25. *Note on Embeddings of Lens Spaces*

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An embedding or immersion of M^n in R^{2n-k} is said to have *efficiency* k (here M^n is an n-dimensional manifold). In [1] and [2] Mahowald and Milgram gave excellent results on efficiency of projective spaces. Their methods are applicable for lens spaces and by using the results in [3] and [4], we can obtain some results on efficiency of embeddings of lens spaces.

Let p be any odd integer≥ 3 and $q_0, \cdots, q_n$ be integers relatively prime to p. The cyclic group Γ of order p with generator t acts on the sphere $S^{2n+1} \subset C^{n+1}$ as follows;

$$t^k(z_0, \cdots, z_n) = (\theta^{kq_0}z_0, \cdots, \theta^{kq_n}z_n),$$

where $(z_0, \cdots, z_n)$ is a complex $(n+1)$-tuple representing a point of S^{2n+1} and $\theta = \exp(2\pi i/p)$. The orbit manifold S^{2n+1}/Γ is a lens space $L^n = L^n(p\,; q_0, \cdots, q_n)$.

Let $L^m = L^m(p\,; q_0, \cdots, q_m) \subset L^{n+m+1} = L^{n+m+1}(p\,; q_0, \cdots, q_{n+m+1})$ be the subspace with the last $n+1$ coordinates 0, while $L^n = L^n(p\,; q_{m+1}, \cdots \cdots, q_{n+m+1})$ is the subspace having the first $m+1$ coodinates 0. A vector bundle $L^{n+m+1} - L^n$ over L^m will be denoted by $L_{n+m+1,m}$.

Let a be an integer such that $a = 4b + c$, $0 \leqq c \leqq 3$, then $j(a) = 8b + 2^c$, and if $d+1 = 2^a e$ with e odd we set $K(d) = j(a) - 1$.

Proposition 1. *Suppose there are differentiable embeddings* $f: L^n \subset R^\alpha$, $g: L^m \subset R^\beta$, $h: L_{n+m+1,m} \subset R^{\beta+\sigma}$ *so that either* (i) $\beta+\sigma > 2(2m+1)$ *or* (ii) $\beta+\sigma = 2(2m+1)$ *and* $2(n+1) \leqq K(2m+1)$, *then if the normal bundle* η_f *of the embedding* f *has* σ *trivial sections, there is a topological embedding* $L^{n+m+1} \subset R^{\alpha+\beta+1}$.

This can be proved by the same way in [1].

Proposition 2. *There are embeddings* $f: L^1 \subset R^6$, $g: L^3 \subset R^{14}$, $h: L^n \subset R^{2(2n+1)} (n \neq 1, 3)$ *and* η_f, η_g, η_h *have* 2, 4, $K(2n+1)$ *sections respectively.*

This is proved by Theorem 2.2 in [1] and (4.1) in [4].

Proposition 3.

(1) *If* $2n \geqq m$, $L_{n+m+1,m} \subset R^{3m+2n+3+\varepsilon}$, $\varepsilon = \frac{1}{2}(1+(-1)^m)$.

(2) *If* $2n < m$, $L_{n+m+1,m} \subset R^{4m+3}$.

(3) *If* $2n < m$ *and* $2(n+1) \leqq K(2m+1)$, $L_{n+m+1,m} \subset R^{4m+2}$.

It is shown in [3] that $L_{n+m+1,m} \subseteq R^{3m+2n+3+\varepsilon}$ and hence we have (1) and (2). (3) is proved by similar way in [1]. (In [3], L^n means $L^n(p\,;1,\cdots,1)$ and p is an odd prime. However, the same conclusions hold for $L^n=L^n(p\,;q_0,\cdots,q_n)$ where p is an odd integer $\geqq 3$.)

Theorem. *Suppose that L^n embeds with efficiency k. We have*

(1) *If $n>4$, $k\geqq 2$.*

(2) *If $n>8$, $k\geqq 3$.*

(3) *If $n>12$, $k\geqq 4$.*

(4) *If $n>13$, $k\geqq 6$.*

(5) *For any n, $k>5\log_3\dfrac{2n+9}{35}+3$.*

Proof. (1), (2), (3) are obtained directly from Propositions $1\sim 3$. By Theorem 4 in [3] we can obtain (4) from (3). By using Theorem 3 in [3], we have that

$$\text{if } \frac{19}{2}3^{\alpha-1}-\frac{9}{2}\leqq n<\frac{9}{2}3^{\alpha}-\frac{9}{2},\ k\geqq 5\alpha-3,$$

$$\text{if } \frac{9}{2}3^{\alpha}-\frac{9}{2}\leqq n<\frac{35}{2}3^{\alpha-1}-\frac{9}{2},\ k\geqq 5\alpha-2,$$

$$\text{if } \frac{35}{2}3^{\alpha-1}-\frac{9}{2}\leqq n<\frac{37}{2}3^{\alpha-1}-\frac{9}{2},\ k\geqq 5\alpha-1, \text{ and}$$

$$\text{if } \frac{37}{2}3^{\alpha-1}-\frac{9}{2}\leqq n<\frac{19}{2}3^{\alpha}-\frac{9}{2},\ k\geqq 5\alpha+1.$$

In any case, $k>5\log_3\dfrac{2n+9}{35}+3$ holds. By similar way in [1] and using only Propositions $1\sim 3$, we have that $k>2\log_2(n+2)-\log_2 18$.[1] Moreover, by using (3.3) in [3], we have that $k>7\log_3\dfrac{2n+17}{43}+3$.

References

[1] M. Mahowald and R. J. Milgram: Embedding real projective spaces. Ann. of Math., **87**, 411–422 (1968).

[2] ——: Embedding projective spaces. Bull. Amer. Math. Soc., **73**, 644–646 (1967).

[3] R. Nakagawa: Embeddings of projective spaces and lens spaces. Sci. Rep. Tokyo Kyoiku Daigaku Sec. A, **9**, 170–175 (1967).

[4] D. Sjerve: Vector bundles over orbit manifolds (to appear).

1) F. Uchida has announced to the author that he obtained by similar methods a result that $k\geqq 2[\log_2(n+1)]-7$.