53. On Bounded Sets of Holomorphic Germs

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Let E, F be separated, complex locally convex spaces, $\mathcal{H}(U;F)$ and $\mathcal{H}(K;F)$ denote the spaces of holomorphic mappings on an open subset U of E and of holomorphic germs on a compact subset K of E, respectively, endowed with their natural topologies (see Barroso [1], Mujica [4], Nachbin [5]). It is interesting to characterize the bounded subsets of $\mathcal{H}(K;F)$ in terms of the successive differentials. Such a characterization would be useful, by example, in the study of sequential convergence in $\mathcal{H}(K;F)$. Let $\mathcal{F}$ be a subset of $\mathcal{H}(K;F)$, Γ a family of seminorms defining the topology of F. We say that $\mathcal{F}$ has an estimate for the differentials in K if there exist a continuous seminorm α on E, real numbers $C>0$, $c>0$ such that for every β in Γ we have:

$$\sup_{x\in K}\left\|\frac{1}{m!}\hat{d}^m f(x)\right\|_{\alpha\beta}\leq Cc^m \qquad \text{for every } \tilde{f}\in\mathcal{F},\ f\in\tilde{f},\ m\in N.$$

One knows that an estimate for the differentials in K is not a sufficient condition for boundedness in $\mathcal{H}(K;F)$, but a bounded subset of $\mathcal{H}(K;F)$ has an estimate for the differentials when E and F are Banach spaces (Chae [2] and Wanderley [6]). Zame [7] showed that, under a weak local connectedness assumption on K, when K is a compact subset of C^n, an estimate for the differentials in K implies boundedness in the space $\mathcal{H}(K)$. The arguments used by Zame can be used in the general case.

Definition 1. Let X be a topological space, K a compact subset of X. We consider the following equivalence relation on X: $x, y\in X$, $x\sim y$ iff $x, y\in K$ or $x=y$. We denote by X/K the quotient space endowed with its natural topology, and K/K the equivalence class of an element of K.

Definition 2. Let X and K be as above. We say that K is of type LQC if, for every $x\in K$, there exists a sequence $K_1\subset\cdots\subset K_n$ of compact connected subsets of K such that $K_1=\{x\}$, K_{i+1}/K_i is locally connected for $i=1,\cdots,n-1$ and K/K_n is locally connected in K_n/K_n. We say that K is of type QC if there exists a sequence $K_1\subset\cdots\subset K_n$ of compact connected subsets of K such that K_1, K_{i+1}/K_i (for $i=1,\cdots,n-1$), K/K_n are locally connected. If K is locally connected then K is a compact of types LQC and QC. The class of compact subsets of E which are of type LQC is different from the class of compact subsets of E which are of type QC.

Now, we generalize the concept of analytic manifold which appears in Gunning and Rossi [3]. We denote $\mathcal{O}(U; F) = {}_U\mathcal{O}(F)$ the shearf of germs of holomorphic F-valued mappings over U.

Definition 3. Let X be a separated topological space, $\mathcal{O}(X; F)$ a sheaf of groups over X, $x \in X$ and $\mathcal{U}$ a basis for the open subsets of X. Denote $\mathcal{U}_x$ the family of those open subsets U of $\mathcal{U}$ such that U contains x. Suppose that for every U in $\mathcal{U}$ there exists a subgroup $\mathcal{S}_U$ of the group $\mathcal{C}(U; F)$ of continuous F-valued mappings in U and denote $\pi: \mathcal{O}(X; F) \to X$ the natural mapping. If $\tilde{g} \in \pi^{-1}(x) = {}_X\mathcal{O}_x(F)$ and there exists a continuous mapping $g: U_0 \to F$, $U_0 \in \mathcal{U}_x$ such that $\tilde{g} = \{f \in \mathcal{S}_U;\ U \in \mathcal{U}_x;\ f = g$ in a neighborhood of x contained in $U \cap U_0\}$ we say that $\tilde{g}$ is a restriction of a germ of continuous F-valued mapping at x. An F-grouped space is a pair $(X, \mathcal{O}(X; F))$ such that, for every $x \in X$ each element in the stalk ${}_X\mathcal{O}_x(F)$ is a restriction of a germ of continuous F-valued mapping at x.

Definition 4. Let $(X, \mathcal{O}(X; F))$ and $(Y, \mathcal{O}(Y; F))$ be F-grouped spaces. We say that these F-grouped spaces are isomorphic if there exists a continuous mapping $f: X \to Y$ such that for every $x \in X$ and $\tilde{h} \in {}_Y\mathcal{O}_{f(x)}(F)$ we have $h \circ \tilde{f} \in {}_X\mathcal{O}_x(F)$, the mapping of ${}_Y\mathcal{O}_{f(x)}(F)$ into ${}_X\mathcal{O}_x(F)$ given by $\tilde{h} \mapsto \tilde{h} \circ f$ is surjective for every x and f is a homeomorphism.

Definition 5. An F-grouped space is an F-analytic manifold modelled on E if, for every $x \in X$, there exists an open neighborhood V of x such that $(V, {}_X\mathcal{O}(F) \mid V)$ is isomorphic to $(U, {}_U\mathcal{O}(F))$, for some open subset U of E.

Definition 6. Let $\mathcal{F}$ be a subset of $\mathcal{H}(K; F)$ and $x \in K$. $\mathcal{F}$ is extendible at x if there exists an open subset U_x of E containing x such that for every $\tilde{f} \in \mathcal{F}$ there is a holomorphic mapping $f^x: U_x \to F$ with the property that $f^x = f$ in some neighborhood of x, for some $f \in \tilde{f}$. $\mathcal{F}$ is extendible if there exist an open subset U of E containing K and a family $\mathcal{F}_U \subset \mathcal{H}(U; F)$ such that $T_U(\mathcal{F}_U) = \mathcal{F}$, where T_U denotes the natural mapping of $\mathcal{H}(U; F)$ into $\mathcal{H}(K; F)$.

Using the concept of F-analytic manifold we get:

Theorem 1. *Let K be a compact subset of E of types LQC or QC. Then $\mathcal{F} \subset \mathcal{H}(K; F)$ is extendible if and only if $\mathcal{F}$ is extendible at every point of K.*

Corollary. *Suppose F is a complete and N-complete space (see Barroso* [1]*), K a compact subset of E of types LQC or QC. Let $\mathcal{F}$ be a subset of $\mathcal{H}(K; F)$ having an estimate for the differentials in K. Then $\mathcal{F}$ is extendible.*

Proposition 1. *Let K and F be as before. Suppose that $\mathcal{F} \subset \mathcal{H}(K; F)$*

has an estimate for the differentials in K, and this estimate is concerned with a family Γ such that, for every $\beta \in \Gamma$, F_β is complete. Then $\mathcal{F}$ is a bounded subset of $\mathcal{H}(K; F)$.

The converse to Proposition 1 is not true in the general case.

Theorem 2. *Let E be a metrizable locally convex space, K a compact subset of E of types LQC or QC, F a Banach space. For every subset $\mathcal{F}$ of $\mathcal{H}(K; F)$, a necessary and sufficient condition for $\mathcal{F}$ to be bounded is that $\mathcal{F}$ has an estimate for the differentials in K.*

It follows from Proposition 1 that the above condition is sufficient. Necessity follows from the characterization of the bounded subsets of $\mathcal{H}(K; F)$ given by Mujica [4].

About sequential convergence in $\mathcal{H}(K; F)$, we get:

Proposition 2. *Let F and K be as in Corollary, Γ a directed family of seminorms generating the topology of F such that F_β is complete for every β in Γ. Let $\{\tilde{f}_n, n \in N\}$ a sequence in $\mathcal{H}(K; F)$. If there exist real numbers $C>0$, $c>0$ and a continuous seminorm α on E such that for every $\varepsilon>0$ we can find a natural number n_ε with the property that:*

$\sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m f_n(x) \right\|_{\alpha\beta} \leq \varepsilon C c^m$ *for every $m \in N$, $\beta \in \Gamma$, $n \geq n_\varepsilon$ and $f_n \in \tilde{f}_n$ (we say that the sequence has an ε-estimate for the differentials in K), then the sequence $\{\tilde{f}_n\}$ converges to zero in $\mathcal{H}(K; F)$.*

Otherwise, by a result from Mujica [4], we get:

Proposition 3. *Let E be a metrizable locally convex space with Condition* B *(see Barroso* [1]*), F a Banach space, K a compact subset of E. If the sequence $\{\tilde{f}_n\}$ converges to zero in $\mathcal{H}(K; F)$, then for every $\varepsilon>0$ the sequence has an ε-estimate for the differentials in K.*

Theorem 3. *Let E, F be as in Proposition* 3 *and K be a compact subset of E of types LQC or QC. Then the sequence $\{\tilde{f}_n\}$ coverges to zero in $\mathcal{H}(K; F)$ if and only if for every $\varepsilon>0$ the sequence $\{\tilde{f}_n\}$ has an ε-estimate for the differentials in K.*

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