38. *Note on the Envelope of Regularity of a Tube-Domain.*

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§1. Introduction.

In the space of n complex variables $(z_1,\dots, z_n)$, there exists a domain B, such that *any* function analytic in B has an analytic continuation over the domain D which is strictly larger than B. Such D is called an *analytic completion* of B. For any domain B, there corresponds a domain $\mathbf{H}(B)$ called its *envelope of regularity*, or *maximal analytic completion*, such that[1]

(i) $\mathbf{H}(B)$ is an analytic completion of B, and

(ii) $\mathbf{H}(B)$ is a domain of regularity, i.e. there exists a function which cannot be continued beyond $\mathbf{H}(B)$.

The geometrically explicit form of the envelope of regularity for a given domain still remains almost unknown. One of the few results concerning this branch is the following due to S. Bochner[2]:

Theorem 1. *The envelope of regularity of a tube-domain T is its convex hull (convex closure) $\mathbf{C}(T)$. Here the tube-domain means the point set which can be written in the form*

$$(1)\quad T = \{(z_j=x_j+iy_j) \mid (x_1,\dots, x_n\in S, \mid y_j\mid<\infty, (j = 1,\dots, n)\}.$$

where S is a domain in the real n-dimensional space $(x_1,\dots,x_n)$, and S is called the base of T.

It seems quite natural that this theorem should be conjectured from the facts that the mapping $w_j = \exp z_j$ transformes T into a covering surface over a Reinhardt domain in (w_j)-space, and that the Reinhardt domain of regularity is convex in logarithmic sense. But his original proof is based upon the expansion of the

1) P. Thullen: Die Regularitätshüllen. Math. Ann. **106** (1932) 64–76.
H. Cartan–P. Thullen: Regularitäts- und Konvergenzbereiche. Math. Ann. **106** (1932) 617–647.

2) S. Bochner: A theorem on analytic continuation of functions in several variables. Annals of Math. **39** (1938) 14–19.
S. Bochner–W.T. Martin. Several complex variables. Princeton 1948, Chap. V.

3) Cf. e.g. H. Cartan: Les fonctions de deux variables complexes et le problème de la représentation analytique.

analytic function in T in multiple Legendre polynomials, which seems to me too much complicated.

In this note, we shall give a brief proof of theorem 1.

§ 2. General considerations.

From the definition of the tube-domain, it is evident that

Lemma 1. *A domain B is a tube if and only if it admits the automorphisms*

$$z_j^* = z_j + i\, c_j \qquad (j = 1, \dots, n) \tag{2}$$

where c_j are arbitrary real constants.

Lemma 2. *The envelope of regularity* $\mathbf{H}(T)$ *of a tube T is also a tube.*

Proof. If a domain B is transformed onto itself by an analytic automorphism, its envelope of regularity $\mathbf{H}(B)$ is also transformed onto itself.[4] Thus applying this fact for the automorphism (2) of T, we see that $\mathbf{H}(T)$ also admits (2), i.e. $\mathbf{H}(T)$ is a tube.

Since the convex tube $\mathbf{C}(T)$ is a domain of regularity,[5] we can conclude that $\mathbf{C}(T) \supseteq \mathbf{H}(T)$. In order to prove the converse, we have only to show that

Lemma 3. *The base Δ of the tube* $\mathbf{H}(T)$ *is convex.*

§ 3. A cross-shaped tube.

First, we consider a special case where $n = 2$, and the base S of our tube T is *cross-shaped*, i.e. it consists of two rectangles $T_a \cup T_b$, where[6]

$$\begin{aligned} T_a &= \{(z_1, z_2) \mid \quad |\, x_j \,| < a_j\} \\ T_b &= \{(z_1, z_2) \mid \quad |\, x_j \,| < b_j\} \\ z_j &= x_j + i y_j\,, \qquad (j = 1,2), \end{aligned} \tag{3}$$

and a_j and b_j are positive constants. Without any loss of generality, we can suppose that

$$0 < b_1 < a_1 \quad \text{and} \quad 0 < a_2 < b_2 \ . \tag{4}$$

Let λ be an arbitrary constant between 0 and 1, and put

$$\begin{aligned} c_j &= \lambda a_j + (1-\lambda)\, b_j\,, \qquad (j = 1,2), \\ T_c &= \{(z_1, z_2) \mid \quad |\, x_j \,| < c_j\} \ . \end{aligned} \tag{5}$$

J. de Math. (9) **10** (1931) 1—114, Chap. V. § 2.

4) H. Cartan—P. Thullen: loc. cit. 1). Theorem 3, Corollary 1.

5) Bochner-Martin: loc. cit. 2). p. 91. This can be proved also by the facts that every bounded convex domain is a domain of regularity (cf. e.g. E.E. Levi: Annali di Mat. (3) **18** (1911) p. 79) and that a finite (not necessarily bounded) domain which is the limit of an increasing sequence of domains of regularity is also a domain of regularity. (cf. K. Oka, Tôhoku Math. J. **49** (1942) p. 27).

6) Of course this is a very special case, but from the results conserning the cross-shaped tube, we can prove lemma 3. (cf. § 4).

Since $\bigcup_\lambda T_c = \mathrm{C}\,(T_a \cup T_b)$, we have only to prove the following:

Theorem 2. *Using the above notations, if a function $f(z_1, z_2)$ is analytic in $T_a \cup T_b$, it is also analytic in T_c.*

Proof. Let h and l be fixed positive constants, and

$$0 < h < \mathrm{Min}\,(b_1, a_2).$$

Denote the elliptic domains by

$$(6) \qquad E_j(a_j\,;\, l) \equiv \left\{ z_j = x_j + iy_j \;\middle|\; \frac{x_j^2}{a_j^2} + \frac{y_j^2}{a_j^2 + l^2} < 1 \right\},$$

$$\Lambda(a\,;\, l) \equiv E_1(a_1\,;\, l) \otimes E_2(a_2\,;\, l),$$

where $\otimes$ means the direct product of two domains. Similarly we define $E_j(b_j\,;\, l)$, $\Lambda(c\,;\, l)$ etc. by substituting a_j for the corresponding letters b_j, c_j respectively. We define further D_j and Ω as the following:

$$(7) \qquad D_j(a_j,\, h\,;\, l) \equiv E_j(a_j\,;\, l) - E_j(h\,;\, l),$$

$$\Omega(a,\, h\,;\, l) \equiv D_1(a_1,\, h\,;\, l) \otimes D_2(a_2,\, h\,;\, l),$$

and $D_j(b_j,\, h\,;\, l)$ and $\Omega(b,\, h\,;\, l)$ are similarly defined. Now, since T_a and T_b contain $\Lambda(a\,;\, l)$ and $\Lambda(b\,;\, l)$ respectively, $f(z_1, z_2)$ is analytic in $\Omega(a,\, h\,;\, l)$ and $\Omega(b,\, h\,;\, l)$. By conformal mappings

$$(8) \qquad w_j = \frac{-i}{l}\left[z_j + (z_j^2 + l^2)^{\frac{1}{2}}\right]^{7)}, \qquad (j = 1,2),$$

the domains $D_j(a_j, h\,;\, l)$ and $D_j(b_j, h\,;\, l)$ are transformed univalently onto the circular rings

$$(9) \qquad \varepsilon < |w_j| < \alpha_j \text{ and } \varepsilon < |w_j| < \beta_j$$

on the w_j-plane respectively, where

$$(10) \qquad \alpha_j = \frac{1}{l}\left[a_j + (a_j^2 + l^2)^{\frac{1}{2}}\right],$$

$$\beta_j = \frac{1}{l}\left[b_j + (b_j^2 + l^2)^{\frac{1}{2}}\right],$$

and

$$\varepsilon = \frac{1}{l}\left[h + (h^2 + l^2)^{\frac{1}{2}}\right].$$

By the transformation (8), $f(z_1, z_2)$ turns into a function $\varphi(w_1, w_2)$ which is analytic in (9). $\varphi(w_1, w_2)$ can be expanded into a Laurent series of w_1 and w_2, and since this series converges[8] in

$$(11) \qquad \varepsilon < |w_j| < \gamma_j(l),$$

where

$$\log \gamma_j(l) = \lambda \log \alpha_j + (1-\lambda) \log \beta_j,$$

$\varphi(w_1, w_2)$ is also analytic in $\varepsilon < |w_j| < \gamma_j(l)$. By the inverse map-

7) For the function $(\zeta)^{\frac{1}{2}}$ we take the branch whose value is $+1$ at $\zeta = 1$.

8) H. Tietze: Über den Bereich absoluter Konvergenz von Potenzreihen mehrerer Veränderlichen. Math. Ann. **99** (1928) 181–182.

pings of (8), $\varphi(w_1, w_2)$ reverses to $f(z_1, z_2)$, which is analytic in $\Omega(c(l), h; l)$ where

$$c_j(l) = \frac{l}{2}\left[\gamma_j(l) - \frac{1}{\gamma_j(l)}\right]. \tag{12}$$

From (10), (11) and (12), $c_j(l)$ satisfies the relation

$$\operatorname{arcsinh} \frac{c_j(l)}{l} = \lambda \operatorname{arcsinh} \frac{a_j}{l} + (1-\lambda) \operatorname{arcsinh} \frac{b_j}{l}. \tag{13}$$

But on the other hand, we see from (4),

$$b_1 < c_1(l) < a_1 \quad \text{and} \quad a_2 < c_2(l) < b_2,$$

then

$$\Theta_1 \equiv E_1(h; l) \otimes E_2(c_2(l); l)$$

and

$$\Theta_2 \equiv E_1(c_1(l); l) \otimes E_2(h; l)$$

are contained completely in T_b and T_a respectively. Therefore $f(z_1, z_2)$ is analytic in

$$\Omega(c(l), h; l) \cup \bar{\Theta}_1 \cup \bar{\Theta}_2 = \Lambda(c(l); l)$$

for arbitrary l, where $\bar{\Theta}$ is the closure of Θ. Now from (13) we obtain

$$\lim_{l \to \infty} c_j(l) = c_j, \tag{14}$$

and

$$\lim_{l \to \infty} \Lambda(c(l); l) = T_c,$$

where c_j is defined in (5). Thus our theorem 2 is proved.

§ 4. The general case.

Now we consider the general case. Owing to the conclusion obtained in the previous section, we have only to prove the following theorem:

Theorem 3. *Let Δ be a domain in a real n-space $(x_1, \dots, x_n)$. Moreover, let us suppose that if Δ contains a cross, it also contains the convex hull of the cross. Here the cross means two line-segments intersecting orthogonally at their centers with each other. Then it is concluded that Δ is convex.*

Proof. First we consider the case $n=2$. If Δ is not convex, there exist three points P_0, P_1, P_2 such that Δ contains the line-segments P_0P_1 and P_1P_2 but not P_0P_2. Let X be a point on P_1P_2 moving from P_1 toward P_2 and further let X_0 be the first point of X, for which there is at least one point on P_0X_0 which is not contained in Δ. Let U be the point on P_0X_0, which is nearest to P_0 and lying on the boundary of Δ. Even if, in the neighbourhood of U, P_0X_0 and the boundary of Δ have a line-segment in common, by a small displacement we can construct a line-segment PVY such that

(i) the segment PV is contained in Δ, except V.

(ii) V is the only boundary point of Δ on PY, in a sufficiently small neighbourbood of V.

Let B be a point on VY sufficiently near to V, then B is an inner point of Δ, and the center M of PB lying on PV is also an inner point of Δ. Therefore we can construct a line-segment AMC completely interior to Δ, orthogonal to $PMVBY$ and $AM=MC$. Such PB and AC build up a *cross*, and V is an inner point of its convex hull, i.e. the rhombus $ABCP$. Rotating this cross around M, we finally obtain a cross $P'B'$ and $A'C'$ completely interior to Δ, and having V as the inner point of its convex hull $A'B'C'P'$. This means that V must be an inner point of Δ, which is a contradiction.

In the case for general n, when Δ contains a cross, the section δ of Δ by the plane Π determined by the cross, satisfies the condition of this theorem for $n=2$; therefore it is convex. Since the plane Π is chosen arbitrarily, the domain Δ has the property that its section δ by any plane Π intersecting to Δ is always convex. It is evident that such Δ is convex, and thus our theorem is proved.

Since the base Δ in Lemma 3 satisfies the condition described in Theorem 3, it is convex. Therefore Lemma 2 and then Theorem 1 is completely proved.