77. The Selberg Zeta Function and the Determinant of the Laplacians

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1. Outline. Let G be a noncompact connected semi-simple Lie group of rank 1 with finite center, and Γ its cofinite discrete subgroup. For such pairs of G and Γ, the Selberg theory is constructed. If we put K to be a maximal compact subgroup of G, then $M:=\Gamma\backslash G/K$ is a Riemannian manifold and the Laplacian Δ over $L^2(M)$ is defined. For compact M, S. Minakshisundaram and Å. Pleijel prove the regularity of the spectral zeta function of Δ at the origin, by which we can define the determinant of Δ. When Γ is torsion-free and cocompact, A. Voros and P. Sarnak show that the Selberg zeta function with local factor is expressed as the determinant of Δ and calculate the local factor explicitly. We will generalize this type of determinant expression for more general G and Γ. Generally, torsion subgroups of Γ cause new local factors of the Selberg zeta function. Moreover, when Γ is not cocompact, that is, M is not compact, the continuous spectrum appears and contributes to the determinant of Δ. In the following sections, we generalize the theorem of Minakshisundaram-Pleijel to some noncompact cases, and give explicit forms of all the local factors of the Selberg zeta function and the contribution of the continuous spectrum.

2. The general program. Let $0=\lambda_0<\lambda_1\leq\lambda_2\leq\cdots$ be the eigen values of Δ. We induce the spectral zeta function generalized by a real variable $s>2\rho_0$; $\zeta(w,s,\Delta):=\sum_{n=0}^{\infty}(\lambda_n-s(2\rho_0-s))^{-w}$, for the purpose of expressing the Selberg zeta function. Here ρ_0 is the constant depending on G, which is defined in [1, p. 4]. For examining poles of $\zeta(w,s,\Delta)$, we use the trace formula of Selberg in the form of general case (Gangolli-Warner [1]). Taking the test function $h(r^2+\rho_0^2):=\exp(-(r^2+(s-\rho_0)^2)t)$ $(t>0)$, the trace formula has the form

$$(1)\qquad \sum_{n=0}^{\infty}\exp(-(\lambda_n-s(2\rho_0-s))t)=I(t)+E(t)+H(t)+P(t)-\mathrm{Tr}_C(t)$$

whose right side has the terms of identity, elliptic, hyperbolic and parabolic conjugacy classes, and the removed trace of the continuous spectrum. The Mellin transformation shows that the behavior of (1) as $t\to 0$ determines the poles of $\zeta(w,s,\Delta)$. Indeed, the behavior t^a (resp. $t^a\log t$) causes the simple (resp. double) pole at $w=-a$ of the function $\Gamma(w)\zeta(w,s,\Delta)$. Studying each term in (1), the main difficulty is the treatment of Tr_C. It has the form

$$(2)\qquad \mathrm{Tr}_C(t):=-(4\pi)^{-1}\int_{-\infty}^{\infty}h(r^2+\rho_0^2)(\varphi'/\varphi)(\rho_0+ir)dr+4^{-1}\lim_{s\to\rho_0}\mathrm{tr}\,\Phi(s)h(\rho_0^2),$$

where $\Phi(s)$ is the scattering matrix and $\varphi(s):=\det\Phi(s)$. When $G=SL(2,\boldsymbol{R})$, A. B. Venkov proves that $\varphi(s)$ can be written in the form $\varphi(s)=(\sqrt{\pi}\,\Gamma(s-(1/2))\Gamma(s)^{-1})^k l(s)$, where k is the number of cusps and $l(s)$ is an absolutely convergent Dirichlet series ($\mathrm{Re}(s)>1$). Using this identity, we have in (2),

(3) $(\varphi'/\varphi)(\rho_0+ir)=k((\Gamma'/\Gamma)(\rho_0-(1/2)+ir)-(\Gamma'/\Gamma)(\rho_0+ir))+(l'/l)(\rho_0+ir).$

We can apply the method of Kurokawa [5] treating the gamma-factors, which produce double poles of $\zeta(w,s,\Delta)$ at $w=(1/2)-n$ $(n=0,1,2,\cdots)$. But as for the last term in (3), we can do little for lack of properties of the function $l(s)$. In the following section, we treat the case when Venkov's type of expression is valid and $l(s)$ can be written explicitly. In those cases, it is proved that $\zeta(w,s,\Delta)$ is regular at $w=0$. Let $\det(\Delta,s)$ be the determinant composed of eigenvalues $\lambda_n-s(2\rho_0-s)$ instead of λ_n and the corresponding continuous spectrum explained below. Assuming the regularity of $\zeta(w,s,\Delta)$ at $w=0$, the contribution to $\det(\Delta,s)$ of the discrete spectrum is defined by

(4) $$\det{}_D(\Delta,s)=\det{}_D(\Delta-s(2\rho_0-s)):=\exp\left[-\frac{\partial}{\partial w}\Big|_{w=0}\zeta(w,s,\Delta)\right].$$

Formally, it is $\prod_{n=0}^{\infty}(\lambda_n-s(2\rho_0-s))$. For defining the contribution of the continuous spectrum and connecting with the Selberg zeta function, we induce an equation

(5) $$\frac{d}{ds}\frac{1}{2(s-\rho_0)}\frac{d}{ds}\log X(s)=\frac{d}{ds}Y(s),$$

where $Y(s)$ is a term in the trace formula, taking another test function

$$h(r^2+\rho_0^2):=(r^2+(s-\rho_0)^2)^{-1}-(r^2+\beta^2)^{-1}\qquad(\beta\gg 0,\ s>2\rho_0).$$

Then from the above definition of $\det_D$, if $Y(s)=\sum_{n=0}^{\infty}h(\lambda_n)$, the solution of (5) is $X(s)=\det_D(\Delta-s(2\rho_0-s))$. So symmetrically, we define the continuous part $\det_D(\Delta,s)$ by the solution of (5) with $Y=\mathrm{Tr}_C$. Next we put $Y=H$. A little calculation shows that the solution of (5) is nothing but the Selberg zeta function $Z(s)$. We obtain its local factors by solving (5) with $Y=I,E,P$. Putting the solutions to be Z_I, Z_E and Z_P, we have determinant expression of $Z(s)$;

(6) $$\hat{Z}(s)=e^{c-c's(2\rho_0-s)}\det(\Delta,s),$$

where $\hat{Z}:=Z\cdot Z_I\cdot Z_E\cdot Z_P$, $\det:=\det_D\cdot\det_C$, and c, c' are constants depending on G and Γ.

3. Examples. In this section we fulfill the above program for two special cases;

Case I: $G=SL_2(\boldsymbol{R})$, $\Gamma=\Gamma_i(N)$ $(i=0,1,2)$, $\rho_0=1/2$,

Case II: $G=SL_2(\boldsymbol{C})$, $\Gamma=SL_2(O_F)$, $\rho_0=1$,

where $\Gamma_2(N)$ is the principal congruence subgroup of level N, and F is an imaginary quadratic field satisfying $F\neq\boldsymbol{Q}(\sqrt{-1})$, $\boldsymbol{Q}(\sqrt{-3})$, with its integer ring O_F.

Theorem 1. *Under the situation of Case I (resp. II), the spectral zeta* *function $\zeta(w,s,\Delta)$ has the analytic continuation to the whole w-plane except the following poles; a simple pole at $w=1$ (resp. $w=3/2$), and double poles at $w=(1/2)-n$ $(n=0,1,2,\cdots)$.*

Proof. First we prove for Case I. As $t\to 0$, by integrating by parts,

$$(7)\quad I(t)=(\mathrm{vol}\,(\Gamma\backslash G/K)/4\pi)\int_{-\infty}^{\infty}\exp(-t(r^2+(s-\rho_0)^2))r\tanh(\pi r)dr=\sum_{n=-1}^{\infty}a_n t^n,$$

with $a_n\in\boldsymbol{R}$. The theorem of Kurokawa [5, Theorem 3(2)] shows that the terms of the gamma-factors in $P(t)-\mathrm{Tr}_C(t)$ can be expanded as

$$(8)\qquad \sum_{n=0}^{\infty}(b_n+c_n\log t)t^{n-(1/2)}+\sum_{n=0}^{\infty}d_n t^n \qquad (b_n,c_n,d_n\in\boldsymbol{R}).$$

The function $l(s)$ in the scattering determinant is obtained by M.N. Huxley. He proves that $l(s)$ can be written as a product of Dirichlet L-functions modulo N. It is easily seen that the concerning term in the trace formula decays exponentially as $t\to 0$, and it causes no poles of $\zeta(w,s,\Delta)$. We can easily expand all the other terms and their expansions does not have different behavior from those in (7) and (8). Next we prove for Case II. As $t\to 0$, the identity component has the expansion;

$$I(t)=(\mathrm{vol}\,(\Gamma\backslash G/K)/4\pi)\int_{-\infty}^{\infty}\exp(-t(r^2+1+s(s-2)))r^2dr=\sum_{n=0}^{\infty}e_n t^{n-(3/2)},$$

with $e_n\in\boldsymbol{R}$. The scattering determinant is given by I. Efrat and P. Sarnak who prove that the function $l(s)$ can be written as a quotient of Dedekind zeta functions of the Hilbert class field H of F. In this case again the concerning term has no influence to the holomorphy of $\zeta(w,s,\Delta)$. The treatment of other terms are the same as in Case I. The proof is completed.

The Selberg zeta function is defined as follows.

$$\text{(Case I)}\qquad Z(s):=\prod_{P\in\boldsymbol{P}}\prod_{n=0}^{\infty}(1-N(P)^{-s-n}),$$

$$\text{(Case II)}\qquad Z(s):=\prod_{P\in\boldsymbol{P}}\prod_{(k,l)}(1-a(P)^{-2k}\overline{a(P)}^{-2l}N(P)^{-s}),$$

where $\boldsymbol{P}$ is the set of all (resp. a certain kind of) primitive hyperbolic conjugacy classes and (k,l) runs through all the pairs of positive integers satisfying a certain congruence relation described in [3]. The norm $N(P)$ is put to be $|a(P)|^2$ where $a(P)$ is the eigen value of P such that $|a(P)|>1$.

Theorem 2. *Under the situation of Case I (resp. II), the Selberg zeta function has the determinant expression* (6) *whose explicit forms are given as follows.*

In Case I;

$$Z_I(s)=(\Gamma_2(s)^2(2\pi)^s/\Gamma(s))^{\mathrm{vol}\,(\Gamma\backslash G/K)/2\pi}$$

$$Z_E(s)=(\Gamma(s/2)^{-1/2}\Gamma((s+1)/2)^{1/2})^{n_2}(\Gamma(s/3)^{-2/3}\Gamma((s+2)/3)^{2/3})^{n_3}$$

$$Z_P(s)=(2^{-s}(s-(1/2))^{1/2}\Gamma(s+(1/2))^{-1})^k$$

$$\det{}_C(\Delta,s)=(A/\pi^k)^s\prod_{\chi}L(2s,\chi)\cdot\Gamma(s)^k(s-(1/2))^{-\mathrm{tr}\,\Phi(1/2)/2},$$

In Case II;

$$Z_I(s)=\exp(-(\mathrm{vol}\,(\Gamma\backslash G/K)/6)(s-1)^3)$$

$$Z_E(s)=\exp\left[\sum_R(\log N(T_0)/2\,\mathrm{ord}\,M(R))\sum_{m=0}^{\nu_R-1}(1-\cos(2m\pi/\nu_R))^{-1}s\right]$$

$$Z_P(s)=\exp\left((h/4\pi)((C/R)+3\log 2-2\gamma)s\right)(s-1)^{h/2}\Gamma(s)^{-h}$$
$$\det_C(\Delta, s)=\omega_F^{-s}\hat{\zeta}_H(s)(s-1)^{\kappa_H+2^{t-2}-1}$$

where all the notations and constants, whose explicit definition is in [2] (*resp.* [3]), *are determined by G and Γ.*

Proof. We only have to solve (5) with Y being terms in the trace formula. We need only straight forward calculation which is in [2] (resp. [3]) in detail. The proof is completed.

Remark 4. Recently, I. Efrat also defines the determinant of the Laplacian composed of both discrete and continuous spectrum. He induces some sequence expressed via poles of $\varphi(s)$, and regards it as "the eigenvalues of the continuous spectrum." Constructing the spectral zeta function including both discrete and continuous spectrum, he defines the determinant by the standard method. By his method, we can treat all the torsion-free cofinite discrete subgroup Γ, but cannot compute the contribution of each type of spectrum separately, which is not symmetric with each other comparing with the trace formula. The explicit relation of his results and the results in this paper is described in [4].

References

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