90. *On the Order of the Absolute Values of a Linear Form, (Third Report.)*

By Seigo FUKAZAWA.

Tokyo Butsuri Gakko.

(Rec. May 31, 1927. Comm. by M. FUJIWARA, M.I.A., June 12, 1927.)

1. In two Notes published in these Proceedings[1] I have reported on my researches on the order of $|\alpha x-y+\beta|$. Especially, in the second of these notes I have given the upper limit of $\liminf |x(\alpha x-y+\beta)|$ for some classes of the irrational number α. Since then I have got for some cases the more precise and best possible limit, which will be shown in the following lines.

In the case iii *l.c.*, that is, when $\liminf q_i=2k$, we have the following theorem :

If $k\geqq 3$ and $\alpha x-y-\beta$ is not equivalent to

$$\sqrt{k^2+1}\,x-y+\frac{\sqrt{k^2+1}-k+1}{2},$$

then

$$\liminf |x(\alpha x-y+\beta)|\leqq \frac{1}{4\left\{\frac{1}{1+\omega}+\frac{1}{2k-1}+\frac{1}{2k+\omega}\right\}}, \tag{A}$$

where
$$\omega=\frac{1}{2k+2}+\frac{1}{2k+4}+\frac{1}{2k+2}+\frac{1}{2k+4}+\cdots\cdots, \quad \text{if } k\geqq 6$$

$$=\frac{1}{2k+2}+\frac{1}{2k+6}+\frac{1}{2k+2}+\frac{1}{2k+6}+\cdots\cdots, \quad \text{if } k=4, 5$$

$$=\frac{1}{8}+\frac{1}{16}+\frac{1}{8}+\frac{1}{16}+\cdots\cdots, \quad \text{if } k=3.$$

The equality in (A) occurs for infinitely many (non-enumerable) forms, and if ε is any positive number smaller than the right-hand side

1) On the extension of Klein's geometrical interpretation of continued fraction, these Proc. 2, 100. On the extension of a theorem of Minkowski, ibid. 2, 305.

of (A), then there are infinitely many (non-enumerable) forms, for which

$$\liminf q_i = 2k,$$

and

$$\liminf |x(\alpha x - y + \beta)| = \varepsilon.$$

In the case iv, that is, when $q_i = 2k$ and $\mu_i = 1$ or $\mu_{i+1} = 1$ for infinitely many i, we have

$$\liminf |x(\alpha x - y + \beta)| \leqq \frac{1}{4\left(1 + \frac{1}{2k-1} + \frac{2}{2k}\right)}. \tag{B}$$

The equality in B occurs for infinitely many (non-enumerable) forms, and if ε is any positive number smaller than the right-hand side of (B), then there exist also infinitely many (non-enumerable) forms with the character above-mentioned, for which

$$\liminf |x(\alpha x - y + \beta)| = \varepsilon.$$

2. Khintchine, in his paper "Uber eine Klasse linearer diophantischer Approximation,"[1] has given some theorems on the order of $|\alpha x - y|$ and $|\alpha x - y + \beta|$. We can prove these theorems by means of our method. Especially his theorem IV — "There exists an absolute constant γ with the following character: for any real number α we can determine β, so that the inequality $|\alpha x - y + \beta| < \frac{\gamma}{x}$ has no solution for integral values of $x > 0$ and y"—can be proved more simply than by his method, and moreover our method gives the precise result

$$\gamma > \frac{1}{457}.$$

We have also the theorem :

" For any real number α we can determine β, such that

$$\liminf |x(\alpha x - y + \beta)| = 0."$$

1) Rend. Parelmo, 50, (1927).