26. *Tychonoff Functor and Product Spaces*

By Shinpei OKA
Shizuoka University

(Communicated by Kôsaku YOSIDA, M. J. A., April 12, 1978)

1. Introduction. In this paper a space means a topological space with no separation axiom unless otherwise specified. We use the term "Tychonoff functor" in the sense of K. Morita [2] and denote it by τ which is the epi-reflective functor from the category of all spaces and continuous maps onto the category of all Tychonoff spaces and continuous maps.

For any spaces X and Y, we denote by $f_{X,Y}$ the unique continuous map from $\tau(X\times Y)$ onto $\tau(X)\times\tau(Y)$ which makes the following diagram commutative, where the symbol Φ_X follows [2].

$$\begin{tikzcd} X\times Y \arrow[rr, "\Phi_X\times\Phi_Y"] \arrow[dr, "\Phi_{X\times Y}"'] & & \tau(X)\times\tau(Y) \\ & \tau(X\times Y) \arrow[ur, "f_{X,Y}"'] & \end{tikzcd}$$

The equality $\tau(X\times Y)=\tau(X)\times\tau(Y)$ means that $f_{X,Y}$ is a homeomorphism. Concerning this equality, the following theorems are known.

Theorem 1 (K. Morita). *$\tau(X\times Y)=\tau(X)\times\tau(Y)$ is valid if and only if every cozero set of $X\times Y$ can be expressed as the union of rectangular cozero sets of $X\times Y$.*

A subset V of $X\times Y$ is called a rectangular cozero set if it is expressed as $V=V_X\times V_Y$, where V_X and V_Y are cozero sets of X and Y respectively.

Theorem 2 (R. Pupier [3]). *If X is a locally compact Hausdorff space, then $\tau(X\times Y)=X\times\tau(Y)$ is valid for any space Y.*

The purpose of this paper is to show that the converse of Theorem 2 is valid in case X is a Tychonoff space. More generally, we can prove the following theorem.

Theorem 3. *Let X be a space. If $\tau(X)$ is not locally compact, then there exists a Hausdorff space Y such that $\tau(X\times Y)\neq\tau(X)\times\tau(Y)$.*

Combining Theorem 3 with Theorem 2, we have the following theorem.

Theorem 4. *Let X be a Tychonoff space. Then the following conditions are equivalent.*

(1) *X is locally compact.*

(2) *$\tau(X\times Y)=X\times\tau(Y)$ for any space Y.*

2. Preliminaries. Hereafter the symbol N denotes the set of all positive integers.

In this section we shall prove the following lemma which is needed to prove Theorem 3.

Lemma 5. *Let X be a Tychonoff space and C a non-compact closed subset of X. Then there exist a (Hausdorff) space Y, a point y_0 of Y and a continuous function $h: X\times Y\to[0,1]$ which satisfy the following conditions.*

(1) *y_0 is not an isolated point of Y.*

(2) *$h(z)=1$ for $z\in X\times\{y_0\}$.*

(3) *$h^{-1}(0)\cap(C\times\{y\})\neq\phi$ for each $y\in Y-\{y_0\}$.*

(In particular the projection p_Y from $X\times Y$ onto Y is not a Z-mapping in the sense of Z. Frolik [1].)

Proof. Since C is a non-compact closed subset of a Tychonoff space X, there exists a collection $\{E_\beta:\beta\in B\}$ of zero sets of X satisfying the condition that $\cap\{E_\beta:\beta\in B\}=\phi$ and $(\cap\{E_\beta:\beta\in\gamma\})\cap C\neq\phi$ for each finite subset γ of B. Let us denote by Γ the set of all finite subsets of B, and put $F_\gamma=\cap\{E_\beta:\beta\in\gamma\}$ for each $\gamma\in\Gamma$. Then we define a space Y as follows:

$$Y=\bigcup\{N_\gamma:\gamma\in\Gamma\}\cup\{y_0\}$$

where $N_\gamma=N$ for each $\gamma\in\Gamma$, with the topology such that

(i) Each point of $\cup\{N_\gamma:\gamma\in\Gamma\}$ is isolated.

(ii) The point y_0 has an open nbd(=neighbourhood) base of the form $\{\cup\{N_\delta^i:\delta\in\Gamma,\ \gamma\subset\delta\}\cup\{y_0\}: i\in N,\ \gamma\in\Gamma\}$, where $N_\delta^i=\{i,i+1,i+2,\cdots\}\subset N_\delta$. Then, clearly, y_0 is not an isolated point of Y.

To construct the function h, we take, for each $\beta\in B$, a countable collection $\{G_\beta^i: i\in N\}$ of cozero sets of X and a countable collection $\{K_\beta^i: i\in N\}$ of zero sets of X such that

$$G_\beta^i\supset K_\beta^i\supset G_\beta^{i+1}\supset K_\beta^{i+1}\qquad\text{for each } i\in N,$$
$$\bigcap\{G_\beta^i: i\in N\}=\bigcap\{K_\beta^i: i\in N\}=E_\beta.$$

Let us put $G_\gamma^i=\bigcap\{G_\beta^i:\beta\in\gamma\}$ and $K_\gamma^i=\bigcap\{K_\beta^i:\beta\in\gamma\}$ for each $\gamma\in\Gamma$ and $i\in N$. Then G_γ^i is a cozero set of X and K_γ^i is a zero set of X such that

$$G_\gamma^i\supset K_\gamma^i\supset G_\gamma^{i+1}\supset K_\gamma^{i+1}\qquad\text{for each } i\in N,$$
$$\bigcap\{G_\gamma^i: i\in N\}=\bigcap\{K_\gamma^i: i\in N\}=F_\gamma.$$

Here we can find, for each $\gamma\in\Gamma$ and $i\in N$, a continuous function $h_\gamma^i: X\to[0,1]$ such that $h_\gamma^i(x)=1$ for $x\in X-G_\gamma^i$ and $h_\gamma^i(x)=0$ for $x\in K_\gamma^i$. Let us now define a function $h: X\times Y\to[0,1]$ as follows:

$$h(z)=1\qquad\text{for } z\in X\times\{y_0\}$$
$$h(z)=h_\gamma^i(z)\qquad\text{for } \gamma\in\Gamma,\ i\in N_\gamma \text{ and } z\in X\times\{i\}.$$

Then it is easily shown that h is a continuous function satisfying the required properties. Thus we complete the proof of Lemma 5.

3. Proof of Theorem 3. We first prove the following theorem.

Theorem 3′. *Let X be a Tychonoff space. If X is not locally compact, then there exists a Hausdorff space Y such that $\tau(X\times Y)\neq X\times\tau(Y)$.*

Proof. Let X be a Tychonoff space which is not lolally compact, and x_0 a point of X which has no compact nbd. Let us fix some open nbd base $\{U_\alpha : \alpha \in A\}$ at x_0, and define a space Y_0 as follows:

$$Y_0 = \bigcup \{\{p_\alpha\} \cup N_\alpha : \alpha \in A\} \cup \{p\},$$

where $N_\alpha = N$ for each $\alpha \in A$, with the topology such that

(1) Each point of $\cup \{N_\alpha : \alpha \in A\}$ is isolated.

(2) The point p has as open nbd base of the form

$$\{\cup \{N_\alpha : \alpha \in A'\} \cup \{p\} : A' \subset A, |A - A'| < \aleph_0\}.$$

(3) The point p_α $(\alpha \in A)$ has an open nbd base of the form $\{\{p_\alpha\} \cup N_\alpha^j : j \in N\}$, where $N_\alpha^j = \{j, j+1, \cdots\} \subset N_\alpha$. The space Y_0 is Hausdorff and satisfies the following condition.

(*) Each cozero set V of Y_0 with $p \in V$ satisfies the inequality

$$|\{\alpha \in A : p_\alpha \in Y_0 - V\}| < \aleph_0.$$

On the other hand, since each $\bar{U}_\alpha$ is a non-compact closed subset of X, there exist, by Lemma 5, a Hausforff space Y_α, a point y_α of Y_α and a continuous function $h_\alpha : X \times Y_\alpha \to [0, 1]$ which satisfy the following conditions.

$(1)_\alpha$ y_α is not an isolated point of Y_α.

$(2)_\alpha$ $h_\alpha(z) = 1$ for $z \in X \times \{y_\alpha\}$.

$(3)_\alpha$ $h_\alpha^{-1}(0) \cap (\bar{U}_\alpha \times \{y\}) \neq \phi$ for each $y \in Y_\alpha - \{y_\alpha\}$.

By identifying the point p_α of Y_0 with the point y_α of Y_α for each $\alpha \in A$, we have a quotient space Y and a quotient map $q : Y_0 \oplus (\oplus \{Y_\alpha : \alpha \in A\}) \to Y$, where the symbol $\oplus$ means the topological sum.

To prove that $\tau(X \times Y) \neq X \times \tau(Y)$, let us define a continuous function $f : X \times Y \to [0, 1]$ as follows:

$$f(z) = 1 \qquad \text{for } z \in X \times q(Y_0)$$

$$f(z) = h_\alpha \circ (1_X \times q_\alpha)^{-1}(z) \qquad \text{for } z \in X \times q(Y_\alpha) \text{ and } \alpha \in A,$$

where q_α is the restriction of q to Y_α.

Suppose that there exists a rectangular cozero set $V = V_X \times V_Y$ of $X \times Y$ such that $(x_0, q(p)) \in V \subset f^{-1}([0, 1])$. Then, by condition (*), we have $|\{\alpha \in A : q(p_\alpha) \in Y - V_Y\}| < \aleph_0$. Let us put $A_0 = \{\alpha \in A : q(p_\alpha) \in V_Y\}$. Then, by $(1)_\alpha$ and $(3)_\alpha$, we have $\bar{U}_\alpha - V_X \neq \phi$ for each $\alpha \in A_0$. Since $|A - A_0| < \aleph_0$, this implies that x_0 is an isolated point of X, which is a contradiction. Thus, according to Theorem 1, we complete the proof of Theorem 3′.

Theorem 3 is a direct consequence of Theorem 3′. (Notice that the image of a cozero set of X by Φ_X is also a cozero set of $\tau(X)$).

Remark. In [3], R. Pupier has proved the following theorem which is a partial converse to Theorem 3: *If $\tau(X)$ is locally compact, then $\tau(X \times Y) = \tau(X) \times \tau(Y)$ is valid for any k-space Y.*

References

[1] Z. Frolik: Applications of complete family of continuous functions to the theory of Q-spaces. Czech. Math. Journ., **11**, 115–133 (1961).
[2] K. Morita: Čech cohomology and covering dimension for topological spaces. Fund. Math., **87**, 31–52 (1975).
[3] R. Pupier: La complexion universelle d'un produit d'espaces complètement réguliers. Publ. Dept. Math. Lyon, **6**, 75–84 (1969).