \hat{\eta}), \eta - \hat{\eta} \rangle) + \psi(y).$$

Then any global maximum point of $V - \Phi_\varepsilon - g$ converges to $(\hat{t}, \hat{\eta}, \hat{y})$, as $\varepsilon \to 0$. So $|\nabla_y \psi(\hat{y})|^2 \leq 0$, because V is a subsolution of (12). Therefore v^* is a viscosity solution of the equation $|\nabla_y v^*|^2 = 0$. Now use Appendix of [1] to complete the proof.

Lemma 3. *V is a unique viscosity solution of the min-max equation*:

$$\frac{\partial V}{\partial t} - \langle A^* \partial V, \eta\rangle - \inf_{y \in \boldsymbol{R}^d} \sup_{u \in \mathscr{A}} \langle \partial V, \beta(\eta, y, u)\rangle - h(\eta) = 0 \tag{13}$$

with initial condition $V(0, \eta) = 0$.

Proof. Since $|\nabla_y \phi|^2 \geq 0$, V_j is a subsolution of the equation:

$$\frac{\partial W}{\partial t} - \langle A^* \partial W, \eta\rangle - \frac{1}{2}\Delta_y W - \sup_{u \in \mathscr{A}} \langle \partial W, \beta(\eta, y, u)\rangle - h(\eta) = 0. \tag{14}$$

Hence, by Lemma 1 we see that V is also a subsolution of (14). Since V is independent of y, we use a test function $\Phi(t, \eta) + |y - y^*|^2$ for any fixed y^*. Then we can easily see that V is a subsolution of (13).

On the other hand, the comparison theorem holds for (13) and its unique viscosity solution coincides with the upper value ν of our differential game, [3]. This fact implies "$V \leq \nu$". Now Theorem 4 completes the proof.

Lemma 4. *Put* $B_n = \{\zeta \in H ; \|\zeta\| \leq n\}$, $S_n = \{y \in \boldsymbol{R}^d ; |y| \leq n\}$ $\|w\|_n = \sup\{|w(t, \eta, y)| ; (t, \eta, y) \in [0, T] \times B_n \times S_n\}$ *and define* m ; $C([0, T] \times H \times \boldsymbol{R}^d) \to [0, 1)$ *by* $m(w) = \sum_{n=1}^{\infty} 2^{-n} \|w\|_n / (1 + \|w\|_n)$. *Then m provides the topology of uniform convergence on bounded sets.*

Now we obtain the following main theorem.

Theorem 5. *If $c \to 0$, V_c converges, uniformly in any bounded set of* $[0, T] \times H \times \boldsymbol{R}^d$, *to the upper value ν of differential game, which is a unique viscosity solution of the min-max equation* (13).

Proof. From Lemmas 1, 2 and 3 we see that this theorem holds for every sequence $c_1, c_2, \ldots$. Now we can use Lemma 4 to prove that it holds for $c \to 0$.

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