47. The Sheaf of Relative Canonical Forms of a Kähler Fiber Space over a Curve

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In this note we announce an improvement of a result in [1]. Details shall be published elsewhere.

A triple $f: M \to S$ of a holomorphic mapping f and compact complex manifolds M, S is called a *Kähler fiber space* if M is Kähler, f is surjective and any general fiber of f is connected. By $\omega_{M/S}$ we denote the relative dualizing sheaf $\omega_M \otimes f^*\omega_S^{\vee} = \mathcal{O}_M[K_M - f^*K_S]$. Then we have the following

Theorem. *Let $f: M \to C$ be a Kähler fiber space over a curve C. Then $f_*\omega_{M/C} \cong \mathcal{O}_C[A \oplus U]$ for an ample vector bundle A and a flat vector bundle U on C.*

For a proof, we show the following lemma and use the criterion of Hartshorne [4].

Lemma. *Let E be the vector bundle such that $f_*\omega_{M/C} \cong \mathcal{O}_C[E]$. Then* $\deg(\det Q) \geqq 0$ *for any quotient bundle Q of E. Moreover, if* $\deg(\det Q) = 0$, *then Q is a direct sum component of E and has a flat connection.*

Outline of the proof of lemma. Let S be the image of singular fibers of f and let $C^o = C - S$. Note that the restriction E_{C^o} of E to C^o is isomorphic to the bundle $\bigcup_{x \in C^o} H^{n,0}(F_x)$, where $F_x = f^{-1}(x)$ and $n = \dim F_x$. Hence E_{C^o} has a natural Hermitian structure. This defines a Hermitian structure of Q_{C^o} in a canonical manner. Let Ω be the Chern De Rham curvature form representing $c_1(Q_{C^o})$. Then we have the following formula: $\deg(\det Q) = \int_{C^o} \Omega + \sum_{p \in S} e_p$, where e_p is the local exponent of $\det Q$ at $p \in S$ (see [3]). Similarly as in [1], we prove that Ω is semi-positive and that $e_p \geqq 0$ for any $p \in S$. So $\deg(\det Q) \geqq 0$. Moreover, if $\deg(\det Q) = 0$, then $\Omega \equiv 0$ and $e_p = 0$ for any p. $\Omega \equiv 0$ implies that the orthogonal complements $\tilde{Q}_x$ $(x \in C^o)$ of $\mathrm{Ker}\,(E_x \to Q_x)$, considered as subspaces of $H^{n,0}(F_x) \subset H^n(F_x; \boldsymbol{C})$, form a flat subbundle of the flat bundle $\bigcup_{x \in C^o} H^n(F_x; \boldsymbol{C})$. So, Q_{C^o} is isomorphic to the vector bundle $\tilde{Q}_o = \bigcup_{x \in C^o} \tilde{Q}_x$ associated with the monodromy action of $\pi_1(C^o, x_o)$ on $\tilde{Q}_{x_o} \subset H^n(F_{x_o}; \boldsymbol{C})$, where x_o is a point on C^o. Now, $e_p = 0$

implies that the local monodromy of $\tilde{Q}_{x_0}$ at p is trivial and that the splitting $Q_{C^0} \cong \tilde{Q}_{C^0} \subset E_{C^0}$ can be extended over p. Thus we prove the lemma.

As an application, we can prove the following

Theorem. *Let $f: M \to C$ be an algebraic fiber space over a curve. Suppose that the canonical bundle of any general fiber of f is trivial. Then $\kappa(M, \omega_{M/C}) \geqq 0$. Moreover, the equality holds if and only if f is birationally equivalent to a holomorphic fiber bundle over C.*

References

[1] T. Fujita: Kähler fiber spaces over curves (to appear in J. Math. Soc. Japan).
[2] ——: On the theory of Kodaira dimension (in Japanese) (to appear in Sugaku).
[3] ——: On local exponents (in preparation).
[4] R. Hartshorne: Ample vector bundles on curves. Nagoya Math. J., **43**, 73–89 (1971).