109. A Remark on Theorems of Stone and Bochner.

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Recent progress of the theory of topological groups generalizes many theorems of real analysis into locally compact abelian groups. Among them, Raikov-Ambrose's extension of the well-known theorem of Stone reads as follows: *Let $g^{\sharp}$ is a continuous unitary representation of a locally compact abelian group G on a Hilbert space H. Then there exists a resolution of the identity $de(\chi)$ which is defined for all Borel sets of the character group G^* such that*

$$(1) \qquad g^{\sharp} = \int \chi(g) de(\chi) .$$

Combining this and the well-known theorem of Gelfand-Raikov which gives the representation of positive definite functions on G, we can deduce Bochner-Weil-Raikov's representation of positive definite functions putting $d\mu(\chi) = d(\xi e(\chi), \xi)$; that is, *each positive definite function $\varphi(g)$ allows a representation*

$$(2) \qquad \varphi(g) = \int \chi(g) d\mu(\chi) ,$$

where $d\mu(\chi)$ means a regular Borel measure on G^*. Therefore if we can prove Stone's theorem directly, then we can prove Bochner's theorem as in the above.

In this note, we shall do this utilizing recent published theorem of N. Dunford [1], which states that *a commutative C^*-algebra A* (in the sense of I. E. Segal [2]; i.e., a uniformly closed operator algebra on a Hilbert space H) *admits the integral representation*

$$(3) \qquad x = \int_{\Omega} x(\omega) de(\omega) ,$$

where Ω is the compact space of all maximal ideals of A (with the Stone topology), $x(\omega)$ is the representation of x on Ω and $de(\omega)$ is the resolution of the identity defined for all Borel sets of Ω. Although Dunford proved the theorem for compact space, it is not difficult to generalize for locally compact case delecting the identity (we will use this generalized form in the below).

Let $R(G)$ be the operator group algebra of G in the sense of I. E. Segal [2], i.e., the uniform closure of the "regular" representation of $L_1(G)$. Furthermore suppose A be the uniform closure of the operators of the form:

$$(4) \qquad x^{\#} = \int x(g) g^{\#} dg$$

where $x(g)$ being an element of $L_1(G)$. In abelian case, it is pointed out already by K. Yosida [3], that the algebra $R(G)$ is isometrically isomorphic to $C_0(G^*)$ where the later means the set of all continuous functions on G^* vanishing at infinity. It is also known for abelian case A is homomorphic to $R(G)$ and whence to $C_0(G^*)$. Hence by Dunford's theorem there exists a resolution of the dentity $de(\chi)$ on G^* with

$$(5) \qquad x^{\#} = \int x(\chi) de(\chi) ,$$

where

$$(6) \qquad x(\chi) = \int x(g) \chi(g) dg .$$

Hence we have

$$(7) \qquad x^{\#} = \iint x(g) \chi(g) dg \, de(\chi) .$$

Using the Fubini Theorem, we have

$$(8) \qquad x^{\#} = \int x(g) \left[\int \chi(g) de(\chi) \right] dg.$$

Comparing (8) and (4) we have (1) by the continuity of the representation $g^{\#}$. This proves the theorem.

References.

1) N. Dunford: *Resolution of the identity for commutative B^*-algebras of operators*, Acta Szeged., **12** (1950), Pars B, 51–56.

2) I. E. Segal: *Irreducible representations of operator algebras*, Bull. Amer. Math. Soc., **53** (1947), 73–88.

3) K. Yosida, *Normed rings and spectral theorems*, V, Proc. Imp. Acad. Tôkyô, **20** (1944), 269–273.