Spacelike Hypersurfaces with Constant Mean Curvature in the Steady State Space

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Abstract

In this paper we obtain height estimates concerning to a compact spacelike hypersurface Σ^n immersed with constant mean curvature H in the Steady State space $\mathcal{H}^{n+1}$, when its boundary is contained into some hyperplane of this spacetime. As a first application of these results, when Σ^n has spherical boundary, we establish relations between its height and the radius of its boundary. Moreover, under a certain restriction on the Gauss map of Σ^n, we obtain a sharp estimate for H. Finally, we also apply our estimates to describe the end of a complete spacelike hypersurface and to get theorems of characterization concerning to spacelike hyperplanes in $\mathcal{H}^{n+1}$.

1 Introduction

Interest in the study of spacelike hypersurfaces in Lorentzian manifolds has increased very much in recent years, from both the physical and mathematical points of view. For example, it was pointed out by J. Marsdan and F. Tipler in [16] and S. Stumbles in [22] that spacelike hypersurfaces with constant mean curvature in arbitrary spacetime play an important part in the relativity theory. They are convenient as initial hypersurfaces for the Cauchy problem in arbitrary spacetime and for studying the propagation of gravitational radiation. From a mathematical point of view, that interest is also motivated by the fact that these hypersurfaces exhibit nice Bernstein-type properties. Actually, E. Calabi in [6], for

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$n \leq 4$, and S.Y. Cheng and S.T. Yau in [8], for arbitrary n, showed that the only complete immersed spacelike hypersurfaces of the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{L}^{n+1}$ with zero mean curvature are the spacelike hyperplanes.

Recently, R. López obtained a sharp estimate for the height of compact spacelike surfaces Σ^2 immersed into the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ with constant mean curvature (cf. [15], Theorem 1). For the case of constant higher order mean curvature, by applying the techniques used by Hoffman, de Lira and Rosenberg in [11], the second author obtained another sharp height estimate for compact spacelike hypersurfaces immersed in the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{L}^{n+1}$ (cf. [14], Theorem 4.2). As an application of this estimate, he studied the nature of the end of a complete spacelike hypersurface of $\mathbb{L}^{n+1}$.

In this paper we deal with a compact spacelike hypersurface Σ^n immersed in the *Steady State space* $\mathcal{H}^{n+1}$, which is a particular model of Robertson-Walker spacetime given by $\mathcal{H}^{n+1} = -\mathbb{R} \times_{e^t} \mathbb{R}^n$ (cf. Section 3). In this setting, by supposing the mean curvature H of Σ^n constant and its boundary $\partial\Sigma$ contained into some hyperplane of $\mathcal{H}^{n+1}$, we obtain estimates for its height function h in terms of H. We prove the following result:

Theorem (Theorem 3.1) *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary $\partial\Sigma$ is contained in the hyperplane $\mathcal{L}^n(\tau)$. Suppose that the mean curvature $H > 1$ is constant and that $\partial\Sigma$ is mean convex. Then, the height h of Σ^n with respect to $\mathcal{L}^n(\tau)$ satisfies the inequality*

$$e^h + h \leq H$$

and, consequently,

$$h \leq \ln H.$$

It is important to point out that, by analyzing López's result in [15] as well as the estimate obtained by the second author in [14] (both of them related to the Lorentz-Minkowski space $\mathbb{L}^{n+1}$), the estimate of the preceding theorem has the great virtue that it does not depend on the geometry of the hypersurface Σ^n, but only on the value of the mean curvature H.

On the other hand, we observe that E. Heinz in [10] discovered that a compact graph Σ^n over a hyperplane Π of the Euclidean space $\mathbb{R}^{n+1}$, with zero boundary values and having constant mean curvature $H > 0$, is at most a height $\frac{1}{H}$ from Π (see also [20] and [21]). A hemisphere in $\mathbb{R}^{n+1}$ of radius $\frac{1}{H}$ shows that this estimate is optimal. If we reinterpret the estimate of Heinz as an upper bound for the mean curvature in terms of the height function h, i.e.,

$$H \leq \frac{1}{h},$$

we note that there exists a duality between our Theorem 3.1 and Heinz's result in the sense that in our theorem we have obtained a lower bound one; as follows:

$$H \geq e^h.$$

In this setting, we note that the hyperplanes realize our estimate.

We also observe that, in connection with our work but using different tools, S. Montiel obtained another height estimate for compact spacelike hypersurfaces with constant mean curvature $H > 1$ in the Steady State space (cf. [18], Theorem 5). By comparing with Montiel's result, the advantage of our corresponding estimates is their independence on the hyperplane where the boundary of the hypersurface is contained. This fact is exactly what enable us to use our estimates for the study of the complete noncompact spacelike hypersurfaces immersed in $\mathcal{H}^{n+1}$ with constant mean curvature (cf. Section 5). At this point notice that, since there is not exist complete noncompact spacelike hypersurface with constant mean curvature $H < 1$ in $\mathcal{H}^{n+1}$ which are umbilical (cf. [18], Section 3), it is natural that we restrict ourselves to the case $H \geq 1$.

Suitable formulae for the Laplacians of h and of a support-like function naturally attached to a spacelike hypersurface (cf. Lemma 2.1) constitute the analytical tools that we use to get our estimate. As a first application of Theorem 3.1, when such a hypersurface has spherical boundary, we establish relations between its height and the radius of its boundary (cf. Corollary 3.3 and Corollary 3.8).

Afterwards, by imposing a certain restriction on the Gauss map of Σ^n, we obtain a sharp estimate for the mean curvature H. More precisely:

Theorem (Theorem 4.1) *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary $\partial\Sigma$ is contained in the hyperplane $\mathcal{L}^n(\tau)$. Suppose that the mean curvature $H \geq 1$ is constant and that the hyperbolic image of Σ^n is contained in the closure of the interior domain enclosed by the horosphere $L^n(\varrho)$. Then,*

$$H \leq \frac{\varrho^2}{\tau^2}$$

and the equality happens only in the case that Σ^n is entirely contained in $\mathcal{L}^n(\tau)$. Moreover, the height h of Σ^n with respect to $\mathcal{L}^n(\tau)$ satisfies

$$h \leq \ln\left(\frac{\varrho}{\tau}\right).$$

Furthermore, we apply our estimates to obtain the following description of the end of a complete spacelike hypersurface:

Theorem (Theorem 5.1) *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface with one end. Suppose that the mean curvature $H > 1$ is constant and that Σ^n is horizontally mean convex. Then, the end of Σ is not divergent.*

We also use our estimates to get theorems of characterization concerning spacelike hyperplanes in $\mathcal{H}^{n+1}$, as the following one:

Theorem (Theorem 5.3) *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface with one end over a hyperplane $\mathcal{L}^n(\tau)$, for some $\tau > 0$. Suppose that the hyperbolic image of Σ^n is contained in the closure of the interior domain enclosed by a horosphere $L^n(\varrho)$, for some $\varrho \geq \tau$. If the mean curvature H is a constant satisfying $H \geq \frac{\varrho^2}{\tau^2}$, then Σ^n is a* *hyperplane* $\mathcal{L}^n(\tilde{\tau})$ *for some* $0 < \tilde{\tau} \leq \tau$.

Finally, we observe that A.L. Albujer and L.J. Alías have recently found interesting Bernstein-type results in the 3-dimensional Steady State space (cf. [1]). Moreover, considering the generalized Robertson-Walker spacetime model of $\mathcal{H}^{n+1}$, they extended their results to a wider family of spacetimes.

2 Preliminaries

Let M^n be a connected, n-dimensional ($n \geq 2$) oriented Riemannian manifold, $I \subset \mathbb{R}$ an interval and $f : I \to \mathbb{R}$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the factors I and M, respectively.

A particular class of Lorentzian manifolds (*spacetimes*) is the one obtained by furnishing $\overline{M}$ with the metric

$$\langle v, w \rangle_p = -\langle (\pi_I)_* v, (\pi_I)_* w \rangle + (f \circ \pi_I)\,(p)^2 \langle (\pi_M)_* v, (\pi_M)_* w \rangle,$$

for all $p \in \overline{M}$ and all $v, w \in T_p\overline{M}$. Such a space is called (following the terminology introduced in [3]) a *Generalized Robertson-Walker* (GRW) spacetime, and in what follows we shall write $\overline{M}^{n+1} = -I \times_f M^n$ to denote it. In particular, when M^n has constant sectional curvature, then $-I \times_f M^n$ is classically called a *Robertson-Walker* (RW) spacetime (cf. [19]).

We recall that a tangent vector field K on a spacetime $\overline{M}^{n+1}$ is said to be conformal if the Lie derivative with respect to K of the metric $\langle , \rangle$ of $\overline{M}^{n+1}$ satisfies:

$$\pounds_K \langle , \rangle = 2\phi \langle , \rangle ,$$

for a certain smooth function $\phi \in \mathcal{D}(\overline{M}^{n+1})$. Since $\mathcal{L}_K(X) = [K, X]$ for all $X \in \mathcal{X}(\overline{M})$, it follows from the tensorial character of $\mathcal{L}_K$ that $K \in \mathcal{X}(\overline{M})$ is conformal if and only if

$$\langle \overline{\nabla}_X K, Y \rangle + \langle X, \overline{\nabla}_Y K \rangle = 2\phi \langle X, Y \rangle,$$

for all $X, Y \in \mathcal{X}(\overline{M})$. In particular, K is a Killing vector field relatively to the metric $\langle , \rangle$ if and only if $\phi \equiv 0$.

We observe that, when $\overline{M}^{n+1} = -I \times_f M^n$ is a GRW spacetime, the vector field

$$K = f \partial_t = (f \circ \pi_I) \partial_t$$

is conformal and closed (in the sense that its dual $1-$form is closed), with conformal factor $\phi = f'$, where the prime denotes differentiation with respect to $t \in I$ (cf. [17]).

A smooth immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ of an n-dimensional connected manifold Σ^n is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on Σ^n, which, as usual, is also denoted by $\langle , \rangle$. In this setting, ∇ stands for the Levi-Civita connection of Σ^n, A the corresponding shape operator and $H = -\mathrm{tr}(A)/n$ the mean curvature.

To close this section, we present the analytical framework that we will use to obtain our estimates. The formulae collected in the following lemma are particular cases of ones obtained by L.J. Alías jointly with the first author (cf. [2], Lemma 4.1 and Corollary 8.5). Here, and for the sake of completeness, we present more direct and specific proofs (for an alternative proof of the item (b) of the following lemma see [7], Proposition 3.1; see also [4], Proposition 3.1).

Lemma 2.1. *Let $\psi : \Sigma^n \to -I \times_f M^n$ be a spacelike hypersurface immersed into a GRW spacetime, with Gauss map N. Then,*

(a) by denoting $h = \pi_I \circ \psi$ the height function of Σ, we have

$$\triangle h = -(\ln f)'(h)(n + |\nabla h|^2) - nH\langle N, \partial_t\rangle;$$

(b) by supposing $\overline{M}^{n+1}$ a RW spacetime with flat Riemannian fiber M^n, we get

$$\triangle\langle N, K\rangle = n\langle\nabla H, K\rangle + nHf'(h) + \langle N, K\rangle\left(|A|^2 - (n-1)(\ln f)''(h)|\nabla h|^2\right),$$

where $K = f\partial_t$ and $|A|$ is the Hilbert-Schmidt norm of A.

Proof. (a) One has

$$\begin{aligned}\nabla h &= \nabla(\pi_{I_{|\Sigma}}) = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top \\ &= -\partial_t - \langle N, \partial_t\rangle N,\end{aligned}$$

where $\overline{\nabla}$ denotes the gradient with respect to the metric of the ambient space and $X^\top$ the tangential component of a vector field $X \in \mathcal{X}(\overline{M})$ on Σ. Now, fixed $p \in M$ and $v \in T_pM$, write $v = w - \langle v, \partial_t\rangle\partial_t$, so that $w \in T_p\overline{M}$ is tangent to the fiber of $\overline{M}$ passing through p. By repeated use of the formulae of item (2) of Proposition 7.35 of [19], we get

$$\begin{aligned}\overline{\nabla}_v\partial_t &= \overline{\nabla}_w\partial_t - \langle v, \partial_t\rangle\overline{\nabla}_{\partial_t}\partial_t = \overline{\nabla}_w\partial_t \\ &= (\ln f)'w = (\ln f)'(v + \langle v, \partial_t\rangle\partial_t).\end{aligned}$$

Thus,

$$\begin{aligned}\nabla_v\nabla h &= \overline{\nabla}_v\nabla h + \langle Av, \nabla h\rangle N \\ &= \overline{\nabla}_v(-\partial_t - \langle N, \partial_t\rangle N) + \langle Av, \nabla h\rangle N \\ &= -(\ln f)'w - v(\langle N, \partial_t\rangle)N + \langle N, \partial_t\rangle Av + \langle Av, \nabla h\rangle N \\ &= -(\ln f)'w + (\langle Av, \partial_t\rangle - \langle N, \overline{\nabla}_v\partial_t\rangle)N + \langle N, \partial_t\rangle Av + \langle Av, \nabla h\rangle N \\ &= -(\ln f)'w + (\langle Av, \partial_t^\top\rangle - \langle N, (\ln f)'w\rangle)N + \langle N, \partial_t\rangle Av + \langle Av, \nabla h\rangle N \\ &= -(\ln f)'w - (\ln f)'\langle v, \partial_t\rangle\langle N, \partial_t\rangle N + \langle N, \partial_t\rangle Av \\ &= -(\ln f)'\{v - \langle v, \partial_t\rangle(-\partial_t - \langle N, \partial_t\rangle N)\} + \langle N, \partial_t\rangle Av \\ &= (\ln f)'(-v + \langle v, \partial_t^\top\rangle\nabla h) + \langle N, \partial_t\rangle Av \\ &= -(\ln f)'(v + \langle v, \nabla h\rangle\nabla h) + \langle N, \partial_t\rangle Av.\end{aligned}$$

Finally, by fixing $p \in \Sigma$ and an orthonormal frame $\{e_i\}$ at $T_p\Sigma$, one gets

$$\begin{aligned}
\triangle h &= \operatorname{tr}(\operatorname{Hess} h) = \sum_{i=1}^{n} \langle \nabla_{e_i} \nabla h, e_i \rangle \\
&= \sum_{i=1}^{n} \langle -(\ln f)'(e_i + \langle e_i, \nabla h \rangle \nabla h) + \langle N, \partial_t \rangle A e_i, e_i \rangle \\
&= -(\ln f)'(n + |\nabla h|^2) + \langle N, \partial_t \rangle \operatorname{tr}(A) \\
&= -(\ln f)'(h)(n + |\nabla h|^2) - nH \langle N, \partial_t \rangle.
\end{aligned}$$

(b) To prove the formula for $\triangle \langle N, K \rangle$, suppose that the Riemannian fiber M^n is flat. Since $\overline{\nabla}_V K = f'(h)V$ for all $V \in \mathcal{X}(\Sigma)$, we easily see that

$$\nabla \langle N, K \rangle = -A(K^\top).$$

Thus, for all $V \in \mathcal{X}(\Sigma)$,

$$\nabla_V(\nabla \langle N, K \rangle) = -(\nabla_V A)(K^\top) - A(\nabla_V K^\top).$$

Then, since $\nabla h = -\partial_t^\top$, by Codazzi equation we get

$$\nabla_V(\nabla \langle N, K \rangle) = -(\nabla_{K^\top} A)(V) - (\overline{R}(V, K^\top)N)^\top - f'(h)A(V) + \langle N, K \rangle A^2(V),$$

where $\overline{R}$ denotes the curvature tensor of the spacetime $\overline{M}^{n+1}$. On the other hand, by taking an orthonormal frame $\{e_i\}$, one has

$$\operatorname{tr}(\nabla_W A) = \sum_{i=1}^{n} \langle (\nabla_W A)e_i, e_i \rangle = \sum_{i=1}^{n} W \langle A e_i, e_i \rangle = -n \langle \nabla H, W \rangle,$$

for all $W \in \mathcal{X}(\Sigma)$.

Therefore, from Proposition 7.42 and Corollary 7.43 of [19], we obtain that

$$\triangle \langle N, K \rangle = n \langle \nabla H, K \rangle + nHf'(h) + \langle N, K \rangle \left(|A|^2 - (n-1)(\ln f)''(h)|\nabla h|^2 \right). \quad \blacksquare$$

3 Height Estimate for Hypersurfaces in the Steady State Space

In what follows we consider a particular model of RW spacetime, the *Steady State space*, namely

$$\mathcal{H}^{n+1} = -\mathbb{R} \times_{e^t} \mathbb{R}^n.$$

We observe that the Steady State space appears naturally in physical context as an exact solution for the Einstein equations, being a cosmological model where matter is supposed to travel along geodesics normal to the horizontal hyperplanes; these, in turn, serve as the initial data for the Cauchy problem associated to those equations (see [9], Chapter 5).

An alternative description of the Steady State space $\mathcal{H}^{n+1}$ (cf. [18]; see also [13]) can be given as follows. Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz-Minkowski space ($n \geq 2$), that is, the real vector space $\mathbb{R}^{n+2}$, endowed with the Lorentz metric

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$:

$$\mathbb{S}_1^{n+1} = \left\{ p \in L^{n+2}; \langle p, p \rangle = 1 \right\}.$$

From the above definition it is easy to show that the metric induced from $\langle \ , \ \rangle$ turns $\mathbb{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature 1. Moreover, for $p \in \mathbb{S}_1^{n+1}$, we have

$$T_p\mathbb{S}_1^{n+1} = \left\{ v \in \mathbb{L}^{n+2}; \langle v, p \rangle = 0 \right\}.$$

Let $a \in \mathbb{L}^{n+2}$ be a nonzero null vector of the null cone with vertex in the origin, such that $\langle a, e_{n+2} \rangle > 0$, where $e_{n+2} = (0, \ldots, 0, 1)$.

It can be shown that the open region

$$\left\{ p \in \mathbb{S}_1^{n+1}; \langle p, a \rangle > 0 \right\}$$

of the de Sitter space $\mathbb{S}_1^{n+1}$ is isometric to $\mathcal{H}^{n+1}$ (cf. [1], Section 4). This open region forms the spacetime for the steady state model of the universe proposed by Bondi and Gold [5] and Hoyle [12], when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous), but at all times (cf. [9], Section 5.2; see also [23], Section 14.8). We will call by *Minkowski model* this alternative description of $\mathcal{H}^{n+1}$. We observe that, in this setting, the boundary of $\mathcal{H}^{n+1}$ is the null hypersurface

$$\left\{ p \in \mathbb{S}_1^{n+1}; \langle p, a \rangle = 0 \right\},$$

whose topology is that of $\mathbb{R} \times \mathbb{S}^{n-1}$. Now, we consider in $\mathcal{H}^{n+1}$ the timelike vector field

$$K = -\langle x, a \rangle\, x + a.$$

We easily see that

$$\overline{\nabla}_V K = -\langle x, a \rangle\, V, \text{ for all } V \in \mathcal{X}(\mathcal{H}^{n+1}).$$

Thus, K is a closed conformal vector field globally defined in $\mathcal{H}^{n+1}$ and, consequently (cf. [17], Proposition 1), determines a foliation on $\mathcal{H}^{n+1}$ by hyperplanes

$$\mathcal{L}^n(\tau) = \left\{ x \in \mathbb{S}_1^{n+1}; \langle x, a \rangle = \tau \right\}, \ \tau > 0,$$

which are totally umbilical hypersurfaces of $\mathcal{H}^{n+1}$. Moreover, each $\mathcal{L}^n(\tau)$ is isometric to the Euclidean space $\mathbb{R}^n$ and has constant mean curvature 1 with respect to the unit normal fields

$$N_\tau(x) = -x + \frac{1}{\tau}a, \quad x \in \mathcal{L}^n(\tau).$$

We note that the hyperplanes $\mathcal{L}^n(\tau)$ approach to the boundary of $\mathcal{H}^{n+1}$ when τ tends to zero and that, when τ tends to ∞, they approach to the spacelike future infinity for timelike and null lines of de Sitter space.

In what follows, considering the Minkowski model of $\mathcal{H}^{n+1}$, we will deal with a compact spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ whose boundary $\partial\Sigma$ is a $(n-1)$-dimensional closed submanifold embedded in a hyperplane $\mathcal{L}^n(\tau)$, for some $\tau > 0$. Moreover, we will suppose that the spacelike hypersurface Σ^n oriented by a unit normal field N in the same time-orientation of K (that is, such that $\langle N, K\rangle < 0$).

In this setting, we will subtend the compositions with the isometry Φ_τ between the Minkowski and RW models of $\mathcal{H}^{n+1}$ which is characterized by

$$\Phi_\tau(\mathcal{L}^n(\tau)) = \{0\} \times \mathbb{R}^n \quad \text{and} \quad (\Phi_\tau)_*(N_\tau) = \partial_t.$$

In order to establish our main estimate for the height function of a spacelike hypersurface of the Steady State space, we regard that an embedded closed hypersurface of a Euclidean space is said to be *mean convex* when its mean curvature with respect to the interior normal field is non-negative.

Theorem 3.1. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary $\partial\Sigma$ is contained in the hyperplane $\mathcal{L}^n(\tau)$. Suppose that the mean curvature $H > 1$ is constant and that $\partial\Sigma$ is mean convex. Then, the height h of Σ^n with respect to $\mathcal{L}^n(\tau)$ satisfies the inequality*

$$e^h + h \leq H$$

and, consequently,

$$h \leq \ln H.$$

Proof. We first see that, as a consequence of the tangency principle in the Steady State space (cf. [18], Theorem 2), Proposition 3 of [18] assures that the height function h is non-negative. Thus, in order to get the above inequalities, we define on Σ^n the function

$$\varphi = c\,h - \langle N, K\rangle,$$

where c is a positive constant to be determined and $K = e^t\partial_t$. Notice that, from the assumptions that Σ^n has constant mean curvature $H > 1$ and that the boundary $\partial\Sigma$ is mean convex, we are in the position to apply the gradient estimates of Montiel (cf. [18], Theorem 7). Then, we have that

$$0 > \langle N, a\rangle \geq -H\tau$$

on Σ^n and, consequently,

$$\langle N, \partial_t\rangle = \langle N, N_\tau\rangle = \langle N, -\psi + \frac{1}{\tau}a\rangle \geq -H$$

on $\partial\Sigma$. Therefore,

$$\varphi_{|\partial\Sigma} \leq H.$$

On the other hand, since $\nabla h = -\partial_t^\top$, one has $|\nabla h|^2 = \langle N, \partial_t\rangle^2 - 1$ and, from Lemma 2.1,

$$\triangle\varphi = c\{1 - \langle N, \partial_t\rangle^2 - n - nH\langle N, \partial_t\rangle\} - nHe^h - \langle N, \partial_t\rangle|A|^2e^h.$$

Now, let S_2 denote the second elementary symmetric function on the eigenvalues of A, and $H_2 = 2S_2/n(n-1)$ denote the mean value of S_2. Elementary algebra gives

$$|A|^2 = n^2H^2 - n(n-1)H_2,$$

which put into the above formula gives

$$\begin{aligned}\triangle\varphi &= c\{1 - \langle N, \partial_t\rangle^2 - n - nH\langle N, \partial_t\rangle\}\\ &\quad + nH(H-1)e^h - n(n-1)(H^2 - H_2)\langle N, \partial_t\rangle e^h.\end{aligned}$$

Since $h \geq 0$, $-1 \geq \langle N, \partial_t\rangle \geq -H$ and (from the Cauchy-Schwarz inequality) $H^2 - H_2 \geq 0$, we get

$$\triangle\varphi \geq nH(H-1)(1-c) + (n-1)(H^2-1)c.$$

Thus, by taking $c = 1$, we get that $\triangle\varphi \geq 0$. Consequently, from the maximum principle, $\varphi \leq H$ on Σ^n. Therefore, by the definition of the function φ,

$$e^h + h \leq H$$

and, consequently,

$$h \leq \ln H. \qquad \blacksquare$$

Remark 3.2. Related to our previous theorem, it is important to observe the following facts:

(i) A result which is similar to the above theorem were obtained by S. Montiel (cf. [18], Theorem 5). Differently of Montiel's result, we note that the above estimate does not depend of the parameter $\tau > 0$ associated to the hyperplane where the boundary of the hypersurface is contained. This fact will allow us to describe the nature of the *end* of a complete spacelike hypersurface immersed in the Steady State space with constant mean curvature $H > 1$ (cf. Section 5).

(ii) Notice that compact spacelike hypersurfaces satisfying the conditions required in Theorem 3.1 actually exist. In fact, Montiel has proved that given a compact domain Γ on a time slice of the Steady State space $\mathcal{H}^{n+1}$ with mean convex boundary and a real number $H \geq 1$, there exists a spacelike graph over Γ with constant mean curvature H and boundary $\partial\Gamma$ (cf. [18], Theorem 9).

As an application of the previous theorem, and using Theorem 2 of [13], we obtain the following result.

Corollary 3.3. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary $\partial\Sigma = \mathbb{S}^{n-1}(\rho)$ is a $(n-1)-$dimensional geodesic sphere of radius ρ into the hyperplane $\mathcal{L}^n(\tau)$. Suppose that the mean curvature $H > 1$ is constant. If $\sqrt{3} - 1 < \rho < 1$ or $\sqrt{2} + 1 < \rho < \sqrt{3} + 1$, then the height h of Σ^n with respect to $\mathcal{L}^n(\tau)$ satisfies the inequality*

$$h < -\ln\left(\rho - \left|1 - \frac{\rho^2}{2}\right|\right).$$

Remark 3.4. In the above corollary, we have a strict inequality because a necessary condition to get $h = -\ln\left(\rho - \left|1 - \frac{\rho^2}{2}\right|\right)$ is that $1 - \frac{\rho^2}{2} = 0$ or, equivalently, $\rho = \sqrt{2}$; but, this implies that $|H| \leq \frac{\sqrt{2}}{2} < 1$ (cf. [13], Corollary 3).

Theorem 3.5. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary $\partial\Sigma$ is contained in the hyperplane $\mathcal{L}^n(\tau)$. Suppose that the mean curvature $H > (n-1) \geq 1$ is constant and that $\partial\Sigma$ is mean convex. Then, the height h of Σ^n with respect to $\mathcal{L}^n(\tau)$ satisfies the inequality*

$$e^h + \frac{nHh}{H-(n-1)} \leq H$$

and, consequently,

$$h \leq \frac{H}{n} + \frac{n-1}{nH} - 1.$$

Proof. To get this another estimate, it's enough to reproduce the proof of Theorem 3.1 by taking

$$c = \frac{nH}{H-(n-1)}.$$ ■

Remark 3.6. We observe that, for example, in the case $H = n$ the estimate in Theorem 3.5 is better than that in Theorem 3.1. On the other hand, fixed the dimension n, since $\frac{\ln H}{H} \to 0$ when $H \to \infty$, for H sufficiently large we have that the estimate in Theorem 3.1 is better than that in Theorem 3.5.

As consequences of our last estimate, we obtain the following results.

Corollary 3.7. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary $\partial\Sigma$ is contained in the hyperplane $\mathcal{L}^n(\tau)$. Suppose that the mean curvature $H > (n-1) \geq 1$ is constant and that $\partial\Sigma$ is mean convex. Then, the height h of Σ^n with respect to $\mathcal{L}^n(\tau)$ satisfies the inequality*

$$h \leq \min\left(\ln H, \frac{H}{n} + \frac{n-1}{nH} - 1\right).$$

Corollary 3.8. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary $\partial\Sigma = \mathbb{S}^{n-1}(\rho)$ is a $(n-1)-$dimensional geodesic sphere of radius ρ into the hyperplane $\mathcal{L}^n(\tau)$. Suppose that the mean curvature $H > (n-1) \geq 1$ is constant. If $\sqrt{3} - 1 < \rho < 1$ or $\sqrt{2} + 1 < \rho < \sqrt{3} + 1$, then the height h of Σ^n with respect to $\mathcal{L}^n(\tau)$ satisfies the inequality*

$$h < \frac{1}{n}\left\{\left(\rho - \left|1 - \frac{\rho^2}{2}\right|\right)^{-1} - 1\right\}.$$

4 A sharp estimate for the mean curvature

In order to establish the next results, we observe that the Gauss map N of a spacelike hypersurface Σ^n immersed in the Steady State space $\mathcal{H}^{n+1}$ can be thought of as a map

$$N : \Sigma^n \to \mathbb{H}^{n+1}$$

taking values in the hyperbolic space

$$\mathbb{H}^{n+1} = \left\{ x \in \mathbb{L}^{n+2}; \langle x, x \rangle = -1, \ \langle x, a \rangle < 0 \right\}.$$

Here a is any non-zero null vector in $\mathbb{L}^{n+2}$, which will be chosen to be the same one in the beginning of this section; that is, $\langle a, e_{n+2} \rangle > 0$, where $e_{n+2} = (0, \cdots, 0, 1)$. In this setting, the image $N(\Sigma)$ is called the *hyperbolic image* of Σ^n.

On the other hand, all the horospheres of $\mathbb{H}^{n+1}$ can be realized in the Minkowski model in the following way:

$$L^n(\varrho) = \left\{ x \in \mathbb{H}^{n+1}; \langle x, a \rangle = -\varrho \right\},$$

where ϱ is a positive number.

We easily see that when Σ^n is a compact spacelike hypersurface whose boundary in some hyperplane $\mathcal{L}^n(\tau)$ and its hyperbolic image is contained in the closure of the interior domain enclosed by some horosphere $L^n(\varrho)$, we must have $\varrho \geq \tau$.

Now, by supposing a certain restriction on the hyperbolic image of the spacelike hypersurface, we state and prove a sharp estimate for its mean curvature.

Theorem 4.1. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary $\partial\Sigma$ is contained in the hyperplane $\mathcal{L}^n(\tau)$. Suppose that the mean curvature $H \geq 1$ is constant and that the hyperbolic image of Σ^n is contained in the closure of the interior domain enclosed by the horosphere $L^n(\varrho)$. Then,*

$$H \leq \frac{\varrho^2}{\tau^2}$$

and the equality happens only in the case that Σ^n is entirely contained in $\mathcal{L}^n(\tau)$. Moreover, the height h of Σ^n with respect to $\mathcal{L}^n(\tau)$ satisfies

$$h \leq \ln\left(\frac{\varrho}{\tau}\right).$$

Proof. Fixe a positive number c and consider on Σ^n the function φ given by

$$\varphi = c\,h - \langle N, K \rangle,$$

where $K = e^t \partial_t$. We observe that, since our hypothesis on the hyperbolic image of Σ^n implies that

$$0 > \langle N, a \rangle \geq -\varrho,$$

we have that

$$-1 \geq \langle N, \partial_t \rangle = \langle N, N_\tau \rangle = \langle N, -\psi + \frac{1}{\tau} a \rangle \geq -\frac{\varrho}{\tau}.$$

Consequently, we get

$$\varphi_{|\partial\Sigma} \leq \frac{\varrho}{\tau}.$$

On the other hand, from Lemma 2.1, we have that

$$\triangle\varphi = c\{1 - \langle N, \partial_t \rangle^2 - n - nH\langle N, \partial_t \rangle\} - nHe^h - \langle N, \partial_t \rangle |A|^2 e^h,$$

where we also have used that $|\nabla h|^2 = \langle N, \partial_t\rangle^2 - 1$; since $-1 \geq \langle N, \partial_t\rangle \geq -\frac{\varrho}{\tau}$ and $|A|^2 \geq nH^2$, we then get

$$\triangle\varphi \geq n\left\{c\left(H - \frac{\varrho^2}{\tau^2}\right) + H(H-1)\right\}.$$

Now, supposing by contradiction that $H > \frac{\varrho^2}{\tau^2}$, we have that $\triangle\varphi \geq 0$. Thus, from the maximum principle, we get that $\varphi \leq \frac{\varrho}{\tau}$ on Σ^n and, consequently,

$$h \leq \frac{\varrho}{c\,\tau}.$$

Since $c > 0$ was taken arbitrarily, we conclude that $h \equiv 0$ and so, $\psi(\Sigma)$ must be a compact domain of the hyperplane $\mathcal{L}^n(\tau)$. However, in this case we must have $H \equiv 1 \leq \frac{\varrho^2}{\tau^2}$, and we arrive at a contradiction. Therefore, $1 \leq H \leq \frac{\varrho^2}{\tau^2}$.

Now, suppose that $H = \frac{\varrho^2}{\tau^2}$. Thus, as in the preceding calculations, we see that in this case we have $h \equiv 0$; so, $\psi(\Sigma)$ is a compact domain of the hyperplane $\mathcal{L}^n(\tau)$. Reciprocally, if $\psi(\Sigma) \subset \mathcal{L}^n(\tau)$, we have that $N = N_\tau = -\psi + \frac{1}{\tau}a$ and, consequently, $\langle N, a\rangle = -\tau$; so, $\varrho = \tau$ and $H = \frac{\varrho^2}{\tau^2} \equiv 1$.

Finally, by applying the same steps of the proof of Theorem 3.1 for the function

$$\varphi = \frac{H(H-1)}{\frac{\varrho^2}{\tau^2} - H}\,h - \langle N, K\rangle,$$

we obtain that

$$h \leq \ln\left(\frac{\varrho}{\tau}\right).$$

∎

We derive from Theorem 4.1 the following corollaries.

Corollary 4.2. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary $\partial\Sigma$ is contained in the hyperplane $\mathcal{L}^n(\tau)$. Suppose that the mean curvature $H > 1$ is constant and that the hyperbolic image of Σ^n is contained in the closure of the interior domain enclosed by the horosphere $L^n(\varrho)$. If $H < \frac{\varrho^2}{\tau^2}$, then the height h of Σ^n with respect $\mathcal{L}^n(\tau)$ satisfies*

$$h \leq \frac{\left(\frac{\varrho}{\tau} - 1\right)\left(\frac{\varrho^2}{\tau^2} - H\right)}{H(H-1)}.$$

Corollary 4.3. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary is contained in a hyperplane $\mathcal{L}^n(\tau)$. Suppose that Σ^n satisfies the following conditions:*

(a) The mean curvature $H \geq 1$ is constant;

(b) The hyperbolic image is contained in the closure of the interior domain enclosed by the horosphere $L^n(\tau)$.

Then, Σ^n is a compact domain of $\mathcal{L}^n(\tau)$.

To close this section, also as a consequence of Theorem 4.1, we present a result that gives us an information about the Gauss map of a compact spacelike hypersurface of the Steady State space.

Corollary 4.4. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface whose boundary $\partial\Sigma$ is contained in the hyperplane $\mathcal{L}^n(\tau)$. Suppose that the mean curvature $H \geq 1$ is constant. If Σ^n is not entirely contained into $\mathcal{L}^n(\tau)$, then the hyperbolic image of Σ^n is contained in the closure of the interior domain enclosed by a horosphere $L^n(\varrho)$ with $\varrho > \tau$.*

5 Complete spacelike hypersurfaces with one end

In this section, considering the RW model of the Steady State space, we deal with complete spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ with *one end* N^n, that is, a hypersurface Σ^n that we can regarded as

$$\Sigma^n = \Sigma^n_t \cup N^n,$$

where Σ^n_t is a compact hypersurface with boundary contained into a horizontal hyperplane $\Pi_t = \{t\} \times \mathbb{R}^n$ of $\mathcal{H}^{n+1}$, and N^n is diffeomorphic to the cylinder $[t, \infty) \times \mathbb{S}^{n-1}$.

Given a complete spacelike hypersurface $\Sigma^n = \Sigma^n_t \cup N^n$ with one end, we say that its end N^n is *divergent* if, considering N^n with coordinates $p = (s, q) \in [t, \infty) \times \mathbb{S}^{n-1}$, we have that

$$\lim_{s\to\infty} h(p) = \infty,$$

where h denotes the height function of the end N^n with respect to ∂_t.

Finally, a complete spacelike hypersurface Σ^n immersed in the Steady State space $\mathcal{H}^{n+1}$ is said to be *horizontally mean convex* when the nonempty intersection of Σ^n with any horizontal hyperplane of $\mathcal{H}^{n+1}$ is mean convex.

Theorem 5.1. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface with one end. Suppose that the mean curvature $H > 1$ is constant and that Σ^n is horizontally mean convex. Then, the end of Σ is not divergent.*

Proof. Suppose, by contradiction, that the end N^n of $\Sigma^n = \Sigma^n_t \cup N^n$ is divergent. Then, since Σ^n_t is a compact hypersurface with mean convex boundary contained into a hyperplane Π_t, from Theorem 3.1 we have that the height of Σ^n_t with respect Π_t is at most equal to $\ln H$.

Now, using the hypothesis of that the end N^n of Σ^n is divergent, we can intersect Σ^n by the hyperplane $\Pi_{t-\ln H}$ and to obtain a compact hypersurface $\Sigma^n_{t-\ln H}$ with constant mean curvature H, with mean convex boundary contained into the hyperplane $\Pi_{t-\ln H}$, and whose height is strictly greater than $\ln H$.

Therefore, we get a contradiction with respect the estimate of Theorem 3.1 and, consequently, we conclude that the end N^n of Σ^n must be not divergent. ■

In order to establish our last results, we need one more definition: we say that a spacelike hypersurface Σ^n immersed in $\mathcal{H}^{n+1}$ is *over a hyperplane* $\mathcal{L}^n(\tau)$ when, taking into accounting the isometry Φ_τ (cf. Section 3), the height function h of Σ^n is such that $h \geq 0$.

Remark 5.2. Related to the preceding definition, we note that:

(i) Since N_τ is past directed, saying that the hyperplane $\mathcal{L}^n(\widetilde{\tau})$ be over $\mathcal{L}^n(\tau)$ means that $0 < \widetilde{\tau} \leq \tau$;

(ii) Theorem 5.1 gives us a sufficient condition for a complete spacelike hypersurface Σ^n with one end be over a hyperplane $\mathcal{L}^n(\tau)$, for some $\tau > 0$.

As a consequence of Theorem 4.1, one can reason as in the previous result to obtain the following characterization concerning spacelike hyperplanes of the Steady State space.

Theorem 5.3. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface with one end over a hyperplane $\mathcal{L}^n(\tau)$, for some $\tau > 0$. Suppose that the hyperbolic image of Σ^n is contained in the closure of the interior domain enclosed by a horosphere $L^n(\varrho)$, for some $\varrho \geq \tau$. If the mean curvature H is a constant satisfying $H \geq \frac{\varrho^2}{\tau^2}$, then Σ^n is a hyperplane $\mathcal{L}^n(\widetilde{\tau})$ for some $0 < \widetilde{\tau} \leq \tau$.*

Proof. Suppose, by contradiction, that Σ^n is not a hyperplane; so, since Σ^n is over $\mathcal{L}^n(\tau)$, (considering the RW model) we can intersect it by two horizontal hyperplanes $\Pi_{\widetilde{t}}$ and Π_t, with $0 < \widetilde{t} < t$.

In this way, we obtain two compact hypersurfaces $\Sigma^m_{\widetilde{t}}$ and Σ^n_t, with $\Sigma^m_{\widetilde{t}} \subset \Sigma^n_t$, both of them with constant mean curvature $H \geq \frac{\varrho^2}{\tau^2}$. But, from Theorem 4.1, we must have that $\Sigma^m_{\widetilde{t}}$ is a compact domain of $\Pi_{\widetilde{t}}$, while Σ^n_t is a compact domain of Π_t.

Therefore, we get a contradiction and, consequently, (returning to the Minkowski model) Σ^n must be a hyperplane $\mathcal{L}^n(\widetilde{\tau})$ for some $0 < \widetilde{\tau} \leq \tau$. ∎

From the previous theorem we derive a nonexistence result concerning to complete spacelike hypersurfaces immersed in the Steady State space with constant mean curvature.

Corollary 5.4. *There exists no complete spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ with one end over a hyperplane $\mathcal{L}^n(\tau)$ and satisfying the following conditions:*

(a) The hyperbolic image of Σ^n is contained in the closure of the interior domain enclosed by a horosphere $L^n(\varrho)$, for some $\varrho \geq \tau$.

(b) The mean curvature H of Σ^n is a constant satisfying $H > \frac{\varrho^2}{\tau^2}$.

The following characterization concerning to spacelike hyperplanes of $\mathcal{H}^{n+1}$ is a consequence of Corollary 4.3.

Theorem 5.5. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface over a hyperplane $\mathcal{L}^n(\tau)$, with constant mean curvature $H \geq 1$. If the hyperbolic image of Σ^n is contained in the closure of the interior domain enclosed by the horosphere $L^n(\tau)$, then Σ^n is a hyperplane $\mathcal{L}^n(\widetilde{\tau})$ for some $0 < \widetilde{\tau} \leq \tau$.*

Proof. To demonstrate this result, by using a similar procedure as in the proof of the previous theorem, it is enough to consider the intersection of Σ^n by a hyperplane over $\mathcal{L}^n(\tau)$ and to apply Corollary 4.3. ∎

Remark 5.6. We observe that A.L. Albujer and L.J. Alías have recently considered in [1] complete spacelike hypersurfaces with constant mean curvature in $\mathcal{H}^{n+1}$. They proved that if the hypersurface is bounded away from the infinity of the ambient space, then the mean curvature must be $H = 1$. Moreover, in the 2-dimensional case they concluded that the only complete spacelike surfaces with constant mean curvature which are bounded away from the infinity are the totally umbilical flat surfaces $\mathcal{L}^2(\tau)$.

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