Abel transforms of positive linear operators on weighted spaces

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Abstract

The classical Korovkin approximation theory deals with the convergence of a sequence of positive linear operators. When the sequence of positive linear operators does not converge it will be useful to use some summability methods. In this paper we use the Abel method, a sequence-to-function transformation, to study a Korovkin type approximation theorem for positive linear operators acting from a weighted space C_{ρ_1} into a weighted space B_{ρ_2}. Moreover using the modulus of continuity we also give rate of Abel convergence.

1 Introduction

The Korovkin theorem provides a criterion for whether a given sequence $\{L_n\}$ of positive linear operators on $C[0,1]$ converges to the identity operator ([2],[12]). Some variations of this result may be found in [13], [15], [18]. If the sequence of positive linear operators does not converge to the identity operator then it might be beneficial to use some summability methods ([1], [8], [14], [17]). Using the Abel convergence method, recently Unver [19] has studied a Korovkin type approximation theorem for the positive linear operators over the space of continuous functions defined on a closed bounded interval. The purpose of this paper is to use the Abel method, a sequence-to-function transformation, to study a Korovkin type approximation of a function f by means of a sequence $\{L_n(f;x)\}$

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of positive linear operators acting from a weighted space C_{ρ_1} into a weighted space B_{ρ_2}.

First of all, we give some basic definitions and notations used in this paper:

A real valued function ρ is called a weight function if it is continuous on $\mathbb{R}$, $\rho(x) \geq 1$ for all $x \in \mathbb{R}$ and

$$\lim_{|x| \to \infty} \rho(x) = \infty$$

where $\mathbb{R}$ denotes the set of all real numbers.

Let ρ be a weight function. The space of real valued functions f defined on $\mathbb{R}$ and for all $x \in \mathbb{R}$ satisfying $|f(x)| \leq K_f \rho(x)$ is called the weighted space and denoted by B_ρ, where K_f is a constant depending on f. The weighted subspace C_ρ of B_ρ is given by

$$\left\{C_\rho := f \in B_\rho : f \textit{ is continuous on } \mathbb{R}\right\}.$$

It is known [9] that the spaces B_ρ and C_ρ are Banach spaces with the norm

$$\|f\|_\rho := \sup_{x \in \mathbb{R}} \frac{|f(x)|}{\rho(x)}.$$

Let $L : C_{\rho_1} \to B_{\rho_2}$ be a linear operator. Then L is called positive if $Lf \geq 0$ whenever $f \geq 0$. If L is a positive linear operator then $f \leq g$ implies that $Lf \leq Lg$ and, $|f| \leq g$ implies $|Lf| \leq Lg$.

The following approximation theorem for a sequence of positive linear operators acting from C_{ρ_1} into B_{ρ_2} may be found in [9] and [10].

Theorem A. Assume that ρ_1 and ρ_2 are weight functions such that

$$\lim_{|x| \to \infty} \frac{\rho_1(x)}{\rho_2(x)} = 0 \tag{1.1}$$

and $\{L_n\}$ is a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then $\lim_n \|L_n f - f\|_{\rho_2} = 0$ for every $f \in C_{\rho_1}$ if and only if $\lim_n \|L_n F_i - F_i\|_{\rho_1} = 0$ for $i = 0, 1, 2$ where

$$F_i = \frac{x^i \rho_1(x)}{1 + x^2}, i = 0, 1, 2.$$

Some analogs of this theorem can be found in [3], [4] and [7].

In the present paper, using the Abel method, we will give another analog of Theorem A.

Let us recall the Abel method:

If the series

$$\sum_{k=0}^{\infty} x_k \alpha^k$$

converges for all $\alpha \in (0, 1)$ and

$$\lim_{\alpha \to 1^-} (1 - \alpha) \sum_{k=0}^{\infty} x_k \alpha^k = L \tag{1.2}$$

then we say that the sequence $x = (x_k)$ is Abel convergent to L.

As $\dfrac{1}{1-\alpha} = \sum_{k=0}^{\infty} \alpha^k$, $0 < \alpha < 1$, (1.2) is equivalent to the following:

$$\lim_{\alpha \to 1^-} (1-\alpha) \sum_{k=0}^{\infty} (x_k - L)\alpha^k = 0.$$

Note that the convergence of a sequence implies the Abel convergence of it, but not conversely ([5], [16]).

Let $\{L_n\}$ be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} such that

$$\sum_{n=0}^{\infty} \left\| L_n(\rho_1) \right\|_{\rho_2} \alpha^n < \infty \tag{1.3}$$

for all $\alpha \in (0,1)$, then for all $f \in C_{\rho_1}$ the series $\sum_{n=0}^{\infty} L_n(f(t); x)\alpha^n$ converges. Hence the operator U_α defined by

$$U_\alpha (f; x) := (1-\alpha) \sum_{n=0}^{\infty} L_n(f(t); x)\alpha^n$$

is a positive linear operator from C_{ρ_1} to B_{ρ_2} which is bounded for all $\alpha \in (0,1)$. Thus

$$\begin{aligned} \|U_\alpha\|_{C_{\rho_1} \to B_{\rho_2}} &= \sup_{\|f\|_{\rho_1}=1} \|U_\alpha f\|_{\rho_2} \\ &\leq \sup_{x \in \mathbb{R}} \frac{(1-\alpha) \left| \sum_{n=0}^{\infty} L_n(\rho_1; x)\alpha^n \right|}{\rho_2} \\ &= \left\| U_\alpha \rho_1 \right\|_{\rho_2} \end{aligned}$$

for all $\alpha \in (0,1)$.

2 Approximation by Abel Method on Weighted Spaces

In this section using the Abel method we study a Korovkin type approximation theorem for a sequence of positive linear operators acting from C_{ρ_1} into B_{ρ_2}.

We need the following lemmas.

Lemma 1. *Let $\{L_n\}$ be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} such that (1.3) holds and let ρ_1 and ρ_2 be weight functions satisfying (1.1). Assume that*

$$\sup_{\alpha \in (0,1)} \|U_\alpha\|_{C_{\rho_1} \to B_{\rho_1}} < \infty. \tag{2.1}$$

If for any $s \in \mathbb{R}$,

$$\lim_{\alpha \to 1^-} \sup_{\|f\|_{\rho_1}=1} \sup_{|x| \leq s} \frac{|U_\alpha (f; x)|}{\rho_1(x)} = 0 \tag{2.2}$$

then

$$\lim_{\alpha \to 1^-} \|U_\alpha\|_{C_{\rho_1} \to B_{\rho_2}} = 0.$$

Proof. The proof can be obtained easily by applying the same arguments used in Lemma 2 [10]. But we include the proof for the sake of completeness.
It follows from (1.1) that, for any $\varepsilon > 0$, there exists a number s_0 such that $\rho_1(x) \leq \varepsilon\rho_2(x)$ for all $|x| > s_0$. By the continuity of ρ_1/ρ_2, there exists $K > 0$ such that $\rho_1(x) \leq K\rho_2(x)$ for all $|x| \leq s_0$. Hence we get

$$\begin{aligned}
\|U_\alpha\|_{C_{\rho_1} \to B_{\rho_2}} &= \sup_{\|f\|_{\rho_1}=1} \|U_\alpha f\|_{\rho_2} \\
&= \sup_{\|f\|_{\rho_1}=1} \sup_{x \in \mathbb{R}} \frac{|U_\alpha(f;x)|}{\rho_2} \\
&\leq \sup_{\|f\|_{\rho_1}=1} \sup_{|x| \leq s_0} \frac{|U_\alpha(f;x)|}{\rho_2} \\
&+ \sup_{\|f\|_{\rho_1}=1} \sup_{|x| > s_0} \frac{|U_\alpha(f;x)|}{\rho_2} \\
&\leq K \sup_{\|f\|_{\rho_1}=1} \sup_{|x| \leq s_0} \frac{|U_\alpha(f;x)|}{\rho_1} + \varepsilon \|U_\alpha\|_{C_{\rho_1} \to B_{\rho_1}}.
\end{aligned}$$

Then from (2.1) and (2.2) we have

$$\lim_{\alpha \to 1^-} \|U_\alpha\|_{C_{\rho_1} \to B_{\rho_2}} = 0$$

which concludes the proof. ∎

Lemma 2. *Let $\{L_n\}$ be a sequence of linear operators from C_{ρ_1} into B_{ρ_2} such that (1.3), (1.1) and (2.1) hold. If for any $s \in \mathbb{R}$,*

$$\lim_{\alpha \to 1^-} \sup_{\|f\|_{\rho_1}=1} \sup_{|x| \leq s} |U_\alpha(f;x) - f(x)| = 0 \tag{2.3}$$

then

$$\lim_{\alpha \to 1^-} \|U_\alpha f - f(x)\|_{\rho_2} = 0$$

for any $f \in C_{\rho_1}$.

Proof. Let I be the identity operator on C_{ρ_1}. Let $T_n := L_n - I$ and consider the operator V_α defined by

$$V_\alpha(f;x) = (1-\alpha) \sum_{n=0}^{\infty} T_n(f(t);x)\alpha^n$$

for all $\alpha \in (0,1)$, which is well defined from (1.3) and belongs to B_{ρ_2}. Since

$$\begin{aligned}
\|V_\alpha\|_{C_{\rho_1}\to B_{\rho_1}} &= \sup_{\|f\|_{\rho_1}=1} \|V_\alpha f\|_{\rho_1} \\
&= \sup_{\|f\|_{\rho_1}=1} \sup_{x\in\mathbb{R}} \frac{|V_\alpha(f;x)|}{\rho_1(x)} \\
&\le \sup_{\|f\|_{\rho_1}=1} \sup_{x\in\mathbb{R}} \frac{|U_\alpha(f;x)|}{\rho_1(x)} \\
&+ \sup_{\|f\|_{\rho_1}=1} \sup_{x\in\mathbb{R}} (1-\alpha) \frac{\left|\sum_{n=0}^{\infty} f(x)\alpha^n\right|}{\rho_1(x)} \\
&= \|U_\alpha\|_{C_{\rho_1}\to B_{\rho 1}} + \sup_{\|f\|_{\rho_1}=1} \|f\|_{\rho_1} (1-\alpha) \sum_{n=0}^{\infty} \alpha^n \\
&= \|U_\alpha\|_{C_{\rho_1}\to B_{\rho 1}} + 1
\end{aligned}$$

it follows from (2.1) that

$$\sup_{\alpha\in(0,1)} \|V_\alpha\|_{C_{\rho_1}\to B_{\rho_1}} < \infty.$$

As $\rho_1 > 1$ we have for any $s \in \mathbb{R}$ that

$$\begin{aligned}
\sup_{\|f\|_{\rho_1}=1} \sup_{|x|\le s} \frac{|V_\alpha(f;x)|}{\rho_1(x)} &= \sup_{\|f\|_{\rho_1}=1} \sup_{|x|\le s} \frac{|U_\alpha(f;x) - f(x)|}{\rho_1(x)} \\
&\le \sup_{\|f\|_{\rho_1}=1} \sup_{|x|\le s} |U_\alpha(f;x) - f(x)|\,. \qquad (2.4)
\end{aligned}$$

Then from (2.3) and (2.4) we have for any $s \in \mathbb{R}$ that

$$\lim_{\alpha\to 1^-} \sup_{\|f\|_{\rho_1}=1} \sup_{|x|\le s} \frac{|V_\alpha(f;x)|}{\rho_1(x)} = 0.$$

Hence it follows from Lemma 1 that

$$\lim_{\alpha\to 1^-} \|V_\alpha\|_{C_{\rho_1}\to B_{\rho_2}} = 0. \qquad (2.5)$$

Now we get for any $f \in C_{\rho_1}$ that

$$
\begin{aligned}
\|U_\alpha f - f\|_{\rho_2} &= \sup_{x\in\mathbb{R}} \frac{\left|(1-\alpha)\sum_{n=0}^{\infty} L_n(f;x)\alpha^n - f(x)\right|}{\rho_2(x)} \\
&= \sup_{x\in\mathbb{R}} \frac{\left|(1-\alpha)\sum_{n=0}^{\infty} \left(L_n(f;x) - f(x)\right)\alpha^n\right|}{\rho_2(x)} \\
&= \sup_{x\in\mathbb{R}} \frac{\left|(1-\alpha)\sum_{n=0}^{\infty} T_n(f;x)\alpha^n\right|}{\rho_2(x)} \\
&\le \|V_\alpha f\|_{\rho_2} \\
&\le \|V_\alpha\|_{C_{\rho_1}\to B_{\rho_2}} \|f\|_{\rho_1}
\end{aligned}
$$

Hence by (2.5) we have

$$\lim_{\alpha\to 1^-} \|U_\alpha f - f\|_{\rho_2} = 0. \qquad \blacksquare$$

Now we are ready to give our main result.

Theorem 1. *Let $\{L_n\}$ be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} such that (1.1) and (1.3) hold. If*

$$\lim_{\alpha\to 1^-} \|U_\alpha F_i - F_i\|_{\rho_1} = 0 \tag{2.6}$$

then for all $f \in C_{\rho_1}$

$$\lim_{\alpha\to 1^-} \|U_\alpha f - f\|_{\rho_2} = 0, \tag{2.7}$$

where $F_i(x) = \dfrac{x^i\rho_1(x)}{1+x^2}$, $i = 0, 1, 2$.

Proof. Let $f \in C_{\rho_1}$ and assume that (2.6) holds. It is obvious that (2.1) holds. Since $f \in C_{\rho_1}$ there exists a constant M_f such that $|f(x)| \le M_f\rho_1(x)$ for all $x \in \mathbb{R}$. By using the same arguments in Theorem 14 of [11], we have for all $\alpha \in (0,1)$ and any $s \in \mathbb{R}$ that

$$
\begin{aligned}
\sup_{\|f\|_{\rho_1}=1} \sup_{|x|\le s} |U_\alpha(f;x) - f(x)| \le K \Big\{ & \varepsilon \|U_\alpha\|_{C_{\rho_1}\to B_{\rho_1}} \\
& + \|U_\alpha F_2 - F_2\|_{\rho_1} \\
& + \|U_\alpha F_1 - F_1\|_{\rho_1} \\
& + \|U_\alpha F_0 - F_0\|_{\rho_1} \Big\}
\end{aligned}
$$

where $K := \max\{1 + K_2K_3, C(K_1 + K_2K_4) + K_2K_3\}$, $K_1 := K_1(s) = \sup_{|x|\le s} |f(x)|$, $K_2 := K_2(s) = \sup_{|x|\le s} H_{\rho_1}(x)$, $K_3 := K_3(s) = \sup_{|x|\le s} \left\{\frac{\rho_1(x)}{F_0(x)}\right\}$,

$K_4 := K_4(s) = \sup_{|x|\le s}\left\{\frac{H_{\rho_1}(x)}{F_0(x)}\right\}$, $H_{\rho_1}(x) = 4M_f\rho_1(x)\left\{1+\frac{1+x^2}{\delta^2}\right\}$ and $C := \max\left\{\sup_{|x|\le s}\rho_1(x), 2\sup_{|x|\le s}|x|\,\rho_1(x), \sup_{|x|\le s}x^2\rho_1(x)\right\}$. Hence from (2.1) and (2.6) we have for any $s\in\mathbb{R}$ that

$$\lim_{\alpha\to 1^-}\sup_{\|f\|_{\rho_1}=1}\sup_{|x|\le s}|U_\alpha(f;x)-f(x)| = 0.$$

Then by Lemma 2 we have

$$\lim_{\alpha\to 1^-}\|U_\alpha f - f\|_{\rho_2} = 0$$

which concludes the proof. ∎

3 Rate of Abel Convergence

We consider the following weighted modulus of continuity

$$\omega_{\rho_1}(f,\delta) = \sup_{|t-x|\le\delta}\left\{\frac{|f(t)-f(x)|}{\rho_1(x)}\right\}$$

where δ is a positive constant and $f\in C_{\rho_1}$. It was shown in [6] that $\omega_{\rho_1}(f,\delta)$ is a weighted modulus of continuity and it is well known that, for all $f\in C_{\rho_1}$ and for all $c>0$,

$$\omega_{\rho_1}(f,c\delta)\le(1+[|c|])\,\omega_{\rho_1}(f,\delta) \tag{3.1}$$

where $[|c|]$ is the greatest integer less than or equal to c.

In this section, using weighted modulus of continuity, we study the rate of Abel convergence.

Lemma 3. *Let $\{L_n\}$ be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} such that (1.3), (1.1) and (2.1) hold and let $L_n\varphi_x$ and L_nF_0 be in C_{ρ_1} for each n where $\varphi_x(t) := (t-x)^2$ and $F_0(t)=1$. Then for any $s>0$ and all $\alpha\in(0,1)$*

$$\begin{aligned}\sup_{\|f\|_{\rho_1}=1}\sup_{|x|\le s}|U_\alpha(f;x)-f(x)| \le K\Bigg\{&\sup_{\|f\|_{\rho_1}=1}\omega_{\rho_1}(f,\psi(\alpha))\\ &+\|U_\alpha F_0-F_0\|_{\rho_1}\Bigg\}\end{aligned} \tag{3.2}$$

holds, where $\psi(\alpha)=\sqrt{\|U_\alpha\varphi_x\|_{\rho_1}}$ and $K := K(s)$ is a positive constant.

Proof. Using the linearity and positivity of U_α, for all $\alpha\in(0,1)$, $\delta>0$ and $f\in C_{\rho_1}$

we have

$$
\begin{aligned}
|U_\alpha(f;x)-f(x)| &\le U_\alpha\left\{|f(t)-f(x)|;x\right\} \\
&+|f(x)|\left|U_\alpha\left(F_0;x\right)-F_0(x)\right| \\
&\le U_\alpha\left(\rho_1(x)\omega_{\rho_1}\left(f,\frac{|t-x|}{\delta}\delta\right);x\right) \\
&+|f(x)|\left|U_\alpha\left(F_0;x\right)-F_0(x)\right| \\
&\le U_\alpha\left(\left(1+\left[\left|\frac{|t-x|}{\delta}\right|\right]\right)\rho_1(x)\omega_{\rho_1}\left(f,\delta\right);x\right) \\
&+|f(x)|\left|U_\alpha\left(F_0;x\right)-F_0(x)\right| \\
&\le \rho_1(x)\omega_{\rho_1}\left(f,\delta\right)U_\alpha\left(1+\frac{(t-x)^2}{\delta^2};x\right) \\
&+|f(x)|\left|U_\alpha\left(F_0;x\right)-F_0(x)\right| \\
&\le \rho_1(x)\omega_{\rho_1}\left(f,\delta\right)\left\{U_\alpha\left(\rho_1;x\right)\right. \\
&\left.+\frac{1}{\delta^2}U\left\{(\varphi_x(t);x);y\right\}\right\} \\
&+|f(x)|\left|U_\alpha\left(F_0;x\right)-F_0(x)\right|.
\end{aligned} \tag{3.3}
$$

Since $\varphi_x \in C_{\rho_1}$, for any $s>0$ and all $\alpha \in (0,1)$, we get from (3.3) that

$$
\begin{aligned}
\sup_{\|f\|_{\rho_1}=1}\sup_{|x|\le s}|U_\alpha(f;x)-f(x)| \le K_1^2 &\sup_{\|f\|_{\rho_1}=1}\omega_{\rho_1}\left(f,\delta\right)\left\{\|U_\alpha\|_{C_{\rho_1}\to B_{\rho_1}}\right. \\
&\left.+\frac{1}{\delta^2}\left\|U_\alpha\varphi_x\right\|_{\rho_1}\right\} \\
&+K_2\left\|U_\alpha F_0-F_0\right\|_{\rho_1},
\end{aligned} \tag{3.4}
$$

where $K_1=\sup_{|x|\le s}\rho_1(x)=1+s^2$ and $K_2=\sup_{|x|\le s}\dfrac{f(x)}{\rho_1(x)}$. By the hypotheses, for all $\alpha \in (0,1)$, we have

$$
\|U_\alpha\|_{C_{\rho_1}\to B_{\rho_1}} \le M.
$$

Now putting $\delta=\psi(\alpha)=\sqrt{\left\|U_\alpha\varphi_x\right\|_{\rho_1}}$ and by (3.4) we have, for all $\alpha \in (0,1)$, that

$$
\begin{aligned}
\sup_{\|f\|_{\rho_1}=1}\sup_{|x|\le s}|U_\alpha(f;x)-f(x)| \le K&\left\{\sup_{\|f\|_{\rho_1}=1}\omega_{\rho_1}(f,\psi(\alpha))\right. \\
&\left.+\left\|U_\alpha F_0-F_0\right\|_{\rho_1}\right\},
\end{aligned}
$$

where $K=\max\left\{K_1^2(1+M),K_2\right\}$. ∎

Theorem 2. *Let $\{L_n\}$ be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} such that (1.3), (1.1) and (2.1) hold and let $L_n\varphi_x$ and L_nF_0 be in C_{ρ_1} for each n where $\varphi_x(t) := (t-x)^2$ and $F_0(t)=1$. If*

i) $\lim\limits_{\alpha \to 1^-} \|U_\alpha F_0 t - F_0\|_{\rho_1} = 0$
ii) $\lim\limits_{\alpha \to 1^-} \sup\limits_{\|f\|_{\rho_1}=1} \omega_{\rho_1}(f, \psi(\alpha)) = 0$
then for any $f \in C_{\rho_1}$ *we get*

$$\lim_{\alpha \to 1^-} \|U_\alpha f - f\|_{\rho_2} = 0.$$

Proof. By (i), (ii) and Lemma 3, we have

$$\lim_{\alpha \to 1^-} \sup_{\|f\|_{\rho_1}=1} \sup_{|x| \le s} |U_\alpha(f; x) - f(x)| = 0.$$

Then for all $f \in C_{\rho_1}$, it follows from Lemma 2 that

$$\lim_{\alpha \to 1^-} \|U_\alpha f - f\|_{\rho_2} = 0. \qquad \blacksquare$$

4 Remarks

Let ρ_1 and ρ_2 be weight functions satisfying (1.1) and $\{T_n\}$ be a sequence of positive linear operators from C_{ρ_1} to B_{ρ_2} satisfying the hypotheses of Theorem A. Now define a sequence $\alpha = (\alpha_n)$ as $\alpha_n = 1$ if n is a perfect square, and $\alpha_n = 0$ otherwise. Note that α is Abel convergent to zero but not convergent. Let $\{L_n\}$ be a sequence of positive linear operators acting from C_{ρ_1} into B_{ρ_2} defined as

$$L_n(f; x) = (1 + \alpha_n) T_n(f)$$

for $f \in C_{\rho_1}$. Observe that the sequence $\{L_n\}$ does not satisfy the hypotheses of Theorem A but it satisfies the hypotheses of our Theorem 1.

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