170. A Pentavalued Logic and its Algebraic Theory

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In this Note, we shall concern with a pentavalued logic by J. Lukasiewicz and its algebraic theory. The fundamental ideas are due to Professor Gr. C. Moisil (see [1] and [2]).

Let L be a set $\{x, y, z, \cdots\}$ of propositions. The truth values we denote by 0, 1, 2, 3, and 4. We introduce the negation Nx of x by

$$\begin{array}{c|c} x & 0, 1, 2, 3, 4 \\ \hline Nx & 4, 3, 2, 1, 0 \end{array}.$$

The disjunction $\vee$ and the conjunction $\wedge$ are defined as follows:

$\vee$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	3	4
2	2	2	2	3	4
3	3	3	3	3	4
4	4	4	4	4	4

$\wedge$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	2	2	2
3	0	1	2	3	3
4	0	1	2	3	4

Then two operations $\vee$, $\wedge$ satisfy the well known axiom of a distributive lattice, hence L is a distributive lattice. On the other hand, consider a commutative ring of characteristic 5, and denote the elements by 0, 1, 2, 3, and 4. Then we have $5x=x+x+x+x+x=0$, and $x^5=xxxxx=x$.

The negation and the modalities have the following algebraic representations.

The negation Nx of x is algebraically denoted by $Nx=4(x+1)$.

The necessity

$$\begin{array}{c|c} x & 0\ 1\ 2\ 3\ 4 \\ \hline \nu x & 0\ 0\ 0\ 0\ 4 \end{array}$$

and the possibility

$$\begin{array}{c|c} x & 0\ 1\ 2\ 3\ 4 \\ \hline \mu x & 0\ 0\ 1\ 1\ 1 \end{array}$$

are denoted by $\nu x=x^4+4x^3+x^2+4x$ and $\mu x=3x^4+4x^3+4x^2+4x$ respectively.

In the pentavalued logic, there are two positive modalities as follows:

$$\begin{array}{c|c} x & 0\ 1\ 2\ 3\ 4 \\ \hline N\mu Nx & 0\ 0\ 0\ 4\ 4 \\ N\nu Nx & 0\ 4\ 4\ 4\ 4 \end{array}$$

These modalities are algebraically denoted by $N\mu Nx = 2x^4 + 2x^3 + x$, $N\nu Nx = 4x^4$.

Further there are following four modalities containing the negation functor N.

x	0	1	2	3	4
$N\nu x$	4	4	4	4	0
$N\mu x$	4	4	0	0	0
νNx	4	0	0	0	0
μNx	4	4	4	0	0

These are denoted by

$$N\nu x = 4x^4 + x^3 + 4x^2 + x + 4,$$
$$N\mu x = 2x^4 + x^3 + x^2 + x + 4,$$
$$\nu Nx = x^4 + 4,$$

and

$$\mu Nx = 3x^4 + 3x^2 + 4x + 4.$$

The conjunction $x \wedge y$ is expressed by the polynomial

$$x \wedge y = (3x^2y^2 + 4xy)(x^2 + y^2) + 3xy(x^3 + y^3) + x^3y^3 + 2x^2y^2 + 3xy.$$

Similarly the disjunction $x \vee y$ is expressed by

$$x \vee y = 2x^2y^2(x^2 + y^2) + 2xy(x^3 + y^3) + xy(x^2 + y^2) + 4x^3y^3 + 3x^2y^2 + 2xy + (x + y).$$

To find the above expressions, we must solve ten linear equations with 10 unknown. The calculations are not so easy, but we shall omit the detail.

References

[1] Gr. C. Moisil: Sur les anneaux de caractéristique 2 ou 3 et leurs applications. Bull. de l'ecole Poly. de Bucarest, **12**, 1–25 (1941).

[2] ——: La logique mathématique et la technique moderne. Les logiques à plusieurs valeurs et les circuits à contacts et relais (in Roumanian), Probleme filozofice ale stiintelor naturii, Bucarest (1960).