On the functoriality of the blow-up construction

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Abstract

We describe an explicit model for the blow-up construction in the smooth (or real analytic) category. We use it to prove the following functoriality property of the blow-up: Let M and N be smooth (real analytic) manifolds, with submanifolds A and B respectively. Let $f\colon M \to N$ be a smooth (real analytic) function such that $f^{-1}(B) = A$, and such that f induces a fiberwise injective map from the normal space of A to the normal space of B. Then f has a unique lift to a smooth (real analytic) map between the blow-ups. In this way, the blow-up construction defines a continuous functor. As an application, we show how an action of a Lie group on a manifold lifts, under minimal hypotheses, to an action on a blow-up.

1 Introduction

The blow-up of a smooth (or real analytic) manifold at a submanifold is an important construction in geometry and topology. While the construction and its basic properties are well-known, they also seem to be somewhat folklore. Since each one of the present authors had an occasion to be frustrated with the search for a reference to a proof of the basic properties of the blow-up construction (especially properties having to do with functoriality in the smooth, or real analytic, category), we decided to write such a reference ourselves.

During the introduction, we will mostly discuss the case of smooth manifolds, but one can substitute "real analytic" for "smooth" in everything that follows.

Let M be a smooth manifold, and let $A \subset M$ be a closed submanifold. Informally speaking, the (projective) blow-up of M at A is a construction that removes

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A from M, and replaces it with the projectivization of the normal bundle of A in M in the most natural way possible. We will denote the blow-up by $\mathrm{B}(M, A)$.

Let us list some of the properties that one expects the construction of $\mathrm{B}(M, A)$ to have. $\mathrm{B}(M, A)$ should be a smooth manifold, there should be a smooth map $\pi\colon \mathrm{B}(M, A) \longrightarrow M$ such that π restricts to a diffeomorphism from $\pi^{-1}(M \setminus A)$ to $M \setminus A$, while $\pi^{-1}(A)$ is $\mathrm{P}_M A$, the projectivization of the normal bundle of A in M. Furthermore, we would like the construction to be as functorial as one may reasonably expect. To wit, suppose N is another smooth manifold, with a submanifold B. Let $f\colon M \longrightarrow N$ be a smooth map such that $f^{-1}(B) = A$, and such that the derivative of f induces a fiberwise injection from the normal bundle of A to the normal bundle of B. Then f restricts to a map from $M \setminus A$ to $N \setminus B$, and f also induces a map from $\mathrm{P}_M A$ to $\mathrm{P}_N B$. Thus, at least on set level, f induces a canonical function from $\mathrm{B}(M, A)$ to $\mathrm{B}(N, B)$. We would like this induced map to be smooth (real analytic, if f is real analytic).

In this paper we present a construction with all these properties. Our approach is standard, insofar that we first deal with the local question of constructing the blow-up of a Euclidean space $\mathbb{R}^{m+n}$ at a linear subspace $\mathbb{R}^m \times \{0\}$. Once we establish the naturality properties of the local construction, it is easy to extend it to general manifolds using charts.

Our treatment of the local construction is somewhat non-standard. Let $\mathbb{R}\mathrm{P}^{n-1}$ be the projective space of lines in $\mathbb{R}^n$. There is a canonical map $q\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}\mathrm{P}^{n-1}$ sending a point x to the line through the origin containing x. The customary way to construct the blow-up of $\mathbb{R}^{m+n}$ at $\mathbb{R}^m$ is to define it to be the closure of the embedding

$$\mathbb{R}^{m+n} \setminus \mathbb{R}^m \hookrightarrow \mathbb{R}^{m+n} \times \mathbb{R}\mathrm{P}^{n-1}.$$

The embedding above is defined by the inclusion map on the first coordinate, and by the composed map

$$\mathbb{R}^{m+n} \setminus \mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^m \times (\mathbb{R}^n \setminus \{0\}) \longrightarrow \mathbb{R}^n \setminus \{0\} \xrightarrow{q} \mathbb{R}\mathrm{P}^{n-1}$$

on the second coordinate. With this definition, the manifold structure of the blow-up and especially its functoriality properties are not entirely obvious, and are somewhat awkward to prove.

Instead, we consider the space $\mathbb{R}^m \times \mathbb{R} \times S^{n-1}$, with the action of the group $\mathbb{Z}_2$ defined by $\tau(x, r, \bar{\theta}) = (x, -r, -\bar{\theta})$, where $\tau \in \mathbb{Z}_2$ is the non-identity element. This is a free, real analytic action on a real analytic manifold, and we define $\mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m)$ to be the quotient space of this action. It is easy to see that this construction produces the same space as the standard construction described above. For the blow-up map, consider the map

$$\mu\colon \mathbb{R}^m \times \mathbb{R} \times S^{n-1} \longrightarrow \mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$$

defined by $\mu(x, r, \bar{\theta}) = (x, r \cdot \bar{\theta})$. Clearly, μ satisfies $\mu(x, r, \bar{\theta}) = \mu(\tau(x, r, \bar{\theta}))$, and thus μ passes to a map $\pi\colon \mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m) \longrightarrow \mathbb{R}^{m+n}$. With this definition, it is immediately obvious that $\mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m)$ is a smooth (in fact, real analytic) manifold. Moreover, given a smooth map $F\colon \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^{k+l}$ satisfying the required hypotheses, it is not difficult to write an explicit formula for a lift

$$\tilde{F}\colon \mathbb{R}^m \times \mathbb{R} \times S^{n-1} \longrightarrow \mathbb{R}^k \times \mathbb{R} \times S^{l-1}$$

of F, and a little calculation shows that $\tilde{F}$ is smooth (real analytic, if F is). See Lemma 2.5, which is really the key lemma of the paper. Additionally, it is easy to see that $\tilde{F}$ is $\mathbb{Z}_2$-equivariant, and so it induces a map

$$\hat{F}\colon \mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m) \longrightarrow \mathrm{B}(\mathbb{R}^{k+l}, \mathbb{R}^k)$$

which is, again, smooth (real analytic if F is).

Having established the functoriality of $\mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m)$, it is easy to extend the construction to manifolds in general, and prove its functoriality. See Theorem 4.1, which is our main theorem, for a precise statement. We certainly consider these results to be "elementary", but we were not able to find in the literature a proof of the functoriality of the blow-up in this generality. In particular, we are not aware of an explicit treatment of the real analytic case (some authors use it, but no source that we saw really constructs it). Note that our approach does not require choosing a Riemannian metric on M. This can be useful for applications to real analytic actions of Lie groups that are not necessarily compact. (While it is known that, for a compact Lie group G, every real analytic G-manifold admits a real analytic G-invariant Riemannian metric, the corresponding result is not known for all real analytic actions of non-compact Lie groups, not even if the actions are assumed to be proper.) We give an equivariant version of our blow-up construction in Section 5. An application of the equivariant blow-up is given in [3].

A variation of the construction is the spherical blow-up of M at A, where one replaces A with the sphere bundle of the normal bundle of A in M, rather than the projectivization. We will show that all our functoriality results hold for spherical blow-ups as well.

Remark 1.1. Our approach has the pleasant feature that locally the blow-up is constructed as the quotient of a smooth manifold by a free $\mathbb{Z}/2$-action; whence the manifold structure on the blow-up comes for free. Beyond that, we do not claim any originality in this paper. Our sole purpose is to provide a convenient reference for some well-known, but not very well-documented properties of the blow-up construction. After we wrote the paper, it was pointed out to us it overlaps with [1, Section 2]. Nevertheless, it seems to us that our treatment is a little more detailed.

2 Local construction

Let $\mathbb{R}$ denote the set of real numbers. Let $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ be the sets of positive and non-negative real numbers respectively. Let $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. Throughout this section, we identify $\mathbb{R}^m$ with $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n}$. Thus, $\mathbb{R}^{m+n} \setminus \mathbb{R}^m$ is canonically identified with $\mathbb{R}^m \times (\mathbb{R}^n \setminus \{0\})$. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. There is a real analytic diffeomorphism

$$\mathbb{R}^m \times \mathbb{R}_{>0} \times S^{n-1} \xrightarrow{\cong} \mathbb{R}^m \times (\mathbb{R}^n \setminus \{0\}) = \mathbb{R}^{m+n} \setminus \mathbb{R}^m$$

defined by the formula $(x, r, \bar{\theta}) \mapsto (x, r \cdot \bar{\theta})$. This diffeomorphism extends to a real analytic map

$$\mu\colon \mathbb{R}^m \times \mathbb{R} \times S^{n-1} \xrightarrow{\cong} \mathbb{R}^{m+n}$$

defined by $\mu(x,r,\bar{\theta}) = (x, r\cdot\bar{\theta})$. Let $\mu_{\geq 0}$ be the restriction of μ to $\mathbb{R}^m \times \mathbb{R}_{\geq 0} \times S^{n-1}$. Moreover, let $\mathbb{Z}_2$ act on $\mathbb{R}^m \times \mathbb{R} \times S^{n-1}$ by $\tau(x,r,\bar{\theta}) = (x,-r,-\bar{\theta})$. Define $\mathrm{B}(\mathbb{R}^{m+n},\mathbb{R}^m)$ to be the quotient space of this action. Clearly, this is a free, real analytic action, and therefore $\mathrm{B}(\mathbb{R}^{m+n},\mathbb{R}^m)$ has a canonical structure of a real analytic manifold. Also clearly, the map μ satisfies $\mu(x,r,\bar{\theta}) = \mu(\tau(x,r,\bar{\theta}))$, and therefore it passes to a real analytic map

$$\pi\colon \mathrm{B}(\mathbb{R}^{m+n},\mathbb{R}^m) \longrightarrow \mathbb{R}^{m+n}.$$

Definition 2.1. We define the spherical blow-up of $\mathbb{R}^{m+n}$ at $\mathbb{R}^m$ to be the pair $(\mathbb{R}^m \times \mathbb{R}_{\geq 0} \times S^{n-1}, \mu_{\geq 0})$. The projective blow-up of $\mathbb{R}^{m+n}$ at $\mathbb{R}^m$ is the pair $(\mathrm{B}(\mathbb{R}^{m+n},\mathbb{R}^m), \pi)$.

In this paper, we will generally use "blow-up" to mean "projective blow-up".

To summarize the relationship between the various constructions that we introduced: there are canonical real analytic maps

$$\mathbb{R}^m \times \mathbb{R}_{\geq 0} \times S^{n-1} \hookrightarrow \mathbb{R}^m \times \mathbb{R} \times S^{n-1} \longrightarrow \mathrm{B}(\mathbb{R}^{m+n},\mathbb{R}^m) \xrightarrow{\pi} \mathbb{R}^{m+n}.$$

Here the first map is the inclusion, the second map is the quotient map, and the composed map is $\mu_{\geq 0}$. The following lemma follows immediately from the definitions.

Lemma 2.2. *Both π and $\mu_{\geq 0}$ are proper, real analytic maps. Each of these maps restricts to a real analytic diffeomorphism between $\mathbb{R}^{m+n} \setminus \mathbb{R}^m$ and its inverse image. In addition, $\pi^{-1}(\mathbb{R}^m) \cong \mathbb{R}^m \times \mathbb{R}\mathrm{P}^{n-1}$ and $\mu_{\geq 0}^{-1}(\mathbb{R}^m) \cong \mathbb{R}^m \times S^{n-1}$.*

For a smooth map $G\colon \mathbb{R}^n \longrightarrow \mathbb{R}^l$ we use $\mathrm{D}_0 G$ to denote the linear map from $\mathbb{R}^n$ to $\mathbb{R}^l$ obtained by differentiating G at 0. We call it the derivative of G at 0.

Now let F be a smooth (real analytic) map $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^l$. We will write that $F = (F_1, F_2)$, where $(F_1,F_2)(x,y) = (F_1(x,y), F_2(x,y))$. Assume that F satisfies the following hypothesis.

Hypothesis 2.3. Let $F\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^l$ be a smooth map. Assume that $F^{-1}(\mathbb{R}^k \times \{0\}) = \mathbb{R}^m \times \{0\}$. Moreover, for every $x \in \mathbb{R}^m$, let $F_{2,x}\colon \mathbb{R}^n \to \mathbb{R}^l$ be the map $y \mapsto F_2(x,y)$. We assume that the derivative $\mathrm{D}_0 F_{2,x}$ is injective for all $x \in \mathbb{R}^m$. In other words, F induces a fiber-wise injective map from the normal bundle of $\mathbb{R}^m \times \{0\}$ to the normal bundle of $\mathbb{R}^k \times \{0\}$.

Remark 2.4. It is easy to see that if F is a diffeomorphism such that $F(\mathbb{R}^m \times \{0\}) = \mathbb{R}^k \times \{0\}$, then F satisfies the hypothesis.

Let F be a map satisfying the hypothesis. Define the map

$$\widetilde{F}\colon \mathbb{R}^m \times \mathbb{R} \times S^{n-1} \longrightarrow \mathbb{R}^k \times \mathbb{R} \times S^{l-1},$$

by the following formula:

$$\widetilde{F}(x,r,\bar{\theta}) = \begin{cases} \left(F_1(x,r\bar{\theta}), \frac{r}{|r|}\|F_2(x,r\bar{\theta})\|, \frac{r}{|r|}\frac{F_2(x,r\bar{\theta})}{\|F_2(x,r\bar{\theta})\|}\right), & \text{if } r \neq 0 \\ \left(F_1(x,r\bar{\theta}), 0, \frac{\mathrm{D}_0 F_{2,x}(\bar{\theta})}{\|\mathrm{D}_0 F_{2,x}(\bar{\theta})\|}\right), & \text{if } r = 0. \end{cases}.$$

We will prove shortly that $\widetilde{F}$ is smooth (real analytic, if F is real analytic) on its entire domain. First, let us make a couple of obvious observations about $\widetilde{F}$. Clearly, $\widetilde{F}$ restricts to a map from $\mathbb{R}^m \times \mathbb{R}_{\geq 0} \times S^{n-1}$ to $\mathbb{R}^k \times \mathbb{R}_{\geq 0} \times S^{l-1}$. We denote this restriction of $\widetilde{F}$ by $\widetilde{F}_{\geq 0}$. Furthermore, it is easy to see that $\widetilde{F}$ is equivariant with respect to the $\mathbb{Z}_2$-action that we defined on the source and the target. Thus $\widetilde{F}$ induces a map

$$\bar{F}\colon \mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m) \longrightarrow \mathrm{B}(\mathbb{R}^{k+l}, \mathbb{R}^k),$$

such that the following diagram commutes (here the vertical maps are the canonical maps defined above).

$$\begin{array}{ccc}
\mathbb{R}^m \times \mathbb{R}_{\geq 0} \times S^{n-1} & \xrightarrow{\widetilde{F}_{\geq 0}} & \mathbb{R}^k \times \mathbb{R}_{\geq 0} \times S^{l-1} \\
\downarrow & & \downarrow \\
\mathbb{R}^m \times \mathbb{R} \times S^{n-1} & \xrightarrow{\widetilde{F}} & \mathbb{R}^k \times \mathbb{R} \times S^{l-1} \\
\downarrow & & \downarrow \\
\mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m) & \xrightarrow{\bar{F}} & \mathrm{B}(\mathbb{R}^{k+l}, \mathbb{R}^k) \\
\downarrow \pi & & \downarrow \pi \\
\mathbb{R}^{m+n} & \xrightarrow{F} & \mathbb{R}^{k+l}
\end{array}.$$

Now, to the main lemma of this paper.

Lemma 2.5. *Suppose that F satisfies Hypothesis 2.3. Then the map $\widetilde{F}$ that we defined above is is smooth. If F is real analytic, then also $\widetilde{F}$ is real analytic.*

Proof. During the proof, we will assume that F is smooth, and prove that $\widetilde{F}$ is smooth. However, if F is real analytic, the same proof will show that $\widetilde{F}$ is real analytic. All one needs to do is to replace the word "smooth" with "real analytic" throughout the proof.

The map $\widetilde{F}$ is defined by means of three coordinate functions, and we need to prove that each one of these functions is smooth. The first coordinate function is $F_1(x, r\bar{\theta})$, and it is obviously smooth, because F_1 is smooth.

The second and third coordinate functions involve the norm function $\| - \|\colon \mathbb{R}^l \longrightarrow \mathbb{R}$, as well as the absolute value function. These functions are not smooth, but the restriction of $\| - \|$ to $\mathbb{R}^l \setminus \{0\}$ is smooth, and the absolute value function is also smooth away from zero. Thus if $G\colon M \longrightarrow \mathbb{R}^l$ is a smooth function, then $\|G\|$ is a smooth function at all points $x \in M$ for which $G(x) \neq 0$.

With this in mind, we claim that $\widetilde{F}$ is clearly smooth on $\mathbb{R}^m \times (\mathbb{R} \setminus \{0\}) \times S^{n-1}$. Indeed, by Hypothesis 2.3, $F^{-1}(\mathbb{R}^k \times \{0\}) = \mathbb{R}^m \times \{0\}$. It follows that when $r \neq 0$, $F_2(x, r\bar{\theta}) \neq 0$ and from here it follows that $\frac{r}{|r|}\|F_2(x, r\bar{\theta})\|$ is smooth as long as $r \neq 0$.

It remains to show that $\widetilde{F}$ is smooth at points of type $(x, 0, \bar{\theta})$, where $x \in \mathbb{R}^m$ and $\bar{\theta} \in S^{n-1}$. Let us study the second coordinate function, which is $\frac{r}{|r|}\|F_2(x, r\bar{\theta})\|$. This function is well-defined for $r \neq 0$, and our goal is to show that if one extends it to have value 0 for $r = 0$ one obtains a smooth function. Consider first the function $F_2(x, r\bar{\theta})$. This is a smooth function of the three variables x, r, and $\bar{\theta}$, where $x \in \mathbb{R}^m$, $r \in \mathbb{R}$, and $\bar{\theta} \in S^{n-1}$ (if one wishes, one may enlarge the domain of $\bar{\theta}$ to be $\mathbb{R}^n \setminus \{0\}$). For a brief moment, we would like to consider x and $\bar{\theta}$ as parameters, and only r as a variable. Therefore, for fixed $x, \bar{\theta}$, define the function $G_{x,\bar{\theta}}\colon \mathbb{R} \longrightarrow \mathbb{R}^l$ by $G_{x,\bar{\theta}}(r) = F_2(x, r\bar{\theta})$. Let us consider the Taylor expansion of this function in the variable r at zero. It follows from Taylor's theorem that

$$F_2(x, r\bar{\theta}) = G_{x,\bar{\theta}}(r) = G_{x,\bar{\theta}}(0) + \frac{dG_{x,\bar{\theta}}}{dr}(0) \cdot r + r^2 \cdot k(x, r, \bar{\theta})$$

where $k(x, r, \bar{\theta})$ is a smooth function. The assumption that $F^{-1}(\mathbb{R}^k \times \{0\}) = \mathbb{R}^m \times \{0\}$ implies that $G_{x,\bar{\theta}}(0) = F_2(x, 0) = 0$. On the other hand, an elementary calculation shows that $\frac{dG_{x,\bar{\theta}}}{dr}(0) = \mathrm{D}_0 F_{2,x}(\bar{\theta})$. We obtain that

$$F_2(x, r\bar{\theta}) = r \cdot \mathrm{D}_0 F_{2,x}(\bar{\theta}) + r^2 \cdot k(x, r, \bar{\theta}).$$

It follows that

$$\|F_2(x, r\bar{\theta})\| = |r| \cdot \|\mathrm{D}_0 F_{2,x}(\bar{\theta}) + r \cdot k(x, r, \bar{\theta})\|$$

and thus

$$\frac{r}{|r|} \cdot \|F_2(x, r\bar{\theta})\| = r \cdot \|\mathrm{D}_0 F_{2,x}(\bar{\theta}) + r \cdot k(x, r, \bar{\theta})\|.$$

Clearly, the right hand side is well-defined for $r = 0$, and its value at $r = 0$ is zero. We need to show that it is smooth. Since $\mathrm{D}_0 F_{2,x}(\bar{\theta}) + r \cdot k(x, r, \bar{\theta})$ is obviously a smooth function of $x, r, \bar{\theta}$, it is enough to show that $\mathrm{D}_0 F_{2,x}(\bar{\theta}) + r \cdot k(x, r, \bar{\theta}) \neq 0$ for all $(x, r, \bar{\theta}) \in \mathbb{R}^m \times \mathbb{R} \times S^{n-1}$. We already saw that $\mathrm{D}_0 F_{2,x}(\bar{\theta}) + r \cdot k(x, r, \bar{\theta}) \neq 0$ whenever $r \neq 0$. For the case $r = 0$, note that the value of this function is $\mathrm{D}_0 F_{2,x}(\bar{\theta})$, and since by Hypothesis 2.3 $\mathrm{D}_0 F_{2,x}$ is an injective linear homomorphism, and $\bar{\theta}$ is a unit vector, it follows that $\mathrm{D}_0 F_{2,x}(\bar{\theta}) \neq 0$. Therefore, $\frac{r}{|r|} \cdot \|F_2(x, r\bar{\theta})\|$ extends to a smooth function at $r = 0$, as claimed.

The proof that $\frac{r}{|r|} \frac{F_2(x, r\bar{\theta})}{\|F_2(x, r\bar{\theta})\|}$ is smooth at $r = 0$ is similar. We have the following identity, valid for $r \neq 0$.

$$\frac{r}{|r|} \frac{F_2(x, r\bar{\theta})}{\|F_2(x, r\bar{\theta})\|} = \frac{r^2 \left(\mathrm{D}_0 F_{2,x}(\bar{\theta}) + r \cdot k(x, r, \bar{\theta})\right)}{|r|^2 \cdot \|\mathrm{D}_0 F_{2,x}(\bar{\theta}) + r \cdot k(x, r, \bar{\theta})\|} = \frac{\mathrm{D}_0 F_{2,x}(\bar{\theta}) + r \cdot k(x, r, \bar{\theta})}{\|\mathrm{D}_0 F_{2,x}(\bar{\theta}) + r \cdot k(x, r, \bar{\theta})\|}.$$

By the same argument as before, the right hand side is a well-defined, smooth function at all points $(x, r, \bar{\theta}) \in \mathbb{R}^m \times \mathbb{R} \times S^{n-1}$. Thus

$$\frac{r}{|r|} \frac{F_2(x, r\bar{\theta})}{\|F_2(x, r\bar{\theta})\|}$$

extends to a smooth function, whose value at $(x, 0, \bar{\theta})$ is

$$\frac{\mathrm{D}_0 F_{2,x}(\bar{\theta})}{\|\mathrm{D}_0 F_{2,x}(\bar{\theta})\|},$$

as claimed. ∎

Corollary 2.6. *Let $F\colon \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^{k+l}$ be a smooth (real analytic) map satisfying Hypothesis 2.3. Then the map $\bar{F}\colon \mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m) \longrightarrow \mathrm{B}(\mathbb{R}^{k+l}, \mathbb{R}^k)$ defined above is a smooth (real analytic) map. Similarly, the map $\widetilde{F}_{\geq 0}$ induces a smooth (real analytic) map between spherical blow-ups.*

Equip $C^\infty(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^k \times \mathbb{R}^l)$ (and other mapping spaces) with the Whitney C^∞-topology (i.e., the strong topology, see Chapter 2 in [2]). Let $A \subset C^\infty(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^k \times \mathbb{R}^l)$ be the set of C^∞-maps f satisfying Hypothesis 2.3. Then A inherits the topology from $C^\infty(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^n \times \mathbb{R}^l)$. The proof of the following lemma is straightforward, using the formula for $\widetilde{F}$.

Lemma 2.7. *The map*

$$A \to C^\infty(\ \mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m), \mathrm{B}(\mathbb{R}^{k+l}, \mathbb{R}^k)\ ),\ f \mapsto \bar{f},$$

is continuous.

Let G be a smooth map $\mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^p \times \mathbb{R}^q$. Assume that G satisfies Hypothesis 2.3 (with superscripts adjusted accordingly), so G induces a map $\overline{G}\colon \mathrm{B}(\mathbb{R}^{k+l}, \mathbb{R}^k) \longrightarrow (\mathbb{R}^{p+q}, \mathbb{R}^p)$. The proof of the following lemma is easy:

Lemma 2.8. *Let F and G be as above. Then $\overline{G \circ F} = \bar{G} \circ \bar{F}$.*

Lemmas 2.7 and 2.8 say that the blow-up construction defines a continuous functor from the category of pairs of the form $(\mathbb{R}^{m+n}, \mathbb{R}^m)$, and smooth (real analytic) maps satisfying Hypothesis 2.3 to the category of smooth (real analytic) manifolds.

3 Construction of the blow-up in the general case

In this section we will construct the blow-up of a smooth (real analytic) manifold M at a closed smooth (real analytic) submanifold A. The manifold M may have a boundary, and in this case we assume that A is *neatly embedded* in M, in the sense of [2]. Recall that this means that $\partial A = A \cap \partial M$ and A is not tangent to ∂M at any point of $A \cap \partial M$.

In what follows, we fix a Euclidean space $\mathbb{R}^{m+n}$, and we identify $\mathbb{R}^m$ with the subspace $\mathbb{R}^m \times \{0\}$ of $\mathbb{R}^{m+n}$. Recall that $\pi\colon \mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m) \longrightarrow \mathbb{R}^{m+n}$ is the blow-up map. We may now define the blow-up at $\mathbb{R}^m$ for arbitrary codimension zero submanifolds of $\mathbb{R}^{m+n}$ in the following way. Let $O \subset \mathbb{R}^{m+n}$ be a codimension zero submanifold. The submanifold O may have a boundary, in which case we require that $O \cap \mathbb{R}^m$ is a neatly embedded submanifold of O. Define the blow-up of O at $O \cap \mathbb{R}^m$ to be the space

$$\mathrm{B}(O, O \cap \mathbb{R}^m) := \pi^{-1}(O)$$

together with the evident map $\mathrm{B}(O, O \cap \mathbb{R}^m) \longrightarrow O$, which we will continue denoting by π. Clearly, it is still true that π restricts to a diffeomorphism $\pi^{-1}(O \setminus O \cap \mathbb{R}^m) \xrightarrow{\cong} O \setminus O \cap \mathbb{R}^m$ and that $\pi^{-1}(O \cap \mathbb{R}^m) \cong (O \cap \mathbb{R}^m) \times \mathbb{R}\mathrm{P}^{n-1}$.

Remark 3.1. It is easy to see from the definition that if U, V are subsets of $\mathbb{R}^{m+n}$ then

$$\mathrm{B}(U, U \cap \mathbb{R}^m) \cap \mathrm{B}(V, V \cap \mathbb{R}^m) = \mathrm{B}(U \cap V, U \cap V \cap \mathbb{R}^m)$$

and

$$\mathrm{B}(U, U \cap \mathbb{R}^m) \cup \mathrm{B}(V, V \cap \mathbb{R}^m) = \mathrm{B}(U \cup V, (U \cup V) \cap \mathbb{R}^m),$$

where the union and the intersection on the left hand sides is taken by considering all these spaces as subspaces of $\mathrm{B}(\mathbb{R}^{m+n}, \mathbb{R}^m)$.

Once again, let M be a smooth (real analytic) $(m+n)$-dimensional manifold, possibly with boundary, and let A be a closed neatly embedded m-dimensional submanifold. Let $\{(U, \alpha)\}$ be a collection of charts of M such that $\alpha(U \cap A) = \alpha(U) \cap \mathbb{R}^m$, and $\alpha(U \cap A)$ is neatly embedded in $\alpha(U)$. Let (U, α) and (V, β) be charts in the collection, such that $U \cap V \neq \emptyset$. It is easy to see, using Remark 2.4 and Corollary 2.6, that the map $\beta \circ \alpha^{-1}$ induces a smooth (real analytic) map $\overline{\beta \circ \alpha^{-1}}$ between the blow-ups such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{B}(\alpha(U \cap V), \alpha(U \cap V \cap A)) & \xrightarrow{\overline{\beta \circ \alpha^{-1}}} & \mathrm{B}(\beta(U \cap V), \beta(U \cap V \cap A)) \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ \alpha(U \cap V) & \xrightarrow{\beta \circ \alpha^{-1}} & \beta(U \cap V) \end{array} \qquad (*).$$

Let T be the disjoint union of all the spaces $\mathrm{B}(\alpha(U), \alpha(U \cap A))$. We define a relation $\sim$ on T by setting $x \sim y$, if $x \in \mathrm{B}(\alpha(U \cap V), \alpha(U \cap V \cap A))$, $y \in \mathrm{B}(\beta(U \cap V), \beta(U \cap V \cap A))$ and $y = \overline{\beta \circ \alpha^{-1}}(x)$. Then $\sim$ is an equivalence relation.

Let $\mathrm{B}(M, A)$ denote the quotient space $T/\sim$, and let $p \colon T \to \mathrm{B}(M, A)$ be the quotient map. We endow $\mathrm{B}(M, A)$ with the quotient topology: a set O is open in $\mathrm{B}(M, A)$ if and only if $p^{-1}(O)$ is open in T. Note that $\sim$ is an open relation, and thus p is an open map. In particular, p restricts to an open embedding of each $\mathrm{B}(\alpha(U), \alpha(U \cap A))$ into $\mathrm{B}(M, A)$. We define a smooth (real analytic) differential structure on $\mathrm{B}(M, A)$ by requiring that the restriction of p to each $\mathrm{B}(\alpha(U), \alpha(U \cap A))$ is a smooth (real analytic) diffeomorphism onto its image $p(\mathrm{B}(\alpha(U), \alpha(U \cap A)))$. The functoriality properties of local blow-ups guarantee that $\mathrm{B}(M, A)$ has the local structure of a smooth (real analytic, if M and A are real analytic) manifold.

We still need to show that $\mathrm{B}(M, A)$ is Hausdorff (of course, we assume that M is Hausdorff). For this, it is enough to show that any two points $x, y \in \mathrm{B}(M, A)$ are contained in an open Hausdorff subspace of $\mathrm{B}(M, A)$. If x and y are both contained in $p(\mathrm{B}(\alpha(U), \alpha(U \cap A)))$ for some chart (U, α) then obviously they are contained in a Hausdorff subspace of $\mathrm{B}(M, A)$, because $p(\mathrm{B}(\alpha(U), \alpha(U \cap A))) \cong \mathrm{B}(\alpha(U), \alpha(U \cap A))$ is Hausdorff. Assume therefore that $x \in p(\mathrm{B}(\alpha(U), \alpha(U \cap A)))$ and $y \in p(\mathrm{B}(\beta(V), \beta(V \cap A)))$, but x and y do not belong to a single set of this form. Consider the image of x in M under the composed map

$$p(\mathrm{B}(\alpha(U), \alpha(U \cap A))) \xrightarrow{p^{-1}} \mathrm{B}(\alpha(U), \alpha(U \cap A)) \xrightarrow{\pi} \alpha(U) \xrightarrow{\alpha^{-1}} U \hookrightarrow M$$

and consider the image of y in M under the analogous map for y. These two images can not be the same, because if they were the same, then both x and y would be in $p(\mathrm{B}(\gamma(O), \gamma(O \cap A)))$ where (O, γ) is any chart containing the common image of x and y. Since x and y are mapped to different points of M, one can choose charts (U, α) and (V, β) containing the images of x and y respectively in M, such that $U \cap V = \emptyset = \alpha(U) \cap \beta(V)$. In this case, we may form the union chart $(U \cup V, \gamma)$, and x, y are both contained in $p(\mathrm{B}(\gamma(U \cup V), \gamma((U \cup V) \cap A)))$, again contradicting our assumption.

In conclusion, we have the following proposition. We have proved all but the last claim of this proposition. The last claim is left as an exercise.

Proposition 3.2. *Let* $\mathrm{B}(M, A)$ *be defined as above. Then* $\mathrm{B}(M, A)$ *is a smooth (real analytic, if M and A are) manifold. The boundary of* $\mathrm{B}(M, A)$ *equals to* $\mathrm{B}(\partial M, \partial M \cap A)$.

The blow-up maps $\pi\colon \mathrm{B}(\alpha(U), \alpha(U \cap A)) \longrightarrow \alpha(U)$ assemble into a smooth (real analytic) map (which we continue to denote by π).

$$\pi\colon \mathrm{B}(M, A) \to M.$$

We call the pair $(\mathrm{B}(M, A), \pi)$ *the blow-up of M at A.* It is now easy to check that π has the following properties.

1. π is a proper map.
2. $\pi^{-1}(A)$ is isomorphic, as a space over A, to the projectivization of the normal bundle of A.
3. π restricts to a smooth (real analytic) diffeomorphism $\pi^{-1}(M \setminus A) \to M \setminus A$.

Remark 3.3. The spherical blow-up of M at A can be defined in an analogous way, using our construction of local spherical blow-up. It has all the analogous properties, that are proved in a similar way. We will omit the details.

4 Functoriality of the blow-up

Let M be a smooth manifold and A a closed submanifold of M. Let $x \in A$. We denote the tangent spaces of M and A at x, by $\mathrm{T}_x(M)$ and $\mathrm{T}_x(A)$, respectively. The normal space of A at x is $\mathrm{N}_x(A) = \mathrm{T}_x(M)/\mathrm{T}_x(A)$.

Our construction of a blow-up satisfies the following property:

Theorem 4.1. *Let M and N be smooth (real analytic) manifolds with neat closed smooth (real analytic) submanifolds A and B, respectively. Let* $f\colon M \to N$ *be a smooth (real analytic) map such that* $f^{-1}(B) = A$. *Moreover, assume that the map* $\mathrm{N}_x(A) \to \mathrm{N}_{f(x)}(B)$, *induced by* f, *is injective for all* $x \in A$. *Then* f *induces a smooth (real analytic) map* $\bar{f}$ *between the blow-ups such that the diagram*

$$\begin{array}{ccc} \mathrm{B}(M, A) & \xrightarrow{\bar{f}} & \mathrm{B}(N, B) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & N \end{array}$$

commutes.

Proof. The proof follows immediately from the local properties of the blow-up construction; Corollary 2.6 and Diagram $(*)$. ∎

Clearly, if we choose $N = M$, $B = A$ and $f = \mathrm{id}\colon M \to M$ in Theorem 4.1, then the induced map $\bar{\mathrm{id}}$ equals the identity map of $\mathrm{B}(M, A)$. As in Lemma 2.8, the composition $g \circ f$ of maps f and g induces the map $\overline{g \circ f} = \bar{g} \circ \bar{f}$. Thus the blow-up construction defines a functor from the category whose morphisms are smooth (or real analytic) maps satisfying the hypothesis of Theorem 4.1 to the category of smooth (or real analytic) manifolds.

Let $\mathrm{C}^\infty((M, A), (N, B))$ denote the set of maps satisfying the properties required for f in Theorem 4.1. Then $\mathrm{C}^\infty((M, A), (N, B))$ inherits the Whitney C^∞-topology from $\mathrm{C}^\infty(M, N)$. Lemma 2.7 implies:

Proposition 4.2. *The map*

$$\mathrm{C}^\infty((M, A), (N, B)) \to \mathrm{C}^\infty(\mathrm{B}(M, A), \mathrm{B}(N, B)),\ f \mapsto \bar{f},$$

is continuous.

Thus, the blow-up construction is in fact a continuous functor.

Proposition 4.3. *Let M be a smooth (real analytic) manifold and let A be a neat closed smooth (real analytic) submanifold of M. Let N be a smooth (real analytic) manifold without boundary. Then there is a natural smooth (real analytic) diffeomorphism*

$$\mathrm{B}(N \times M, N \times A) \to N \times \mathrm{B}(M, A).$$

Proof. Let $\mathrm{pr}_M\colon N \times M \to M$ and $\mathrm{pr}_N\colon N \times M \to N$ be the projections. For every $y \in N$, let $\mathrm{i}_y\colon M \to N \times M$, $x \mapsto (y, x)$. By Theorem 4.1, the diagram

$$\begin{tikzcd}
\mathrm{B}(M, A) \arrow[r, "\bar{\mathrm{i}}_y"] \arrow[d, "\pi"] & \mathrm{B}(N \times M, N \times A) \arrow[r, "\bar{\mathrm{pr}}_M"] \arrow[d, "\pi"] & \mathrm{B}(M, A) \arrow[d, "\pi"] \\
M \arrow[r, "\mathrm{i}_y"] & N \times M \arrow[r, "\mathrm{pr}_M"] & M
\end{tikzcd}$$

commutes and the induced maps $\bar{\mathrm{i}}_y$ and $\bar{\mathrm{pr}}_M$ are smooth (real analytic). Since $\mathrm{pr}_M \circ \mathrm{i}_y$ equals the identity map of M, for every y, the induced map $\overline{\mathrm{pr}_M \circ \mathrm{i}_y} = \bar{\mathrm{pr}}_M \circ \bar{\mathrm{i}}_y$ equals the identity map of $\mathrm{B}(M, A)$, for every y. Let

$$h\colon \mathrm{B}(N \times M, N \times A) \to N \times \mathrm{B}(M, A),\ z \mapsto (\mathrm{pr}_N \circ \pi(z), \bar{\mathrm{pr}}_M(z)).$$

Then h is a smooth (real analytic) map. The map h is also a bijection, with the inverse map given by

$$\mathrm{i}\colon N \times \mathrm{B}(M, A) \to \mathrm{B}(N \times M, N \times A),\ (y, x) \mapsto \bar{\mathrm{i}}_y(x).$$

Notice that both h and i are induced by the identity map $\mathrm{id}\colon N \times M \to N \times M$, and that the diagram

$$\begin{tikzcd}
\mathrm{B}(N \times M, N \times A) \arrow[r, "h"] \arrow[d, "\pi"] & N \times \mathrm{B}(M, A) \arrow[r, "i"] \arrow[d, "\mathrm{id}_N \times \pi"] & \mathrm{B}(N \times M, N \times A) \arrow[d, "\pi"] \\
N \times M \arrow[r, "\mathrm{id}"] & N \times M \arrow[r, "\mathrm{id}"] & N \times M
\end{tikzcd}$$

commutes. It follows immediately from the definition of the local blow-up that h is a diffeomorphism when M, N and A are Euclidean spaces. It follows that in general h is a local diffeomorphism as well as a bijection. Consequently, h is a smooth (real analytic) diffeomorphism. ■

5 Equivariant properties

Let G be a Lie group and let M be a smooth (real analytic) manifold on which G acts smoothly (real analytically). The action of G on M is called proper if the map

$$G \times M \to M \times M, \quad (g, x) \mapsto (gx, x),$$

is proper. In this case we call M a *proper smooth (real analytic) G-manifold*.

It is well-known that the action of G on M is proper if and only if for every two points x and y in M there are neighbourhoods U and V of x and y, respectively, such that the closure of the set

$$G(U, V) = \{g \in G \mid gU \cap V \neq \emptyset\}$$

is compact.

The action of G is called *Cartan*, if every $x \in M$ has a neighbourhood U such that the set $G(U, U)$ has a compact closure. Clearly, every proper action is Cartan.

Theorem 5.1. *Let G be a Lie group and let M be a smooth (or real analytic) G-manifold. Let A be a neat closed smooth (real analytic) G-invariant submanifold of M. Then the blow-up* $\mathrm{B}(M, A)$ *is a smooth (real analytic) G-manifold and the canonical projection* $\pi \colon \mathrm{B}(M, A) \to M$ *is a smooth (real analytic) G-equivariant map. If the action of G on M is proper (Cartan), then also the action of G on* $\mathrm{B}(M, A)$ *is proper (Cartan).*

Proof. We already know that $\mathrm{B}(M, A)$ is a smooth (real analytic) manifold and that π is a smooth (real analytic) map. Therefore, to prove the first claim, it suffices to show that G acts smoothly (real analytically) on $\mathrm{B}(M, A)$ and that π is G-equivariant.

By Theorem 4.1, each map $\bar{g}$ induced by $g \in G$, is a smooth (real analytic) diffeomorphism of $\mathrm{B}(M, A)$ and the diagram

$$\begin{array}{ccc} \mathrm{B}(M, A) & \xrightarrow{\bar{g}} & \mathrm{B}(M, A) \\ \big\downarrow \pi & & \big\downarrow \pi \\ M & \xrightarrow{g} & M \end{array}$$

commutes. Thus π is G-equivariant.

We want to show that the action

$$G \times \mathrm{B}(M, A) \to \mathrm{B}(M, A), \ (g, x) \mapsto \bar{g}(x), \qquad (**)$$

is smooth (real analytic). Let $\phi \colon G \times M \to M$ denote the action map on M, and let $\mathrm{i} \colon G \times \mathrm{B}(M, A) \to \mathrm{B}(G \times M, G \times A)$ be as in Proposition 4.3. Then the diagram

$$\begin{array}{ccccc} G \times \mathrm{B}(M, A) & \xrightarrow{\mathrm{i}} & \mathrm{B}(G \times M, G \times A) & \xrightarrow{\bar{\phi}} & \mathrm{B}(M, A) \\ \big\downarrow \mathrm{id}\times\pi & & \big\downarrow \pi & & \big\downarrow \pi \\ G \times M & \xrightarrow{\mathrm{id}} & G \times M & \xrightarrow{\phi} & M \end{array}$$

commutes. The map $\bar{\phi} \circ \mathrm{i}$ is smooth (real analytic) and it defines the action in $(**)$.

Assume G acts properly on M. Let $x, y \in \mathrm{B}(M, A)$. Then $\pi(x), \pi(y) \in M$. Since G acts properly on M, $\pi(x)$ and $\pi(y)$ have neighbourhoods U and V, respectively, such that the set $G(U, V)$ is relatively compact. Now, $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are neighbourhoods of x and y. For $g \in G$,

$$g\pi^{-1}(U) \cap \pi^{-1}(V) = \pi^{-1}(gU \cap V) \neq \emptyset$$

if and only if $gU \cap V \neq \emptyset$. Thus $G(\pi^{-1}(U), \pi^{-1}(V))$ has compact closure and the action of G on $\mathrm{B}(M, A)$ is proper. Similarly, we see that if the action of G on M is Cartan, then the action of G on $\mathrm{B}(M, A)$ is also Cartan. This completes the proof. ■

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