33. On A Characterization of Abelian Varieties

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Let G, G' be two group varieties, f_0 a rational homomorphism of G into G', and a a point of G'. Then $f(x)=f_0(x)\cdot a$ for the point x of G, is a rational mapping of G into G'. We shall write more simply $f=f_0\cdot a$. (The same rational mapping f can be also expressed in the form $f=a\cdot f'_0$, where $f'_0=a^{-1}\cdot f_0\cdot a$ is another rational homomorphism of G into G'.) We shall call a rational mapping f which is expressible in the form $f_0\cdot a$ (or $a\cdot f'_0$) a *mapping of type HT* (homomorphism plus translation).

One of the fundamental theorems on abelian varieties asserts that every rational mapping of an abelian variety A into another abelian variety B is a mapping of type HT (cf. [1] Theorem 9). In this theorem, the abelian variety A can be replaced by any group variety G, as was shown by S. Lang [2]. In the present note, we shall prove the converse of this fact in the following sense: Let G, G' be two group varieties. If every rational mapping of G into G' is of type HT, then G' must be an abelian variety.

We shall use the following terminologies and notations. A *homomorphism* of a group variety into a group variety will always mean a rational homomorphism. A *linear group* will always mean a linear algebraic group. A *biregular isomorphism* between group varieties is a group isomorphism defined by a birational mapping which we shall denote by $\cong$. G_a denotes an affine line with the law of composition $z=x+y$, and G_m an affine line, from which the origin is excluded, with the law of composition $z=x\cdot y$. A connected linear group of dimension 1 is isomorphic to G_a or G_m (cf. [1] p. 69). G_a and G_m can be defined over any field k, and their generic points over k are those which have transcendental elements over k as their coordinates. We denote the characteristic of the universal domain by p.

We shall begin with some lemmas.

LEMMA 1. *Every linear group L of dimension $n>0$ has a linear subgroup of dimension 1.*

PROOF. We may assume L as connected. Let L_0 be the Borel subgroup, i.e. the maximal closed solvable connected subgroup, of L, then L/L_0 is a projective variety (cf. Borel [3] Theorem 16.5), so dim $L_0>0$. As L_0 is solvable, L_0 has a linear subgroup of dimension 1.

LEMMA 2. 1) *Let L_1 and L_2 be connected linear groups of* *dimensions* n_1 *and* n_2 *respectively.* $n_1, n_2 > 0$. *Then there exists a rational mapping from* L_1 *into* L_2 *which is not of type HT.*

2) *Let* A *be an abelian variety of dimension* m, L *a connected linear group of dimension* n. $m, n > 0$. *Then there exists a rational mapping from* A *into* L *which is not of type HT.*

PROOF. 1) Let G be a connected linear subgroup of dimension 1 of L_2, k a common field of definition for L_1, L_2, G and the birational correspondence between G and G_a (or G_m), which is not a prime field. Let x be a generic point of L_1 over k, then $k(x)$ is a regular extension of k of dimension >0. Therefore there exists an element t in $k(x)$ such that $k(t)$ is a purely transcendental extension of k of dimension 1. So we can choose a generic point y of G over k to satisfy the equation $k(y)=k(t)$. Then we may write $g(x)=y$, where g is a non-constant rational mapping from L_1 into L_2.

Now assume that every rational mapping from L_1 into L_2 is of type HT, so that above rational mapping g is expressible as $g_0 \cdot a$, where g_0 is a homomorphism and a is a fixed point of L_2. Let e_1, e_2 be the unit of L_1, L_2 respectively, then $g(e_1)=g_0(e_1)\cdot a=e_2\cdot a=a$. Therefore a is a point of G, and $g_0=g\cdot a^{-1}$ is a generically subjective homomorphism of L_1 into G.

First we consider the case in which $G \cong G_a$. Let q be a natural number which is neither p^s nor 1 (where s is a natural number). Let t be a coordinate of a generic point of G_a over k, then there is a generic point of G_a which has t^q as its coordinate. Let y' be a point of G which corresponds to the point t^q. Next consider the case in which $G \cong G_m$. Let r be an element of k which is neither -1 nor 0. Let t be a coordinate of a generic point of G_m over k, then there is a generic point of G_m which has $t+r/1+r$ as its coordinate. Let y' be a point of G which corresponds to a point $t+r/1+r$. In both cases, we shall write $h(y)=y'$. Then h is a non-constant rational mapping of G into G, and $h(e_2)=e_2$ for the units e_2 of G. But h is not a homomorphism.

As $h\cdot g_0$ is a rational mapping of L_1 into L_2, it must be of type HT by our assumption, and $h\cdot g_0(e_1)=h(e_2)=e_2$, i.e. $h\cdot g_0$ is a homomorphism, therefore so should be h, which is a contradiction.

2) Let G be a connected linear subgroup of dimension 1 of L, k a common field of definition for A and L. Let x be a generic point of A over k, then $k(x)$ is a regular extension of k of dimension >0. Therefore there exists an element t in $k(x)$ such that $k(t)$ is a purely transcendental extension of k of dimension 1. So we can choose a generic point y of G over k to satisfy the equation $k(y)=k(t)$. Then we may write $g(x)=y$, where g is a non-constant rational mapping from A into L. On the other hand it is known

that any homomorphism of an abelian variety into a linear group is trivial (cf. Rosenlicht [4] Theorem 11). So g is not of type HT.

THEOREM. *Let G and G' be given group varieties, and the dimension n of G be >0. Every rational mapping of G into G' is of type HT if and only if G' is an abelian variety.*

PROOF. We have only to prove the only-if part.

First we consider the case in which G is a connected linear group. Let L' be a maximal connected linear normal algebraic subgroup of G', then G'/L' is an abelian variety (cf. Rosenlicht [4] Theorem 16). If the dimension of L' is not zero, there exists a rational mapping from the linear group G into L' which is not of type HT (Lemma 2, 1)). This is a contradition. So $L'=\{e'\}$, i.e. G' is an abelian variety.

Next consider the case in which G is an abelian variety. Let L' be a maximal connected linear normal algebraic subgroup of G'. Then G'/L' is an abelian variety. If the dimension of L' is not zero, there exists a rational mapping from the abelian variety G into L' which is not of type HT (Lemma 2. 2)). This is a contradiction. So $L'=\{e'\}$, i.e. G' is an abelian variety.

The general case. Let L be a maximal connected linear normal algebraic subgroup of G, then G/L is an abelian variety which we call A. Let τ be a canonical homomorphism of G onto A. If the dimension of A is zero, $G=L$, i.e. G is a connected linear group. This was treated as the first case. If the dimension of A is not zero, let f be a rational mapping of A into G'. Then $f\cdot\tau$ is a rational mapping from G into G'. Therefore, by the assumption, $f\cdot\tau$ is of type HT, and so must be f. This was treated as the second case. Thus G' is an abelian variety in every case.

Remark. In the above theorem, we can not replace "Every rational mapping of G into G'" by "Any everywhere defined rational mapping of G into G'". For example, let G be an abelian variety, G' be G_a, then everywhere defined rational mapping of G into G' is a constant, which is of course of type HT.

References

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