41. Probability-theoretic Investigations on Inheritance. VIII$_2$. Further Discussionson Non-Paternity Problems.

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2$^{\text{bis}}$. Sub-probability with respect to a type of wife.

We have hitherto considered a probability with respect to each fixed couple. If a frequency of mating is also to be taken into account, the probability has only to be multiplied by a respective mating-frequency; the resulting probability will be, corresponding to (2.2) of VII, denoted by

$$W(ij,\ hk) \equiv \bar{A}_{ij}\bar{A}_{hk}U(ij,\ hk) \qquad (i, j, h, k=1,\dots, m;\ i \leqq j;\ h \leqq k). \tag{2.11}$$

We put further, corresponding to (2.3) of VII,

$$W(ij) = \sum_{h,k} W(ij,\ hk), \tag{2.12}$$

the summation extending over all possible sets of suffices, i.e., $h, k=1,\dots, m;\ h \leqq k$. The quantity $W(ij)$ thus defined represents the sub-probability of proving non-paternity with respect to the fixed type A_{ij} of wives. As already noticed in § 1, it must coincide just with the quantity introduced in (2.3) of VII; namely, the identical relation holds:

$$W(ij) = P(ij). \tag{2.13}$$

We shall now verify in a direct manner the validity of the identity (2.13), to make sure. For that purpose, we first consider a homozygotic wife A_{ii}. We then get, corresponding to (2.12) of VII,

$$W(ii) = W(ii, ii) + \sum_{h \neq i}(W(ii, ih) + W(ii, hh)) + \sum_{h, k \neq i}{}' W(ii, hk). \tag{2.14}$$

Substituting the respective values of (2.11) obtained by (2.2) to (2.5) into the right-hand side of (2.14) and then remembering the first relation (1.16) of VII, we get

$$\begin{aligned} W(ii) &= p_i^4(1-p_i) + \sum_{h \neq i}(2p_i^3p_h(1-p_i-p_h) + p_i^2p_h^2(1-p_h)) \\ &\qquad + \sum_{h, k \neq i}{}' 2p_i^2p_hp_k(1-p_h-p_k) \\ &= p_i^2\{p_i^2(1-p_i) + 2p_i((1-p_i)^2-(S_2-p_i^2)) + S_2 - p_i^2 - (S_3-p_i^3) \\ &\qquad + 1 - 2S_2 - 2p_i(1-p_i-S_2) - (S_2-2S_3) + 2p_i^2(1-2p_i)\} \\ &= p_i^2(1-2S_2+S_3), \end{aligned} \tag{2.15}$$

S_ν denoting the power-sum defined in (1.2) of VII.

Next, for a heterozygotic wife $A_{ij}(i\neq j)$, we get, corresponding to (2.15) of VII,

$$\begin{aligned}(2.16)\quad W(ij)=&W(ij,ii)+W(ij,jj)+W(ij,ij)\\&+\sum_{h\neq i,j}(W(ij,ih)+W(ij,jh)+W(ij,hh))+\sum_{h,k\neq i,j}{}'W(ij,hk)\quad (i\neq j).\end{aligned}$$

Hence, we get similarly from the values obtained in (2.6) to (2.10), by taking the second relation (1.16) of VII into account,

$$\begin{aligned}(2.17)\quad W(ij)=&2p_i^3p_j(1-p_i-\tfrac{1}{2}p_j)+2p_ip_j^3(1-\tfrac{1}{2}p_i-p_i)+4p_i^2p_j^2(1-p_i-p_j)\\&+\sum_{h\neq i,j}(4p_i^2p_jp_h(1-p_i-\tfrac{1}{2}p_j-p_h)+4p_ip_j^2(1-\tfrac{1}{2}p_i-p_j-p_h)\\&+2p_ip_jp_h^2(1-p_h))+\sum_{h,k\neq i,j}{}'4p_ip_jp_hp_k(1-p_h-p_k)\\=&p_ip_j\{p_i^2(2-2p_i-p_j)+p_j^2(2-p_i-2p_j)+4p_ip_j(1-p_i-p_j)\\&+2p_i((2-2p_i-p_j)(1-p_i-p_j)-2(S_2-p_i^2-p_j^2))\\&+2p_j((2-p_i-2p_j)(1-p_i-p_j)-2(S_2-p_i^2-p_j^2))\\&+2(S_2-p_i^2-p_j^2-(S_3-p_i^3-p_j^3))+2(1-2S_2)\\&-4p_i(1-p_i-S_2)-4p_j(1-p_j-S_2)-2(S_2-2S_3)\\&+4p_i^2(1-2p_i)+4p_j^2(1-2p_j)+4p_ip_j(1-p_i-p_j)\}\\=&p_ip_j\{2(1-2S_2+S_3)-4p_ip_j+3p_ip_j(p_i+p_j)\}\quad (i\neq j).\end{aligned}$$

It has thus been verified that (2.15) and (2.17) coincide just with (2.14) and (2.18) of VII, respectively, and hence the idendity (2.13) is valid in general. Consequently, as shown in § 2 of VII, the *whole probability of proving non-paternity* given by

$$(2.18)\qquad W=\sum_{i=1}^{m}W(ii)+\sum_{i,j}{}'W(ij)$$

is nothing but the quantity P obtained in (2.20) of VII, that is,

$$(2.19)\qquad W=1-2S_2+S_3-2S_2^2+2S_4+3S_2S_3-3S_5.$$

The results similar to (3.1) to (3.3), (3.5) and (3.7) of VII can also be derived. First, the partial sum over all couples consisting of a wife and her husband both of the same homozygote becomes

$$(2.20)\qquad \sum_{i=1}^{m}W(ii,ii)=\sum_{i=1}^{m}p_i^4(1-p_i)=S_4-S_5.$$

The partial sum over all couples consisting of a homozygotic wife and her heterozygotic husband possessing a gene in common with her becomes

$$\begin{aligned}(2.21)\quad \sum_{i=1}^{m}\sum_{h\neq i}W(ii,ih)&=\sum_{i=1}^{m}\sum_{h\neq i}2p_i^3p_h(1-p_i-p_h)\\&=\sum_{i=1}^{m}2p_i^3((1-p_i)^2-(S_2-p_i^2))=2S_3-4S_4-2S_2S_3+4S_5.\end{aligned}$$

The partial sum over couples of different homozygotes becomes

$$\sum_{i=1}^{m}\sum_{h\neq i}W(ii,hh)=\sum_{i=1}^{m}\sum_{h\neq i}p_i^2p_h^2(1-p_h)$$
$$=\sum_{i=1}^{m}p_i^2(S_2-p_i^2-(S_3-p_i^3))=S_2^2-S_4-S_2S_3+S_5. \tag{2.22}$$

The partial sum over couples consisting of a homozygotic wife and her husband possessing no gene in common with her becomes

$$\sum_{i=1}^{m}\sum_{h,k\neq i}'W(ii,hk)=\sum_{i=1}^{m}\sum_{h,k\neq i}'2p_i^2p_hp_k(1-p_h-p_k)$$
$$=\sum_{i=1}^{m}p_i^2(1-2S_2-2p_i(1-p_i-S_2)-(S_2-2S_3)+2p_i^2(1-2p_i)) \tag{2.23}$$
$$=S_2-2S_3-3S_2^2+4S_4+4S_2S_3-4S_5.$$

To the partial sums corresponding to heterozygotic wives, similar considerations can be applied, which lead to the following results:

$$\sum_{i,j}'(W(ij,ii)+W(ij,jj))$$
$$=\sum_{i,j}'p_ip_j(p_i^2(2-2p_i-p_j)+p_j^2(2-p_i-2p_j)) \tag{2.24}$$
$$=2S_3-4S_4-S_2S_3+3S_5,$$

$$\sum_{i,j}'W(ij,ij)=\sum_{i,j}'4p_i^2p_j^2(1-p_i-p_j)$$
$$=2S_2^2-2S_4-4S_2S_3+4S_5, \tag{2.25}$$

$$\sum_{i,j}'\sum_{h\neq i,j}(W(ij,ih)+W(ij,jh))$$
$$=\sum_{i,j}'\sum_{h\neq i,j}(4p_i^2p_jp_h(1-p_i-\tfrac{1}{2}p_j-p_h)+4p_ip_j^2p_h(1-\tfrac{1}{2}p_i-p_j-p_h))$$
$$=2\sum_{i,j}'(p_i((2-2p_i-p_j)(1-p_i-p_j)-2(S_2-p_i^2-p_j^2)) \tag{2.26}$$
$$+p_j((2-p_i-2p_j)(1-p_i-p_j)-2(S_2-p_i^2-p_j^2)))$$
$$=4S_2-12S_3-10S_2^2+22S_4+16S_2S_3-20S_5,$$

$$\sum_{i,j}'\sum_{h\neq i,j}W(ij,hh)=\sum_{i,j}'\sum_{h\neq i,j}2p_ip_jp_h^2(1-p_h)$$
$$=2\sum_{i,j}'p_ip_j(S_2-p_i^2-p_j^2-(S_3-p_i^3-p_j^3)) \tag{2.27}$$
$$=S_2-3S_3-S_2^2+4S_4+S_2S_3-2S_5,$$

$$\sum_{i,j}'\sum_{h,k\neq i,j}'W(ij,hk)=\sum_{i,j}'\sum_{h,k\neq i,j}'4p_ip_jp_hp_k(1-p_h-p_k)$$
$$=2\sum_{i,j}'p_ip_j(1-2S_2-2p_i(1-p_i-S_2)-2p_i(1-p_j-S_2)$$
$$-(S_2-2S_3)+2p_i^2(1-2p_i)+2p_j^2(1-2p_j)+2p_ip_j(1-p_i-p_j)) \tag{2.28}$$
$$=1-8S_2+14S_3+9S_2^2-18S_4-10S_2S_3+12S_5.$$

The relations (2.20) to (2.28) constitute just our desired results. That the total sum of these quantities gives again the whole probability given in (2.19) is a matter of course. It will also be noticed that the sum of coefficients in every expression for partial sum is equal to zero. The reason may be explained quite similarly as in § 3 of VII.

The results derived in the present section will be listed in the following table. Usual agreement is here also made with respect to suffices.

Wife	Husband	Child of wife denied by husband	Freq. of mating	Freq. of deniable child
A_{ii}	A_{ii}	$A_{ij}\,(j\neq i)$	p_i^4	$1-p_i$
	A_{ih}	$A_{ij}\,(j\neq i,\,h)$	$2p_i^3p_h$	$1-p_i-p_h$
	A_{hh}	$A_{ij}\,(j\neq h)$	$p_i^2p_h^2$	$1-p_h$
	A_{hk}	$A_{ij}\,(j\neq h,\,k)$	$2p_i^2p_hp_k$	$1-p_h-p_k$.
A_{ij}	A_{ii}	$A_{ik}(k\neq i,\,j),\ A_{jk}(k\neq i)$	$2p_i^3p_j$	$1-p_i-\frac{1}{2}p_j$
	A_{jj}	$A_{ik}(k\neq j),\ A_{jk}(k\neq i,\,j)$	$2p_ip_j^3$	$1-\frac{1}{2}p_i-p_j$
	A_{ij}	$A_{ik}(k\neq i,\,j)\ A_{jk}(k\neq i,\,j)$	$4p_i^2p_j^2$	$1-p_i-p_j$
	A_{ih}	$A_{ik}(k\neq i,\,j,\,h),\ A_{jk}(k\neq i,\,h)$	$4p_i^2p_jp_h$	$1-p_i-\frac{1}{2}p_j-p_h$
	A_{jh}	$A_{ik}(k\neq j,\ h),\ A_{jk}(k\neq i,\,j,\,h)$	$4p_ip_j^2p_h$	$1-\frac{1}{2}p_i-p_j-p_h$
	A_{hh}	$A_{ik}(k\neq h),\ A_{jk}(k\neq h)$	$2p_ip_jp_h^2$	$1-p_h$
	A_{hk}	$A_{il}(l\neq h,\,k),\ A_{jl}(l\neq h,\,k)$	$4p_ip_jp_hp_k$	$1-p_h-p_k$

Sub-prob. against each mating	Sub-prob. against each wife	Partial sum over matings
$p_i^4(1-p_i)$	$p_i^2(1-2S_2+S_3)$	S_4-S_5
$2p_i^3p_h(1-p_i-p_h)$		$2S_3-4S_4-2S_2S_3+4S_5$
$p_i^2p_h^2(1-p_h)$		$S_2^2-S_4-S_2S_3+S_5$
$2p_i^2p_hp_k(1-p_h-p_k)$		$\begin{cases} S_2-2S_3-3S_2^2 \\ +4S_4+4S_2S_3-4S_5 \end{cases}$
$p_i^3p_j(2-2p^i-p_j)$	$p_ip_j(2(1-2S_2+S_3)$ $-4p_ip_j+3p_ip_j(p_i+p_j))$	$2S_2-4S_4-S_2S_3+3S_5$
$p_ip_j^3(2-p_i-2p_j)$		
$4p_i^2p_j(1-p_i-p_j)$		$2S_2^2-2S_4-4S_2S_3+4S_5$
$2p_i^2p_jp_h(2-2p_i-p_j-2p_h)$		$\begin{cases} 4S_2-12S_3-10S_2^2 \\ +22S_4+16S_2S_3-20S_5 \end{cases}$
$2p_ip_j^2p_h(2-p_i-2p_j-2p_h)$		
$2p_ip_jp_h^2(1-p_h)$		$\begin{cases} S_2-3S_3-S_2^2 \\ +4S_4+S_2S_3^2-2S_5 \end{cases}$
$2p_ip_jp_hp_k(1-p_h-p_k)$		$\begin{cases} 1-8S_2+14S_3+9S_2^2 \\ -18S_4-10S\ S_3+12S_5 \end{cases}$

$$W=1-2S_2+S_3-2S_2^2+2S_4+3S_2S_3-3S_5$$

—To be continued—