23. *On the Exponentially Asymptotic Stability of a Perturbed Nonlinear System*

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1. Introduction. Consider the following system of ordinary differential equations, (N), and its perturbed system, (P):

$$(N) \qquad \dot{x}=f(t, x),$$

$$(P) \qquad \dot{y}=f(t, y)+g(t, y),$$

where $f(t, x)$ is continuous, a Lipschitzian with respect to x and $f(t, 0)=0$. Moreover, $g(t, y)$ is continuous and $g(t, 0)=0$. On (N), we assume that the zero solution, $x=0$, has some properties on the stability.

Many authors have studied above systems under the conditions on $g(t, y)$ so that (P) preserves the stability of (N) (cf. Hahn [1], Yoshizawa [2], Strauss and Yorke [3], [4], etc.). In this paper, we give an attention to the exponentially asymptotic stability. A well-known result on this stability is as follows:

Theorem 1.1. *Suppose that the zero solution of (N) is exponentially asymptotically stable. Moreover, suppose that $\|g(t, y)\| \leqq u(t)\|y\|$ in some sets and $\int_0^\infty u(t)dt < +\infty$. Then the zero solution of (P) is exponentially asymptotically stable.*

Our purpose in this paper is to extend conditions on $u(t)$ to more general ones.

2. Definitions and lemmas. Let R^n be the n-dimensionl real Euclidean space and $\|\cdot\|$ denotes the norm on R^n. Let $B_h=\{x \in R^n : \|x\| \leq h\}$ for any $h>0$, and let $R^+=\{t \in R : t \geq 0\}$. $C[X ; Y]$ denotes the set of all continuous functions from X to Y, where X and Y are topological spaces. We also write $C[X]$ instead of $C[X ; Y]$. Let $\mathrm{Lip}(x, L, D)=\{f \in C[R^+ \times D] : \|f(t, x)-f(t, x')\| \leq L\|x-x'\|$ in $R^+ \times D\}$, where D is a domain in R^n, and $x(\cdot\,; t_0, x_0)$, $y(\cdot\,; t_0, y_0)$ denote any solutions of (N), (P) passing through (t_0, x_0), (t_0, y_0), respectively.

Definition 2.1. The zero solution of (N) is exponentially asymptotically stable ([Exp. A.S]) if there exist $h>0$, $K>0$ and $c>0$ such that $\|x(t ; t_0, x_0)\| \leq K\|x_0\| \exp(-c(t-t_0))$ for all $(t_0, x_0) \in R^+ \times B_h$ and $t \geq t_0$.

If the zero solution of (N) is [Exp. A.S.], then we obtain the following lemmas.

Lemma 2.2. *Suppose that $f \in C[R^+ \times B_h ; R^n] \cap \mathrm{Lip}(x, L, B_h)$ and the zero solution of (N) is* [Exp. A.S]. *Then there exist a Liapunov function*

$V(t, x)$ for (N), $h'>0$ with $h'<h$, $K>0$, $c>0$ and $M>0$ which satisfies the following conditions:

(i) $V \in C[R^+ \times B_{h'}; R^+]$,

(ii) $\|x\| \leq V(t, x) \leq K\|x\|$ *in* $R^+ \times B_{h'}$,

(iii) $\dot{V}_{(N)}(t, x) \leq -cV(t, x)$ *in* $R^+ \times B_{h'}$, *where*

$$\dot{V}_{(N)}(t, x) = \limsup_{\delta \to +0} \frac{V(t+\delta, x+\delta f(t, x)) - V(t, x)}{\delta},$$

(iv) $|V(t, x) - V(t, x')| \leq M\|x - x'\|$ *in* $R^+ \times B_{h'}$.

Lemma 2.3. *Suppose that $f \in C[R^+ \times R^n; R^n] \cap \mathrm{Lip}(x, L, R^n)$ and there exist $K>0$ and $c>0$ such that $\|x(t; t_0, x_0)\| \leq K\|x_0\| \exp(-c(t-t_0))$ for all $(t_0, x_0) \in R^+ \times R^n$ and $t \geq t_0$. Then there exist a Liapunov function $V(t, x)$ for (N), $K'>0$, $c'>0$ with $c'<c$ and $M>0$ which satisfies the following conditions:*

(i) $V \in C[R^+ \times R^n; R^+]$,

(ii) $\|x\| \leq V(t, x) \leq K'\|x\|$ *in* $R^+ \times R^n$,

(iii) $\dot{V}_{(N)}(t, x) \leq -c'V(t, x)$ *in* $R^+ \times R^n$,

(iv) $|V(t, x) - V(t, x')| \leq M\|x - x'\|$ *in* $R^+ \times R^n$.

Proofs are omitted. Refer to Theorem 19.2 and its corollary in Yoshizawa [2].

Definition 2.4. Let $u(\cdot) \in C[R^+; R^+]$. We call $u(\cdot)$ diminishing if $u(\cdot)$ satisfies that $U(t) \equiv \int_t^{t+1} u(s)ds \to 0$ as $t \to +\infty$.

Lemma 2.5. *Suppose that $u(\cdot) \in C[R^+; R^+]$ is diminishing and let $U(t) \equiv \int_t^{t+1} u(s)ds$. Then $\int_t^T u(s)ds \leq \int_{t-1}^T U(s)ds$ for all $T \geq t \geq 1$.*

Proof is also omitted. Refer to Lemma 3.4 in Strauss and Yorke [3].

3. Theorems. As extentions of Theorem 1.1, we get the following results.

Theorem 3.1. *Suppose that $f \in C[R^+ \times B_h; R^n] \cap \mathrm{Lip}(x, L, B_h)$ and the zero solution of (N) is* [Exp. A.S]. *Moreover, suppose that $g \in C[R^+ \times B_h; R^n]$ and $\|g(t, y)\| \leq u(t)\|y\|$ in $R^+ \times B_h$, where $u(\cdot) \in C[R^+; R^+]$ is diminishing. Then the zero solution of (P) is* [Exp. A.S].

Theorem 3.2. *Suppose that $f \in C[R^+ \times R^n; R^n] \cap \mathrm{Lip}(x, L, R^n)$ and there exist $K>0$ and $c>0$ such that $\|x(t; t_0, x_0)\| \leq K\|x_0\| \exp(-c(t-t_0))$ for all $(t_0, x_0) \in R^+ \times R^n$ and $t \geq t_0$. Moreover, suppose that $g \in C[R^+ \times R^n; R^n]$ and $\|g(t, y)\| \leq u(t)\|y\|$ in $R^+ \times R^n$, where $u(\cdot) \in C[R^+; R^+]$ is diminishing. Then there exist $K'>0$ and $c'>0$ with $c'<c$ such that $\|y(t; t_0, y_0)\| \leq K'\|y_0\| \exp(-c'(t-t_0))$ for all $(t_0, y_0) \in R^+ \times R^n$ and $t \geq t_0$.*

4. Proofs. *Proof of Theorem* 3.1. By the assumptions, there exists a Liapunov function $V(t, x)$ which satisfies the conditions in Lemma 2.2. Then, the total derivative of $V(t, x)$ along the system (P) satisfies

$$\begin{aligned} (1) \qquad \dot{V}_{(P)}(t, y) &\leq -cV(t, y) + M\|g(t, y)\| \leq -cV(t, y) + Mu(t)\|y\| \\ &\leq (-c + Mu(t))V(t, y) \end{aligned}$$

in $R^+ \times B_{h'}$. Let $y(t) \equiv y(t; t_0, x_0)$, and suppose that $\|x(t)\| \leq h'$ on $[t_0; t_1]$.

Then, by the comparison theorem, we have

$$\text{(2)}\quad V(t, y(t)) \leq V(t_0, y_0) \exp\left\{\int_{t_0}^{t} (-c+Mu(s))ds\right\}$$
$$\leq K\|y_0\| \exp(-c(t-t_0)) \cdot \exp\left(\int_{t_0}^{t} Mu(s)ds\right) \qquad \text{on } [t_0, t_1].$$

Let $U(t) \equiv \int_t^{t+1} u(s)ds$, then $U(t) \to 0$ as $t \to +\infty$. Thus, there exist some constants $N>0$ and $T>1$ such that $\sup_{t\in R^+}|U(t)| \leq N < +\infty$ and $MU(t) \leq (1/2)c$ for all $t \geq T$.

Let $F(t; t_0) \equiv \int_{t_0}^{t} Mu(s)ds$, and make an estimate on $F(t; t_0)$.

First, assume that $0 \leq t_0 \leq 1$. If $t \geq T$, by Lemma 2.5, we have

$$F(t; t_0) = \int_{t_0}^{1} Mu(s)ds + \int_{1}^{t} Mu(s)ds \leq M\int_0^1 u(s)ds + \int_0^t MU(s)ds$$
$$\leq MN + \int_0^T MU(s)ds + \int_T^t MU(s)ds \leq MN(1+T) + \frac{1}{2}c(t-T)$$
$$\leq MN(1+T) + \frac{1}{2}c(t-t_0).$$

For $F(t; t_0)$ is monotone increasing in t and $F(T; t_0) \leq MN(1+T)$, we have

$$\text{(3)}\qquad F(t; t_0) \leq MN(1+T) + \frac{1}{2}c(t-t_0) \qquad \text{for all } t \geq t_0.$$

Next, assume that $1 \leq t_0 \leq T$. If $t \geq T$, by Lemma 2.5, we have

$$F(t; t_0) = \int_{t_0}^{t} Mu(s)ds \leq \int_{t_0-1}^{t} MU(s)ds$$
$$\leq M\int_{t_0-1}^{T} U(s)ds + \int_T^t MU(s)ds \leq MN(T-t_0+1) + \frac{1}{2}c(t-T)$$
$$\leq MN(1+T) + \frac{1}{2}c(t-t_0).$$

For the same reason as the first case, we have the same estimate as (3).

Finally, assume that $T \leq t_0$. Then we have

$$F(t; t_0) = \int_{t_0}^{t} Mu(s)ds \leq \int_{t_0-1}^{t} MU(s)ds$$
$$= M\int_{t_0-1}^{t_0} U(s)ds + \int_{t_0}^{t} MU(s)ds \leq MN + \frac{1}{2}c(t-t_0)$$
$$\leq MN(1+T) + \frac{1}{2}c(t-t_0) \qquad \text{for all } t \geq t_0.$$

By the above estimates, we have

$$\text{(4)}\qquad F(t; t_0) \leq MN(1+T) + \frac{1}{2}c(t-t_0) \qquad \text{for all } t \geq t_0 \geq 0.$$

Therefore, by (2) and the condition (ii) of Lemma 2.2, we have

$$\|y(t)\| \leq V(t, y(t)) \leq K\|y_0\| \exp(-c(t-t_0)) \cdot \exp\left\{MN(1+T) + \frac{1}{2}c(t-t_0)\right\}$$
$$= K'\|y_0\| \exp\left(-\frac{1}{2}c(t-t_0)\right) \qquad \text{on } [t_0, t_1],$$

where $K' \equiv K \exp(MN(1+T))$.

Let $h''=(h'/K')$, then we have $\|y(t;t_0,y_0)\|\le K'\|y_0\|\exp(-(1/2)c(t-t_0))$ for all $(t_0,y_0)\in R^+\times B_{h''}$ and $t\ge t_0$. This implies [Exp. A.S] of the zero solution of (P) and completes the proof. Q.E.D.

Proof of Theorem 3.2. By the assumptions, there exists a Liapunov function $V(t,x)$ which satisfies the conditions of Lemma 2.3. Then, by the same way as the proof of Theorem 3.1, we have

$$\|y(t;t_0,y_0)\|\le K'\|y_0\|\exp\left(-\frac{1}{2}c'(t-t_0)\right)$$

$$\text{for all } (t_0,y_0)\in R^+\times R^n \quad\text{and}\quad t\ge t_0$$

and this completes the proof. Q.E.D.

Remark. Consider the following 1-dimensional linear ordinary differential equation:

$$(L)\qquad \dot{x}=(-a+b(t))x,$$

where $a>0$ is a constant and $b(\cdot)\in C[R^+;R^+]$. The necessary and sufficient condition for [Exp. A.S] of the zero solution of (L) is that a and $b(\cdot)$ satisfy that

$$\limsup_{(t,v)\to(\infty,\infty)}\frac{1}{v}\int_t^{t+v}b(s)ds<a.$$

(Onuchic [5]). By applying the comparison theorem to (1) and (L), we see that the condition on $u(\cdot)$ in Theorem 3.1 and 3.2 can be replaced by

$$\limsup_{(t,v)\to(\infty,\infty)}\frac{1}{v}\int_t^{t+v}u(s)ds=0.$$

References

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