56. On Quasi-Denjoy Integration

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1. Introduction and terminology. We are concerned with showing that the Denjoy-Khintchine process of integration for functions of one real variable is capable of an essential generalization. The gist of our theory consists in replacing the class of the generalized absolutely continuous functions by a broader one composed of the functions called generalized highly continuous in our phrasing.

As regards general terminology (and notation), we shall conform on the whole to the *Theory of the Integral* by Saks, except in certain minor points. The mentioned treatise will be quoted hereafter simply as Saks for short. The conventions that follow will be valid throughout. By *sets* and *intervals*, by themselves, we shall always understand linear sets and linear non-degenerate intervals respectively, where intervals may be infinite (i.e. unbounded). The epithets *open* and *closed* for intervals will as usual be applied only to finite intervals. The term *function* will stand exclusively for a point-function defined on the whole real line and assuming finite real values, unless another meaning is obvious from the context. Finally, a *sequence* will mean a nonvoid countable one, finite or infinite.

2. Semiabsolutely and strongly semiabsolutely continuous functions. Let $F(x)$ be a function, E a set, and α a number such that $0<\alpha\leqq 1$. We say that F is *semiabsolutely continuous* (α) on E, or briefly SC(α) on E, iff (i.e. if and only if) given any $\varepsilon>0$ there is an $\eta>0$ such that for every finite sequence of non-overlapping closed intervals $I_1,\cdots,I_n$ whose extremities belong to E, the inequality

$$|I_1|^\alpha+\cdots+|I_n|^\alpha<\eta \quad \text{implies} \quad |F(I_1)|+\cdots+|F(I_n)|<\varepsilon.$$

Remember that whenever I is a closed interval, $F(I)$ denotes the increment of the function $F(x)$ over I, while the image of I under F will be written $F[I]$ (see Saks, p. 99 and p. 100). When especially $\alpha=1$, the notion just introduced plainly reduces to the absolute continuity on E of the function F (Saks, p. 223).

We say further that F is *strongly semiabsolutely continuous* (α), or SSC(α), on E, iff F is SC(α) on E and moreover the set E is of α-dimensional volume zero (Saks, p. 54). When this is the case, E is evidently a set of measure zero (in the Lebesgue sense).

The reference to the exponent α will be omitted in the above two notions when we are interested only in the existence of α, not in its peculiar value. Thus a function F is called to be SC or SSC on a set E, iff there is in the interval $(0, 1]$ an exponent α (depending in general on both the function and the set) for which F is respectively SC(α) or SSC(α) on E.

Let us enumerate some simple properties of semiabsolutely continuous functions. The exponent α will be kept fixed. (i) *Every function which is* SC(α) *on a set* E *is so too on all the subsets of* E *and is further* SC(β) *on* E *for all the exponents* β *of the interval* $(0, \alpha)$. (ii) *Every function which is continuous on a nonvoid set* E *and which is* SC(α) *on a subset everywhere dense in* E, *is* SC(α) *on the whole set* E. (iii) *A function which is* SC(α) *on a bounded set is always bounded on this set.* (iv) *Every linear combination, with constant coefficients, of two functions which are* SC(α) *on a bounded set, and the product of such functions, are themselves* SC(α) *on the same set.*

3. Highly and generalized highly continuous functions. We shall term a function $F(x)$ to be *highly continuous* (or HC) on a set E, iff F is either AC (absolutely continuous) on E or SSC on E. Again, F will be called to be *generalized highly continuous* (or GHC) on E, iff (a) the function F is continuous on E and (b) the set E is expressible as the union of a sequence of sets on each of which F is HC.

Proposition (iv) of the preceding section has now the following analogue: *Every linear combination of two functions which are* HC [*or* GHC] *on a bounded set, and the product of such functions, are themselves* HC [*or* GHC] *on this set.* The proof is immediate.

THEOREM. *Every function which is* GHC *on a measurable set is approximately derivable at almost all points of this set.*

PROOF. If we replace, in the enunciation of the theorem, the symbol GHC by GAC (i.e. generalized absolutely continuous), the result is a known proposition (Saks, p. 223). Keeping this in mind, suppose that a function F is GHC on a measurable set E and choose, as we clearly can, a subset E_0 of E such that F is GAC on E_0 and that $E-E_0$ is of measure zero. Since E_0 is then measurable together with E, it follows at once that F is approximately derivable at almost every point of the set E_0, and hence, at almost every point of E itself.

4. An auxiliary theorem. We begin with a simple definition. The union of the first n elements of a sequence of sets is called its *n-th partial union.* Needless to say, if the sequence is finite and consists of k terms, the number n can only range over $1, 2, \cdots, k$.

LEMMA. *Given a distinct sequence* Δ *of open intervals whose union* $[\Delta]$ *is an interval, it is always possible to extract from among the elements of* Δ *a distinct sequence* $\Delta^* = \langle A_1, A_2, \cdots \rangle$ *which* *covers $[\Delta]$ already and each of whose partial unions is an interval.*

REMARK. The sequence Δ^* need not be a subsequence of Δ.

PROOF. In the first place we choose for A_1 any interval of the sequence Δ. If it happens that $A_1=[\Delta]$, the singletonic sequence $\langle A_1\rangle$ will clearly serve our purpose. If the contrary is the case, there must exist in Δ one or more intervals intersecting A_1 and not lying in A_1. For by hypothesis the union $[\Delta]$ is connected. Let A_2 be the first one (in the order of Δ) among such intervals. If then $A_1\smile A_2=[\Delta]$, the sequence $\langle A_1, A_2\rangle$ has the required property. In the opposite case we choose from Δ the first interval A_3 intersecting the interval $A_1\smile A_2$ and not contained in $A_1\smile A_2$. And so on we proceed as long as this procedure is practicable. If our construction comes at an end after a finite number of steps, we have nothing more to prove. We may therefore assume in what follows that the sequence $\Delta^*=\langle A_1, A_2,\cdots\rangle$ thus obtained is infinite. Since evidently Δ^* is distinct and all its partial unions are intervals, it only remains to ascertain that $[\Delta^*]=[\Delta]$.

Suppose, if possible, that the contrary is true. By connectedness of $[\Delta]$ there then exists in Δ an interval, say A, which intersects $[\Delta^*]$ without being covered by Δ^*. Therefore, for sufficiently large n, say for $n\geqq n_0$, the n-th partial union of Δ^* always intersects A without containing A. Recalling our construction of Δ^* we conclude that for $n>n_0$ the interval A_n cannot appear in Δ later than A. But this plainly contradicts the distinctness of the sequence Δ^*, and the proof is complete.

THEOREM. *Given a set E and a positive number ξ, suppose that $\Lambda_\xi(E)$, i.e. the ξ-dimensional volume of E, is finite. Then for any $\varepsilon>0$ we can cover the set E by a non-overlapping sequence of closed intervals $I_1, I_2,\cdots$ with diameters $<\varepsilon$ and such that*

$$|I_1|^\xi+|I_2|^\xi+\cdots<\Lambda_\xi(E)+\varepsilon.$$

PROOF. By definition of $\Lambda_\xi(E)$ the set E admits a covering by a distinct sequence Θ of open intervals $J_1, J_2,\cdots$ with diameters $<\varepsilon$, in such a way that $|J_1|^\xi+|J_2|^\xi+\cdots<\Lambda_\xi(E)+\varepsilon$. As is readily seen, we may suppose here that each interval of the sequence Θ is maximal in Θ, i.e. that no J_p is contained in any other J_q. Consider an arbitrary connected component, say D, of the union $[\Theta]$, so that D is an endless interval, i.e. an interval which is an open set. We denote by Δ the subsequence of Θ which consists of all the intervals of Θ that are contained in D. Since plainly $[\Delta]=D$, we can apply the above lemma to the sequence Δ. Thus D is already covered by a distinct sequence $\Delta^*=\langle A_1, A_2,\cdots\rangle$ consisting exclusively of intervals of Δ and all whose partial unions $B_1, B_2,\cdots$ are intervals. Here the sequence $B_1, B_2,\cdots$ may without loss of generality be assumed to be strictly ascending.

Now write $C_1=B_1$ and further $C_n=B_n-B_{n-1}$ for $n>1$ as long as B_n exists. It follows from our choice of Δ^* that $C_1, C_2, \cdots$ then constitute a disjoint sequence of finite intervals which lie respectively in $A_1, A_2, \cdots$ and which together cover D. Replace each of these intervals $C_1, C_2, \cdots$ by its closure and denote by Ψ_D the resulting non-overlapping sequence of closed intervals, the subscript D indicating dependence on the interval D. Writing I generically for an interval which appears in the sequence Ψ_D for some D, we arrange all the intervals I in a distinct sequence $I_1, I_2, \cdots$ and find at once that this sequence conforms to the assertion.

5. Further theorems on GHC functions. Whenever we speak henceforward of an *exponent*, let it be tacitly understood that its value shall belong, just as in §2, to the half-open interval $(0, 1]$.

A function $F(x)$ is said to be *Lusinian* (α) on a set E, where α is an exponent, iff for every set $X \subset E$ of α-dimensional volume zero, the image $F[X]$ of X under the function F is of measure zero. When in particular $\alpha=1$, this condition agrees with the condition (N) of Lusin (see Saks, p. 224). Clearly, *a function is Lusinian* (α) *on the union of a sequence of sets whenever the function is so on each of the constituent sets.* Again, *every function which is Lusinian* (α) *on a set is Lusinian* (β) *on this set for all the exponents* β *of the interval* $(0, \alpha)$.

As is stated and proved on p. 225 of Saks, (a) *every function which is* GAC *on a set fulfils the condition* (N) *on this set*, and (b) *if the approximate derivative of a function which is* GAC *on a closed interval* I *is nonnegative almost everywhere on* I*, then the function is monotone non-decreasing on* I. These two important theorems will now be extended in what follows to the class of GHC functions. Our notion of functions Lusinian on a set is intended to be a machinery subservient to this purpose.

LEMMA. *A function which is* SC(α) *on a set for an exponent* α *is necessarily Lusinian* (α) *on this set.*

PROOF. This may be established as for proposition (a) quoted above, only we make use of the theorem of §4 in the proof.

THEOREM (i). *Every function* F *which is* GHC *on a set* E *fulfils on* E *the condition* (N) *of Lusin.*

PROOF. Without loss of generality we may suppose the function F to be highly continuous on E, so that F is either AC on E or else SSC(α) on E for some exponent α. If the first alternative takes place, F satisfies the condition (N) on E by proposition (a) above. On the other hand, the second alternative means that F is SC(α) on E and that, moreover, E is of α-dimensional volume zero. Then, by the above lemma, the image $F[E]$ must be a set of measure zero. This completes the proof.

THEOREM (ii). *Every function which is* GHC *on a closed interval* I *and whose approximate derivative is nonnegative almost everywhere on* I*, is monotone non-decreasing on* I*. In particular therefore, if the approximate derivatives of two functions which are* GHC *on* I *coincide almost everywhere on* I*, then the functions themselves coincide on* I *identically apart from an additive constant.*

PROOF. Using Theorem (i) just obtained, we may prove this as for proposition (b) quoted above from Saks.

6. Descriptive definition of the quasi-Denjoy integral. Let $f(x)$ be an extended-real function defined over the real line. We term f to be $\mathbf{Q}$-*integrable* on a closed interval I iff there exists a function $F(x)$ which is GHC on I and which has $f(x)$ for its approximate derivative almost everywhere on I (so that f must be finite almost everywhere on I by the theorem of §3). We then say that, on the interval I, the function F is an *indefinite* $\mathbf{Q}$-*integral* of f. Its increment $F(I)$ over I is called *definite* $\mathbf{Q}$-*integral* of f over I and is denoted by $\mathbf{Q}(f; I)$.

The symbol $\mathbf{Q}$ has been used above as an abbreviation for the term *quasi-Denjoy*. For uniformity of notation, we shall also employ the symbols $\mathbf{L}$ and $\mathbf{D}$ to mean the epithets *Lebesgue* and *Denjoy-Khintchine* respectively.

The definite integral $\mathbf{Q}(f; I)$, when existent, is uniquely determined. For, by Theorem (ii) of the foregoing §, any two functions which are, on the interval I, indefinite integrals of f, can only differ on I by an additive constant. More generally, *if two extended-real functions are equal almost everywhere on a closed interval* I *and the one is* $\mathbf{Q}$-*integrable on* I*, then so is the other and the two functions have the same definite integral on* I.

It is evident that every function which is GAC on a set is GHC on the same set. In consequence, *an extended-real function* f *which is* $\mathbf{D}$-*integrable on a closed interval* I *is always* $\mathbf{Q}$-*integrable on* I *and we have* $\mathbf{Q}(f; I)=\mathbf{D}(f; I)$. Furthermore, we deduce easily with the aid of Theorem (ii) just quoted that *every extended-real function which is* $\mathbf{Q}$-*integrable and almost everywhere nonnegative on a closed interval is* $\mathbf{L}$-*integrable on this interval.*

Another property of our integral worth while mentioning is that it is a linear functional of the integrand. Thus, *if two finite functions* g *and* h *are both* $\mathbf{Q}$-*integrable on a closed interval* I*, the same is true of any linear combination* $ag+bh$*, with constant coefficients, of these functions and we have*

$$\mathbf{Q}(ag+bh; I)=a\cdot\mathbf{Q}(g; I)+b\cdot\mathbf{Q}(h; I).$$

7. Example. We proceed now to ascertain, by constructing a concrete family of GHC functions, that our integration is actually more comprehensive than Denjoy-Khintchine integration. Let δ be a fixed exponent other than 1 (so that $0<\delta<1$) and let us consider any closed interval $I=[a, b]$. We take in the interior of I a strictly increasing infinite sequence of points $a_2<a_3<\cdots$ tending to the point b, and we write $a_1=a$ for uniformity of notation. It is easy to see that if this sequence $a_2<a_3<\cdots$ is suitably chosen, then there exists a nonnegative continuous function $P(x)$ vanishing outside the interval I and fulfilling the following five conditions (the letter n stands for a natural number throughout this section):

(i) The function P is a constant on each of the closed intervals $[a_{2n-1}, a_{2n}]$; (ii) $P(x)$ is linear in x, but not a constant, on each of the intervals $[a_{2n}, a_{2n+1}]$; (iii) $P(x)$ is not of bounded variation on I; (iv) we have the inequality $|P(J)|<|J|^{\delta}$ for every closed interval J (which need not lie in I); (v) the sum, for all n, of the power $(a_{2n}-a_{2n-1})^{\delta}$ is less than $(1/2)\cdot|I|^{\delta}$.

Suppose that for each $I=[a, b]$ the sequence $a_2<a_3<\cdots$ and the function $P(x)$ have been uniquely chosen so as to conform to the above stipulation. To avoid ambiguity, we shall write $P(I; x)$ instead of $P(x)$ henceforth. It is convenient to introduce here a temporary concept. Given any continuous function $F(x)$, let K denote generically a maximal closed interval situated in the unit interval $U=[0, 1]$ and on which $F(x)$ is a constant. (Of course such intervals K need not always exist.) By the *indentation* of the function F we then understand the sum $G(x)=\sum P(K; x)$, where the summation extends over all the intervals K and where a possible void sum means zero. As we may verify at once, the indentation G thus associated with F is a continuous nonnegative function vanishing outside U and fulfilling $|G(J)|<|J|^{\delta}$ for every closed interval J. We therefore have also $G(x)<1$ for every x.

Now, we construct an infinite sequence of nonnegative continuous functions $P_1(x), P_2(x), \cdots$ by induction as follows: $P_1(x)$ means the identically vanishing function, and for each n the function $P_{n+1}(x)$ means the sum $P_n(x)+2^{-n}H_n(x)$, where H_n stands for the indentation of P_n. When $n\to+\infty$, the function $P_n(x)$ plainly tends uniformly to a definite limiting function nonnegative and continuous, which we denote by $P_0(x)$. *The function* P_0 *thus defined is* GHC *on* U (*and even on the whole real line*) *without being* GAC *on* U, *and so the approximate derivative of* P_0 *is* **Q**-*integrable on* U *without being* **D**-*integrable on* U. The proof for this will be given elsewhere.