On semi-typically real functions which are generated by a fixed semi-typically real function

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Abstract

Let $\mathcal{A}$ denote the family of all functions that are analytic in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

In this paper, we investigate the class $\mathcal{T}_G$ defined as follows

$$\mathcal{T}_G := \left\{ \sqrt{F(z)G(z)} : F \in \mathcal{T} \right\}, \quad G \in \mathcal{T},$$

where $\mathcal{T}$ denotes the class of all semi-typically real functions i.e. $\mathcal{T} := \{F \in \mathcal{A} : F(z) > 0 \iff z \in (0,1)\}$. We find the sets $\bigcup_{G\in\mathcal{T}} \mathcal{T}_G$ and $\bigcap_{G\in\mathcal{T}} \mathcal{T}_G$, the set of all extreme points of $\mathcal{T}_G$ and the set of all support points of $\mathcal{T}_G$. Moreover, for the fixed G, we determine the radii of local univalence, of starlikeness and of univalence of $\mathcal{T}_G$.

1 Some properties of the class $\mathcal{T}$.

Let $\mathcal{A}$ denote the family of all functions that are analytic in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let A be a subclass of $\mathcal{A}$ and let $A^{(2)} := \{f \in A : f(z) = -f(-z) \text{ for } z \in \Delta\}$.

Let T denote the well-known class of all typically real functions, i.e. T is the subclass of $\mathcal{A}$ consisting of functions f such that $\operatorname{Im} z \operatorname{Im} f(z) \geq 0$, $z \in \Delta$. From the

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definition we conclude that $\mathrm{T} = \{f \in \mathcal{A} : f(z) \in \mathbb{R} \iff z \in (-1,1)\}$. Robertson in [7] gave the explicit relation between a function $f \in \mathrm{T}$ and a probability measure μ defined on $[-1,1]$. Namely

$$(1) \qquad f \in \mathrm{T} \iff f(z) = \int_{-1}^{1} k_t(z)d\mu(t), \quad \text{where} \quad k_t(z) = \frac{z}{1-2tz+z^2}.$$

The class of semi-typically real functions was considered in [5] and was defined as follows

$$\mathcal{T} := \{F \in \mathcal{A} : F(z) > 0 \iff z \in (0,1)\}.$$

For simplicity, instead of h or $z \mapsto h(z)$ we will use $h(z)$. We know that for $F \in \mathcal{T}$ we have $\frac{F(z)}{z} \neq 0$. Thus for $F, G \in \mathcal{T}$ let us define

$$F^{\varepsilon}(z)\, G^{1-\varepsilon}(z) := z\left(\frac{F(z)}{z}\right)^{\varepsilon}\left(\frac{G(z)}{z}\right)^{1-\varepsilon}, \ \varepsilon \in [0,1],\ 1^{\varepsilon} = 1.$$

Let us recall some properties of the class $\mathcal{T}$ as the following lemma (see [5]).

Lemma 1.

$$(2) \qquad F \in \mathcal{T} \iff \sqrt{F(z^2)} \in \mathrm{T}^{(2)}.$$

$$(3) \qquad F \in \mathcal{T} \iff \frac{\sqrt{z\,F(z)}}{1+z} \in \mathrm{T}.$$

P. Todorov in [9] gave the estimation for the operator $\mathrm{Re}\frac{zf'(z)}{f(z)}$ for $f \in \mathrm{T}$. Namely

Theorem 1. [P.G. Todorov] *For each typically real function we have:*

(i)

$$\mathrm{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-6r^2+r^4}{1-r^4}, \quad \text{for} \quad 2-\sqrt{3} \leq r = |z| < 1$$

with equality for the function $f(z) = \frac{z(1+z^2)}{(1-z^2)^2} = \frac{1}{2}\,k_1(z) + \frac{1}{2}\,k_{-1}(z)$ *at the points* $z = \pm ir$.

(ii)

$$\mathrm{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-r}{1+r}, \quad \text{for} \quad 0 \leq r = |z| \leq 2-\sqrt{3}$$

with equality for the functions $k_1(z) = \frac{z}{(1-z)^2}$ *and* $k_{-1}(z) = \frac{z}{(1+z)^2}$ *at the points* $-r$ *and* r*, respectively.*

Now let us prove that for odd typically real functions the following estimation is satisfied.

Theorem 2. *For $f \in \mathrm{T}^{(2)}$ we have*

$$\mathrm{Re}\frac{z f'(z)}{f(z)} \geq \frac{1-6r^2+r^4}{1-r^4} \quad \textit{for} \quad z \in \Delta,\ r=|z|$$

with equality for the function $f(z)=\frac{z(1+z^2)}{(1-z^2)^2}$ at the points $z=\pm ir$.

Proof. For $r \geq 2-\sqrt{3}$ the above estimation is an obvious corollary from the Todorov Theorem. So let us prove it for $r < 2-\sqrt{3}$.

Suppose that $f \in \mathrm{T}^{(2)}$. Then $f(z)=\frac{(1+z^2)}{z}\,h(z^2)$ for some $h \in \mathrm{T}$ (see [6]). Thus

$$\frac{z f'(z)}{f(z)} = z\left(\frac{2z}{1+z^2}+\frac{2z\,h'(z^2)}{h(z^2)}-\frac{1}{z}\right) = -\frac{1-z^2}{1+z^2}+\frac{2z^2\,h'(z^2)}{h(z^2)}.$$

We have $|z|^2 < 2-\sqrt{3}$ and $\left|\frac{1-z^2}{1+z^2}\right| \leq \frac{1+r^2}{1-r^2}$. From these and the Todorov Theorem we get

$$\begin{aligned}\mathrm{Re}\frac{z f'(z)}{f(z)} &= -\mathrm{Re}\frac{1-z^2}{1+z^2}+2\mathrm{Re}\frac{z^2\,h'(z^2)}{h(z^2)}\\ &\geq -\frac{1+r^2}{1-r^2}+2\frac{1-r^2}{1+r^2} = \frac{1-6r^2+r^4}{1-r^4}\end{aligned}$$

and the proof is complete. ■

From (2) we have $F \in \mathcal{T} \iff F(z^2)=f^2(z),\ f \in \mathrm{T}^{(2)}$. This relation and Theorem 2 give us

$$\mathrm{Re}\frac{z^2\,F'(z^2)}{F(z^2)} = \mathrm{Re}\frac{z f'(z)}{f(z)} \geq \frac{1-6r^2+r^4}{1-r^4}\,, \quad r=|z|.$$

Hence, $\mathrm{Re}\frac{z F'(z)}{F(z)} \geq \frac{1-6r+r^2}{1-r^2}$ and we get the following corollary.

Corollary 1. *For $F \in \mathcal{T}$ we have*

$$\mathrm{Re}\frac{z F'(z)}{F(z)} \geq \frac{1-6r+r^2}{1-r^2} \quad \textit{for} \quad z \in \Delta,\ r=|z|$$

with equality for the function $F(z)=\frac{z(1+z)^2}{(1-z)^4}$ at the points $z=-r$.

In this paper, we determine the radii of starlikeness r_{ST}, of local univalence r_{LU} and of univalence r_S in certain classes of $\mathcal{T}$. Let us recall some definitions. Hereafter, let A be a given subclass of $\mathcal{A}$.

Definition 1. We say that $r_{ST}(\mathrm{A})$ is the radius of starlikeness in the class A, if it is the maximum of the numbers $r \in (0,1]$, such that the inequality $\mathrm{Re}\frac{z f'(z)}{f(z)} > 0$ holds in the disk $|z| < r$ for each function $f \in \mathrm{A}$.

Definition 2. We say that $r_S(\mathrm{A})$ ($r_{LU}(\mathrm{A})$) is called the radius of univalence (local univalence) in the class A, if it is the maximum of numbers $r \in (0,1]$, such that every function $f \in \mathrm{A}$ is univalent (local univalent) in $|z| < r$.

In the class A the following inequalities are satisfied

$$r_{ST}(A) \leq r_S(A) \leq r_{LU}(A). \tag{4}$$

Definition 3. A set $G \subset \Delta$ is called the set of local univalence in the class A, if $\forall_{f\in \mathrm{A}}\ \forall_{z\in G}\ f'(z) \neq 0$ and $\forall_{z\in\Delta\setminus G}\ \exists_{f\in A}\ f'(z) = 0$. We denote the set of local univalence in the class A by $G_{LU}(\mathrm{A})$.

Definition 4. The class A is convex, if $\forall_{f_1,f_2\in \mathrm{A}}\forall_{\varepsilon\in[0,1]}\ \varepsilon f_1 + (1-\varepsilon)f_2 \in \mathrm{A}$.

2 Some properties of the class $\mathcal{T}_G$.

For typically real functions f, g and $\varepsilon \in [0,1]$ we know that $f^\varepsilon g^{1-\varepsilon} \in \mathrm{T}$. Analogously for functions $f, g \in \mathrm{T}^{(2)}$ and $\varepsilon \in [0,1]$ we have $f^\varepsilon g^{1-\varepsilon} \in \mathrm{T}^{(2)}$. Because $\mathcal{T} = \left\{F : \sqrt{F(z^2)} \in \mathrm{T}^{(2)}\right\}$, thus for semi-typically real functions F, G we get $F^\varepsilon\, G^{1-\varepsilon} \in \mathcal{T}$, $\varepsilon \in [0,1]$. In this paper, we investigate functions $F^\varepsilon\, G^{1-\varepsilon}$ for $\varepsilon = \frac{1}{2}$, i.e. $\sqrt{F\,G}$. Denote

$$\mathcal{T}_G := \left\{\sqrt{F(z)\,G(z)} : F \in \mathcal{T}\right\} \quad \text{for some fixed function } G \in \mathcal{T}. \tag{5}$$

Observe that the class $\mathcal{T}_G$ is not empty, because the function G belongs to $\mathcal{T}_G$. In the next few theorems we introduce successive important properties of the class $\mathcal{T}_G$.

Theorem 3. $\mathcal{T}_G = \left\{F(z) : F(z^2) = \sqrt{G(z^2)}\, f(z),\ f \in \mathrm{T}^{(2)}\right\}$.

Proof. Let $G \in \mathcal{T}$. Thus from (5) and the fact that $H \in \mathcal{T} \iff \sqrt{H(z^2)} = f(z)$, $f \in \mathrm{T}^{(2)}$ we get

$$\begin{aligned}\mathcal{T}_G &= \left\{F(z) : F(z^2) = \sqrt{H(z^2)\,G(z^2)},\ H \in \mathcal{T}\right\}\\ &= \left\{F(z) : F(z^2) = f(z)\,\sqrt{G(z^2)},\ f \in \mathrm{T}^{(2)}\right\}.\end{aligned} \qquad \blacksquare$$

Theorem 4. $\mathcal{T}_G = \left\{(1+z)\sqrt{\frac{G(z)}{z}}\, f(z) :\ f \in \mathrm{T}\right\}$.

Proof. Assume that $G \in \mathcal{T}$. Therefore from (5) and the fact that $F \in \mathcal{T} \iff F(z) = \frac{(1+z)^2}{z} f^2(z)$, $f \in \mathrm{T}$ we obtain

$$\mathcal{T}_G = \left\{\sqrt{\frac{(1+z)^2}{z} f^2(z)\ G(z)} : f \in \mathrm{T}\right\} = \left\{(1+z)\,f(z)\sqrt{\frac{G(z)}{z}} :\ f \in \mathrm{T}\right\}. \qquad \blacksquare$$

Since the class T is convex, then from Theorem 4 we get the following corollary.

Corollary 2. *For all $G \in \mathcal{T}$, the class $\mathcal{T}_G$ is convex.*

We know that (see for example [2] and [3]):

$$\begin{aligned}
\mathcal{E}\mathrm{T} &= \{k_t : t \in [-1,1]\}, \\
\sigma\mathrm{T} &= \left\{\sum_{i=1}^{n} \varepsilon_i k_{t_i} : \varepsilon_i \in [0,1], \sum_{i=1}^{n} \varepsilon_i = 1, t_i \in [-1,1]\right\},
\end{aligned}$$

where $\mathcal{E}A$ is the set of all extreme points of A, σA is the set of all support points of A and the function k_t is given by (1). Hence for the class $\mathcal{T}_G$ we have:

$$\begin{aligned}
\mathcal{E}\mathcal{T}_G &= \left\{(1+z)\sqrt{\frac{G(z)}{z}}\, f(z) : f \in \mathcal{E}\mathrm{T}\right\} \\
&= \left\{(1+z)\sqrt{\frac{G(z)}{z}}\, k_t(z) : t \in [-1,1]\right\}, \\
\sigma\mathcal{T}_G &= \left\{(1+z)\sqrt{\frac{G(z)}{z}}\, f(z) : f \in \sigma\mathrm{T}\right\} \\
&= \left\{(1+z)\sqrt{\frac{G(z)}{z}}\, \sum_{i=1}^{n} \varepsilon_i k_{t_i}(z) : \varepsilon_i \in [0,1], \sum_{i=1}^{n} \varepsilon_i = 1, t_i \in [-1,1]\right\}.
\end{aligned}$$

Theorem 5.

(i) $\displaystyle\bigcup_{G \in \mathcal{T}} \mathcal{T}_G = \mathcal{T}.$

(ii) $\displaystyle\bigcap_{G \in \mathcal{T}} \mathcal{T}_G = \left\{\frac{z}{(1-z)^2}\right\}.$

Proof. Notice that $\mathcal{T}_G \subset \mathcal{T}$. Hence, $\bigcup_{G\in\mathcal{T}} \mathcal{T}_G \subset \mathcal{T}$. Moreover, $G \in \mathcal{T}_G$, so $\bigcup_{G\in\mathcal{T}} \mathcal{T}_G \supset \bigcup_{G\in\mathcal{T}}\{G\} = \mathcal{T}$. From these facts we conclude that $\bigcup_{G\in\mathcal{T}} \mathcal{T}_G = \mathcal{T}$.

Now we prove the second part of Theorem 5.

Assume that $g_1(z) = \frac{z\,(1+z^2)}{(1-z^2)^2}$ and $g_2(z) = \frac{z}{1+z^2}$. Since $g_1, g_2 \in \mathrm{T}^{(2)}$, so functions $F_1(z) = \frac{z(1+z)^2}{(1-z)^4}$, $F_2(z) = \frac{z}{(1+z)^2}$ belong to $\mathcal{T}$.

First we prove that $\mathcal{T}_{F_1} \cap \mathcal{T}_{F_2} = \left\{\frac{z}{(1-z)^2}\right\}$. Let $F \in \mathcal{T}_{F_1} \cap \mathcal{T}_{F_2}$. Therefore from Theorem 4 we have

$$F(z) = (1+z)\sqrt{\frac{F_1(z)}{z}}\, f_1(z) = (1+z)\sqrt{\frac{F_2(z)}{z}}\, f_2(z), \quad \text{where } f_1, f_2 \in \mathrm{T}.$$

Suppose that $f_1(z) = z + a_2 z^2 + \ldots$ and $f_2(z) = z + b_2 z^2 + \ldots$. Then

$$F(z) = (1 + 4z + 8z^2 + \ldots)(z + a_2 z^2 + \ldots) = z + (4 + a_2) z^2 + \ldots$$

and
$$F(z) = f_2(z) = z + b_2 z^2 + \ldots \quad .$$
Because $f_1, f_2 \in \mathrm{T}$, so $-2 \le a_2 \le 2$ and $-2 \le b_2 \le 2$. From these and the equality $4 + a_2 = b_2$ we conclude that $a_2 = -2$ (and $b_2 = 2$). Thus $f_1(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 + \ldots$ (and $f_2(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots$). Hence $F(z) = \frac{z}{(1-z)^2}$.

We obtain that $\bigcap\limits_{G \in \mathcal{T}} \mathcal{T}_G \subset \mathcal{T}_{F_1} \cap \mathcal{T}_{F_2} = \left\{ \frac{z}{(1-z)^2} \right\}$.

Now we prove that $\frac{z}{(1-z)^2} \in \mathcal{T}_G$ for all $G \in \mathcal{T}$. From [8] we know the Rogosinski representation

$$h \in \mathrm{T} \iff h(z) = \frac{z\,p(z)}{1-z^2}, \quad p \in P_R, \tag{6}$$

where P_R consists of all analytic functions p such that $\operatorname{Re} p(z) > 0$, $p(0) = 1$ and having real coefficients. From (6) and the fact that $p \in P_R \iff \frac{1}{p} \in P_R$ we get

$$\tfrac{1}{p(z)} = \frac{z}{(1-z^2)\,h(z)} \in P_R.$$

Let $f(z) = \frac{z}{1-z^2}\frac{1}{p(z)} = \left(\frac{z}{1-z^2}\right)^2 \frac{1}{h(z)}$. From the above relations $f \in \mathrm{T} \iff h \in \mathrm{T}$.

From Theorem 4 we get

$$\mathcal{T}_G = \left\{ \frac{(1+z)}{z}\sqrt{z\,G(z)}\; f(z) :\; f \in \mathrm{T} \right\} = \left\{ \frac{(1+z)^2}{z}\, h(z)\; f(z) :\; f \in \mathrm{T} \right\},$$

where $h(z) = \frac{\sqrt{z\,G(z)}}{1+z}$. From (3) we know that $h \in \mathrm{T}$.

For $f(z) = \left(\frac{z}{1-z^2}\right)^2 \frac{1}{h(z)}$ we know that the function $\frac{(1+z)^2}{z}\, h(z)\; f(z) = \frac{z}{(1-z)^2}$ is in $\mathcal{T}_G$. ■

3 Some properties of the class $\mathcal{T}_{id}$.

Let us consider the class $\mathcal{T}_G$, where $G(z) = z$. Denote this class by $\mathcal{T}_{id}$. Then $H \in \mathcal{T}_{id} \iff H(z) = \sqrt{z\,F(z)}$, $F \in \mathcal{T}$. Hence

$$\frac{z\,H'(z)}{H(z)} = \frac{1}{2}\left(\frac{z\,F'(z)}{F(z)} + 1\right), \quad F \in \mathcal{T}. \tag{7}$$

From (7) and Corollary 1 we have

$$2\operatorname{Re}\frac{z\,H'(z)}{H(z)} = \operatorname{Re}\left(\frac{z\,F'(z)}{F(z)} + 1\right) \ge \frac{1-6r+r^2}{1-r^2} + 1 = \frac{2-6r}{1-r^2}$$

for $z \in \Delta$, $r = |z|$. Therefore, $\mathrm{Re}\frac{zH'(z)}{H(z)} > 0$ for $0 \le r < \frac{1}{3}$. So $r_{ST}(\mathcal{T}_{id}) \ge \frac{1}{3}$. Observe that $\min \mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right)$ is reached by the function F given in Corollary 1, thus $\min \mathrm{Re}\left(\frac{zH'(z)}{H(z)}\right)$ is reached by the function $H_0(z) = \frac{z(1+z)}{(1-z)^2}$ for $z = -r$.

Furthermore, we have $H_0'\left(-\frac{1}{3}\right) = 0$. This implies $r_{LU}(\mathcal{T}_{id}) \le \left|-\frac{1}{3}\right| = \frac{1}{3}$. From these and (4) we get inequalities $\frac{1}{3} \le r_{ST}(\mathcal{T}_{id}) \le r_{LU}(\mathcal{T}_{id}) \le \frac{1}{3}$, which finally lead us to equalities $r_{ST}(\mathcal{T}_{id}) = r_S(\mathcal{T}_{id}) = r_{LU}(\mathcal{T}_{id}) = \frac{1}{3}$.

We have proved the following theorem.

Theorem 6. $r_{ST}(\mathcal{T}_{id}) = r_S(\mathcal{T}_{id}) = r_{LU}(\mathcal{T}_{id}) = \frac{1}{3}$.

4 Some properties of the class $\mathcal{T}_G$ for $G(z) = \frac{z}{(1-z)^2}$.

Let us study $\mathcal{T}_G$, where $G(z) = \frac{z}{(1-z)^2}$. Thus $H \in \mathcal{T}_G \iff H(z) = \frac{\sqrt{zF(z)}}{1-z}$, $F \in \mathcal{T}$. Therefore

$$\frac{zH'(z)}{H(z)} = \frac{1}{2}\left(\frac{zF'(z)}{F(z)} + \frac{1+z}{1-z}\right), \quad F \in \mathcal{T}. \tag{8}$$

Taking into account (8) and Corollary 1 we obtain

$$\begin{aligned} 2\,\mathrm{Re}\frac{zH'(z)}{H(z)} &= \mathrm{Re}\left(\frac{zF'(z)}{F(z)} + \frac{1+z}{1-z}\right) \\ &\ge \frac{1-6r+r^2}{1-r^2} + \frac{1-r}{1+r} = \frac{2(1-4r+r^2)}{1-r^2}, \end{aligned}$$

for $z \in \Delta$, $r = |z|$. Then, $\mathrm{Re}\frac{zH'(z)}{H(z)} > 0$ for $0 \le r < 2-\sqrt{3}$. Hence $r_{ST}(\mathcal{T}_G) \ge 2-\sqrt{3}$. Since $\min \mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right)$ is reached by the function F given in Corollary 1, so $\min \mathrm{Re}\left(\frac{zH'(z)}{H(z)}\right)$ is reached by the function $H_0(z) = \frac{z(1+z)}{(1-z)^3}$ for $z = -r$.

Moreover, $H_0'\left(-2+\sqrt{3}\right) = 0$. Thus $r_{LU}(\mathcal{T}_G) \le \left|-2+\sqrt{3}\right| = 2-\sqrt{3}$. The inequality (4) and the above facts give us $2-\sqrt{3} \le r_{ST}(\mathcal{T}_G) \le r_{LU}(\mathcal{T}_G) \le 2-\sqrt{3}$, so finally $r_{ST}(\mathcal{T}_G) = r_S(\mathcal{T}_G) = r_{LU}(\mathcal{T}_G) = 2-\sqrt{3}$.

We have proved the following theorem.

Theorem 7. *For* $G(z) = \frac{z}{(1-z)^2}$ *we have* $r_{ST}(\mathcal{T}_G) = r_S(\mathcal{T}_G) = r_{LU}(\mathcal{T}_G) = 2-\sqrt{3}$.

5 Some properties of the class $\mathcal{T}_G$ for $G(z) = \frac{z}{(1+z)^2}$.

Let us investigate the class $\mathcal{T}_G$, where $G(z) = \frac{z}{(1+z)^2}$. Hence from Theorem 4 we get the following theorem.

Theorem 8. *For* $G(z) = \frac{z}{(1+z)^2}$ *we have* $\mathcal{T}_G = \mathrm{T}$.

Theorem 8 and also [1] and [4] give us the following corollary.

Corollary 3. *For* $G(z) = \frac{z}{(1+z)^2}$ *have:*

(i) $r_{ST}(\mathcal{T}_G) = r_S(\mathcal{T}_G) = r_{LU}(\mathcal{T}_G) = \sqrt{2} - 1$.

(ii) $G_{LU}(\mathcal{T}_G) = \{z \in \Delta : 2|z| < |1+z^2|\} = \left\{z : |z+i| < \sqrt{2}\right\} \cap \left\{z : |z-i| < \sqrt{2}\right\}$.

6 Some properties of the class $\mathcal{T}_G$ for $G(z) = \frac{z(1+z)^2}{(1-z)^4}$.

Let us consider $\mathcal{T}_G$, where $G(z) = \frac{z(1+z)^2}{(1-z)^4}$. This implies $H \in \mathcal{T}_G \iff H(z) = \frac{(1+z)\sqrt{zF(z)}}{(1-z)^2}$, $F \in \mathcal{T}$. Therefore

$$\frac{zH'(z)}{H(z)} = \frac{1}{2}\left(\frac{zF'(z)}{F(z)} + \frac{1+6z+z^2}{1-z^2}\right), \quad F \in \mathcal{T}. \tag{9}$$

Relation (9) and Corollary 1 give us

$$\begin{aligned} 2\,\mathrm{Re}\frac{zH'(z)}{H(z)} &= \mathrm{Re}\left(\frac{zF'(z)}{F(z)} + \frac{1+6z+z^2}{1-z^2}\right) \\ &\geq \frac{1-6r+r^2}{1-r^2} + \frac{1-6r+r^2}{1-r^2} = 2\,\frac{1-6r+r^2}{1-r^2}, \end{aligned}$$

for $z \in \Delta$, $r = |z|$. Thus, $\mathrm{Re}\frac{zH'(z)}{H(z)} > 0$ for $0 \leq r < 3 - 2\sqrt{2} = (\sqrt{2}-1)^2$. Therefore $r_{ST}(\mathcal{T}_G) \geq (\sqrt{2}-1)^2$. Due to the fact that $\min \mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right)$ is reached by the function F given in Corollary 1, so $\min \mathrm{Re}\left(\frac{zH'(z)}{H(z)}\right)$ is reached by the function $G(z) = \frac{z(1+z)^2}{(1-z)^4}$ for $z = -r$ $(G \in \mathcal{T}_G)$.

Apart from these, $G'\left(-3+2\sqrt{2}\right) = 0$. Then $r_{LU}(\mathcal{T}_G) \leq \left|-3+2\sqrt{2}\right| = 3 - 2\sqrt{2} = (\sqrt{2}-1)^2$. From these and (4) we get inequalities $(\sqrt{2}-1)^2 \leq r_{ST}(\mathcal{T}_G) \leq r_{LU}(\mathcal{T}_G) \leq (\sqrt{2}-1)^2$, so finally $r_{ST}(\mathcal{T}_G) = r_S(\mathcal{T}_G) = r_{LU}(\mathcal{T}_G) = (\sqrt{2}-1)^2$.

We have proved the following theorem.

Theorem 9. *For* $G(z) = \frac{z(1+z)^2}{(1-z)^4}$ *we have* $r_{ST}(\mathcal{T}_G) = r_S(\mathcal{T}_G) = r_{LU}(\mathcal{T}_G) = (\sqrt{2}-1)^2$.

7 Some properties of the class $\mathcal{T}_G$ for $G(z) = z(1+z)^2$.

Let us study the class $\mathcal{T}_G$, where $G(z) = z(1+z)^2$. Thus, $H \in \mathcal{T}_G \iff H(z) = (1+z)\sqrt{z\,F(z)}$, $F \in \mathcal{T}$. Then

$$\frac{z\,H'(z)}{H(z)} = \frac{1}{2}\left(\frac{z\,F'(z)}{F(z)} + \frac{1+3z}{1+z}\right), \quad F \in \mathcal{T}. \tag{10}$$

Taking into account (10) and Corollary 1 we conclude

$$\begin{aligned} 2\,\mathrm{Re}\frac{z\,H'(z)}{H(z)} &= \mathrm{Re}\left(\frac{z\,F'(z)}{F(z)} + \frac{1+3z}{1+z}\right) \\ &\geq \frac{1-6r+r^2}{1-r^2} + \frac{1-3r}{1-r} = 2\,\frac{1-4r-r^2}{1-r^2}, \end{aligned}$$

for $z \in \Delta$, $r = |z|$. Therefore, $\mathrm{Re}\frac{z\,H'(z)}{H(z)} > 0$ for $0 \leq r < \sqrt{5}-2$. Hence $r_{ST}(\mathcal{T}_G) \geq \sqrt{5}-2$. Since $\min \mathrm{Re}\left(\frac{z\,F'(z)}{F(z)}\right)$ is reached by the function F given in Corollary 1, then $\min \mathrm{Re}\left(\frac{z\,H'(z)}{H(z)}\right)$ is reached by the function $H_0(z) = \frac{z(1+z)^2}{(1-z)^2}$ for $z = -r$.

Moreover, $H_0'\left(2-\sqrt{5}\right) = 0$. This implies $r_{LU}(\mathcal{T}_G) \leq \left|2-\sqrt{5}\right| = \sqrt{5}-2$. From these and (4) we have $\sqrt{5}-2 \leq r_{ST}(\mathcal{T}_G) \leq r_{LU}(\mathcal{T}_G) \leq \sqrt{5}-2$. These finally lead us to equalities $r_{ST}(\mathcal{T}_G) = r_S(\mathcal{T}_G) = r_{LU}(\mathcal{T}_G) = \sqrt{5}-2$.

We have proved the following theorem.

Theorem 10. *For $G(z) = z(1+z)^2$ we have $r_{ST}(\mathcal{T}_G) = r_S(\mathcal{T}_G) = r_{LU}(\mathcal{T}_G) = \sqrt{5}-2$.*

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