16. Quantum Orthogonal and Symplectic Groups and their Embedding into Quantum GL

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We use quantum R matrices [3] to define quantum orthogonal and symplectic groups in the same way as quantum GL and SL of type A [2, 4, 7]. We also consider embedding the quantum orthogonal and symplectic groups $O_q(n)$ and $Sp_q(n)$ into some q-analogues of $GL(n)$. It seems difficult to embed into $GL_q(n)$ of type A. We suggest there are two other types (orthogonal and symplectic) of q-analogues of $GL(n)$, and explain the embedding of $O_q(3)$ into $GL_q^o(3)$, the quantum $GL(3)$ of orthogonal type, in detail.

We work over a field k, and fix an element $q\neq 0$ in k. Let $\mathcal{M}_n$ be the free associative k-algebra on indeterminates x_{ij}, $i, j=1, \cdots, n$, with the following bialgebra structure:

$$\Delta(x_{ik})=\sum_j x_{ij}\otimes x_{jk}, \qquad \varepsilon(x_{ik})=\delta_{ik}.$$

Let X denote the $n\times n$ matrix (x_{ij}) with entries in $\mathcal{M}_n$.

1. Quantum orthogonal groups. For $1\leq i\leq n$, put $i'=n+1-i$ and

$$\bar{i}=\begin{cases} i-(n/2) & \text{if } i<i', \\ 0 & \text{if } i=i', \\ i-(n/2)-1 & \text{if } i>i'. \end{cases}$$

We assume q has a square root $q^{1/2}$ in k when n is odd. Let T denote the following symmetric $n^2\times n^2$ matrix.

$$q\sum_{i\neq i'} e_{ii}\otimes e_{ii}+\sum_{i\neq j,j'} e_{ij}\otimes e_{ji}+(q-q^{-1})\sum_{i<j,i\neq j'} e_{jj}\otimes e_{ii}+\sum_{i'\leq k} a_{ik}e_{ik}\otimes e_{i'k'}$$

where e_{ij} denote matrix units and

$$a_{ik}=\begin{cases} 1 & \text{if } i=i'=k, \\ q^{-1} & \text{if } i\neq i'=k, \\ (q-q^{-1})(\delta_{ik}-q^{-\bar{i}-\bar{k}}) & \text{if } i'<k. \end{cases}$$

We have

$$(T-q)(T+q^{-1})(T-q^{1-n})=0.$$

Definition 1. Define bialgebras $M_q(n)$ and $A_q(n)$ by

$$M_q(n)=\mathcal{M}_n/(X^{(2)}T=TX^{(2)}), \qquad A_q(n)=M_q(n)/(XX'=I=X'X),$$

where $X^{(2)}=(X\otimes I)(I\otimes X)$, and $X'=(q^{\bar{j}-\bar{i}}x_{j'i'})_{ij}$.

Proposition 2. (a) *$A_q(n)$ is a Hopf algebra,* i.e., *has an antipode.*

(b) *If $q\neq\pm 1$, there is a central group-like element γ in $M_q(n)$ such that $XX'=\gamma I=X'X$. The localization $M_q(n)[\gamma^{-1}]$ (with γ^{-1} group-like) is a Hopf algebra, and $A_q(n)$ coincides with the quotient Hopf algebra*

$$M_q(n)/(\gamma-1).$$

The *quantum orthogonal group* $O_q(n)$ is defined as the quantum group corresponding to the Hopf algebra $A_q(n)$. When $q=1$, this reduces to the classical orthogonal group.

2. Quantum symplectic groups. Definition is similar as above. We use T^- instead of T. Assume n is even. For $1\leq i\leq n$, put

$$\begin{cases} \bar{i}=i-(n/2)-1, & \varepsilon_i=1 \qquad \text{if } i<i', \\ \bar{i}=i-(n/2), & \varepsilon_i=-1 \qquad \text{if } i>i'. \end{cases}$$

We define T^- by the same formula as T by using

$$a_{ik}=\begin{cases} q^{-1} & \text{if } i'=k, \\ (q-q^{-1})(\delta_{ik}-\varepsilon_{i'}\varepsilon_k q^{-\bar{i}-\bar{k}}) & \text{if } i'<k. \end{cases}$$

We have

$$(T^- - q)(T^- + q^{-1})(T^- + q^{-1-n})=0.$$

Define quotient bialgebras $M_q^-(n)$ and $A_q^-(n)$ of $\mathcal{M}_n$ by using T^- and $X'=(\varepsilon_i\varepsilon_j q^{\bar{j}-\bar{i}}x_{j'i'})$ in Def. 1. Proposition 2 holds for $M_q^-(n)$ and $A_q^-(n)$. The *quantum symplectic group* $Sp_q(n)$ is defined as the quantum group corresponding to the Hopf algebra $A_q^-(n)$.

3. Quantum exterior and symmetric algebras. Manin [4] uses these algebras to reformulate $GL_q(n)$. We define their orthogonal and symplectic analogues. Let $V=k^n$ with canonical base $v_1, \cdots, v_n$. We identify T and T^- as linear endomorphisms of $V\otimes V$. Assume T (resp. T^-) has *three distinct eigenvalues* q, $-q^{-1}$, q^{1-n} (resp. q, $-q^{-1}$, $-q^{-1-n}$). We put

$$W_e=\operatorname{Ker}(T-q)\oplus\operatorname{Ker}(T-q^{1-n}), \qquad W_s=\operatorname{Ker}(T+q^{-1})$$

$$(\text{resp.}\quad W_e^-=\operatorname{Ker}(T^- - q), \qquad W_s^-=\operatorname{Ker}(T^- + q^{-1})\oplus\operatorname{Ker}(T^- + q^{-1-n})).$$

Definition 3. We put

$$\wedge_q(V)=T(V)/(W_e), \qquad S_q(V)=T(V)/(W_s)$$

$$(\text{resp.}\quad \wedge_q^-(V)=T(V)/(W_e^-), \qquad S_q^-(V)=T(V)/(W_s^-)).$$

When $q=1$, these reduce to the usual exterior and symmetric algebras.

Proposition 4. (a) *The k-algebra $\wedge_q(V)$ is defined by n generators $v_1, \cdots, v_n$ and the following relations:*

i) $v_i^2=0$, if $i\neq i'$,

ii) $v_j v_i=-q^{-1}v_i v_j$, if $i<j$, $i\neq j'$,

iii) $v_{i'}v_i=-v_i v_{i'}+(q^{-1}-q)\sum_{k<i} q^{i-k-1}v_k v_{k'}$, if $i<i'$,

iv) $v_{n_0}^2=(q^{-1/2}-q^{1/2})\sum_{k<n_0} q^{n_0-k-1}v_k v_{k'}$,

where $n_0=(n+1)/2$, and iv) *is required only when n is odd.*

(b) *The products $v_{i_1}\cdots v_{i_r}$ with $i_1<\cdots<i_r$ form a base for $\wedge_q(V)$.*

Proposition 5. (a) *The k-algebra $S_q(V)$ is defined by n generators $v_1, \cdots, v_n$ and the following relations:*

i) $v_j v_i=q v_i v_j$, if $i<j$, $i\neq j'$,

ii) $v_{i'}v_i=v_i v_{i'}+(q^{-1}-q)\sum_{i<k<n_0} q^{i+1-k}v_k v_{k'}$
$\qquad +q^{i+1-n_0}(q^{-1/2}-q^{1/2})v_{n_0}^2$, if $i<i'$,

where $n_0=(n+1)/2$, and the last term in ii) *is required only when n is odd.*

(b) *The products $v_{i_1}\cdots v_{i_r}$ with $i_1\leq\cdots\leq i_r$ form a base for $S_q(V)$.*

The diamond lemma [1] is used to prove (b) of both propositions. Similar facts hold for $\wedge_q^-(V)$ and $S_q^-(V)$.

Take the usual coaction ρ: $T(V)\rightarrow T(V)\otimes\mathcal{M}_n$, $\rho(v_j)=\sum_i v_i\otimes x_{ij}$. Let J be the smallest bi-ideal of $\mathcal{M}_n$ such that ρ induces homomorphisms of quotient algebras

$$\wedge_q(V)\longrightarrow\wedge_q(V)\otimes\mathcal{M}_n/J \quad\text{and}\quad S_q(V)\longrightarrow S_q(V)\otimes\mathcal{M}_n/J$$

(cf. [4]). Similarly, a bi-ideal J^- is associated with $\wedge_q^-(V)$ and $S_q^-(V)$.

Proposition 6. We put $\tilde{M}_q(n)=\mathcal{M}_n/J$ and $\tilde{M}_q^-(n)=\mathcal{M}_n/J^-$.

When $q=1$, both reduce to the polynomial algebra in x_{ij}.

Proposition 7. (a) $M_q(n)$ (resp. $M_q^-(n)$) *is a quotient bialgebra of* $\tilde{M}_q(n)$ (resp. $\tilde{M}_q^-(n)$).

(b) *We have*

$A_q(n)=\tilde{M}_q(n)/(XX'=I=X'X)$ *and* $A_q^-(n)=\tilde{M}_q^-(n)/(XX'=I=X'X)$.

Since $\wedge_q(V)$ and $\wedge_q^-(V)$ are quantum grassmannian algebras of dimension n [4], some group-like elements $\det_q$ in $\tilde{M}_q(n)$ and $\det_q^-$ in $\tilde{M}_q^-(n)$ are determined by the 1-dimensional n-th components. We call them the *quantum determinants of orthogonal and symplectic types.* It is likely that they are central and the localizations $\tilde{M}_q(n)[\det_q^{-1}]$ and $\tilde{M}_q^-(n)[(\det_q^-)^{-1}]$ are Hopf algebras. If this is the case we can well-define new q-analogues of GL, $GL_q^O(n)$ and $GL_q^S(n)$ of orthogonal and symplectic types, as the quantum groups represented by the Hopf algebras.

4. Presentation of $\tilde{M}_q(3)$. Write the generating matrix of $\mathcal{M}_3$ as

$$X=\begin{pmatrix} x & y & z \\ u & v & u' \\ z' & y' & x' \end{pmatrix}.$$

The defining relation for $\tilde{M}_q(3)$ consists of five types. Each type consists of several equations of the same form.

I. $yx=qxy$

and 7 similar ones for (y, z), (x, u) etc. as (x, y),

II. $zx=xz-ty^2$ with $t=q^{1/2}-q^{-1/2}$

and 3 similar ones for (z', y', x'), (x, u, z') etc. as (x, y, z),

III. $\begin{pmatrix} uy \\ vx \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & q-q^{-1} \end{pmatrix}\begin{pmatrix} xv \\ yu \end{pmatrix}$

and 3 similar ones for $(y, z; v, u')$ etc. as $(x, y; u, v)$,

IV. $\begin{pmatrix} u'x \\ vy \\ uz \end{pmatrix}=\begin{pmatrix} q-1 & -t & 1 \\ -t & 2-q^{-1} & t \\ 1 & t & q-1 \end{pmatrix}\begin{pmatrix} zu \\ yv \\ xu' \end{pmatrix}$

and 3 similar ones for $(u, v, u'; z', y', x')$ as $(x, y, z; u, v, u')$,

V. $\begin{pmatrix} u'u \\ y'y \\ x'x \\ z'z \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 & t \\ 0 & 0 & 1 & 0 & t \\ 0 & 1 & -t & -t & -t^2 \\ 1 & 0 & t & -t & 0 \end{pmatrix}\begin{pmatrix} zz' \\ xx' \\ yy' \\ uu' \\ W \end{pmatrix}$

with $W=xx'-zz'-tyy'-v^2$ (consisting of a single equation).

$\tilde{M}_q(3)$ is a non-commutative polynomial algebra, i.e., the ordered products of entries of X (relative to an appropriate order) form a base.

There is the following "cofactor matrix"

$$\tilde{X}=\begin{pmatrix} x'v-qy'u' & zy'-qyx' & -zv+qyu' \\ -ux'+q^{-1}z'u' & W+v^2 & uz-qxu' \\ -z'v+q^{-1}y'u & -xy'+q^{-1}yz' & xv-q^{-1}yu \end{pmatrix}.$$

This means we have $X\tilde{X}=\det_q I=\tilde{X}X$ in $\tilde{M}_q(3)$. Hence the quantum determinant $\det_q$ (of orthogonal type) is central and $\tilde{M}_q(3)[\det_q^{-1}]$ has an antipode. (The same is true for $n=5$.) Thus we can well-define $GL_q^o(3)$, and the quantum group $O_q(3)$ is its closed subgroup defined by the equation $XX'=I=X'X$ with

$$X'=\begin{pmatrix} x' & q^{1/2}u' & qz \\ q^{-1/2}y' & v & q^{1/2}y \\ q^{-1}z' & q^{-1/2}u & x \end{pmatrix}.$$

We put

$$SO_q(3)=O_q(3)\cap SL_q^o(3)$$

where $SL_q^o(3)$ is the quantum subgroup defined by $\det_q=1$.

There is an interesting relation between quantum groups $SL_q(2)$ (of type A) and $SO_{q^2}(3)$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the generating matrix of $A(SL_q(2))$, the Hopf algebra of $SL_q(2)$.

Proposition 8. *The algebra map* $f\colon \mathcal{M}_3\to A(SL_q(2))$,

$$f(X)=\begin{pmatrix} a^2 & q^{1/2}(q+q^{-1})ab & -(q+q^{-1})b^2 \\ q^{1/2}ac & ad+qbc & -q^{1/2}(q+q^{-1})bd \\ -c^2/(q+q^{-1}) & -q^{1/2}cd & d^2 \end{pmatrix}$$

(*which is essentially the matrix* W_1 *of* [5], (10)) *induces a Hopf algebra map* $A_{q^2}(3)\to A(SL_q(2))$ *sending* $\det_{q^2}$ *into* 1.

Thus we have a homomorphism of quantum groups $SL_q(2)\to SO_{q^2}(3)$. This is epimorphic, i.e., the corresponding Hopf algebra map is injective if $\mathrm{char}\,(k)=0$ and q is not a root of 1.

During preparation of the work, the author had a chance to attend a talk by L. Takhtajan, where similar constructions and results were presented independently. For instance, our quantum group $O_q(n)$ was introduced under the symbol $SO_q(n)$. On the other hand, the notion of $\wedge_q(V)$ or $\det_q$ (of orthogonal or symplectic type) does not seem contained in his work. His results will appear in the paper by N. Reshetikhin, L. Takhtajan and L. Faddeev (in Russian) in Algebra and Analysis, vol. 1, 1989.

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