The Arens regularity of certain Banach algebras related to compactly cancellative foundation semigroups

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Abstract

We study in this paper the space $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ of a locally compact semigroup $\mathcal{S}$. That space consists of all μ-measurable ($\mu \in M_a(\mathcal{S})$) functions vanishing at infinity, where $M_a(\mathcal{S})$ denotes the algebra of all measures with continuous translations. We introduce an Arens multiplication on the dual $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ of $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ under which $M_a(\mathcal{S})$ is an ideal. We then give some characterizations for Arens regularity of $M_a(\mathcal{S})$ and $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$. As the main result, we show that $M_a(\mathcal{S})$ or $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ is Arens regular if and only if $\mathcal{S}$ is finite.

1 Introduction

Let $\mathcal{S}$ denote a *locally compact semigroup*, that is a semigroup with a locally compact Hausdorff topology under which the binary operation of $\mathcal{S}$ is jointly continuous. As usual, $C_0(\mathcal{S})$ denotes the space of all continuous complex-valued functions on $\mathcal{S}$ vanishing at infinity, and $M(\mathcal{S})$ denotes the Banach space of all bounded complex-valued regular Borel measures on $\mathcal{S}$ with the total variation norm. The convolution multiplication $*$ is defined on $M(\mathcal{S})$ as the dual of $C_0(\mathcal{S})$

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by the equation

$$\langle \mu * \nu, g \rangle = \int_{\mathcal{S}} \int_{\mathcal{S}} g(xy)\, d\mu(x)\, d\nu(y)$$

for all

$$\mu, \nu \in M(\mathcal{S}) \quad \text{and} \quad g \in C_0(\mathcal{S});$$

then $M(\mathcal{S})$ with this multiplication is a Banach algebra. It is well-known from Wong [19] that the latter equality also holds for all

$$\mu, \nu \in M(\mathcal{S}) \quad \text{and} \quad g \in L^1(|\mu| * |\nu|).$$

The space of all measures $\mu \in M(\mathcal{S})$ for which the maps

$$x \longmapsto \delta_x * \mid \mu \mid \quad \text{and} \quad x \longmapsto \mid \mu \mid * \delta_x$$

from $\mathcal{S}$ into $M(\mathcal{S})$ are weakly continuous is denoted by $M_a(\mathcal{S})$ (or $\widetilde{L}(\mathcal{S})$ as in [1]), where δ_x denotes the Dirac measure at x. Then $M_a(\mathcal{S})$ is a closed two-sided L-ideal of $M(\mathcal{S})$; see Baker and Baker [1]. The locally compact semigroup $\mathcal{S}$ is called *foundation* if the set $\bigcup\{\text{supp}(\mu) : \mu \in M_a(\mathcal{S})\}$ is dense in $\mathcal{S}$.

A complex-valued function g on $\mathcal{S}$ is said to be $M_a(\mathcal{S})$*-measurable* if it is μ-measurable for all $\mu \in M_a(\mathcal{S})$. Denote by $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ the set of all bounded $M_a(\mathcal{S})$-measurable functions on $\mathcal{S}$ formed by identifying functions that agree μ-almost everywhere for all $\mu \in M_a(\mathcal{S})$. For each $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$, define

$$\| g \|_\infty = \sup\{ \|g\|_{\infty,|\mu|} : \mu \in M_a(\mathcal{S}) \},$$

where $\|.\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. Observe that $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with the complex conjugation as involution, the pointwise operations and the norm $\|.\|_\infty$ is a commutative C^*-algebra with the constant function one as identity. The duality

$$\langle g, \mu \rangle := \int_{\mathcal{S}} g\, d\mu$$

defines a linear mapping from $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ into the dual space $M_a(\mathcal{S})^*$ of $M_a(\mathcal{S})$. It is well-known from Sleijpen [17] that if $\mathcal{S}$ is a foundation semigroup with identity, then $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ can be identified with $M_a(\mathcal{S})^*$.

Note that a function $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ vanishes at infinity if for each $\varepsilon > 0$, there is a compact subset K of $\mathcal{S}$ for which

$$\|g\chi_{\mathcal{S}\setminus K}\|_\infty \leq \varepsilon;$$

that is,

$$|g(x)| \leq \varepsilon$$

for μ-almost all $x \in \mathcal{S} \setminus K$ $(\mu \in M_a(\mathcal{S}))$. We denote by $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ the C^*-subalgebra of $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ consisting of all functions in $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ that vanish at infinity. Note that $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ is the $\|.\|_\infty$-closure of $L_{00}^\infty(\mathcal{S}, M_a(\mathcal{S}))$, the space of all functions $f \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with compact support.

In the case where $\mathcal{G}$ is a locally compact group, $L_0^\infty(\mathcal{G}, M_a(\mathcal{G}))$ is the space $L_0^\infty(\mathcal{G})$ of all essentially bounded measurable functions that vanish at infinity.

In this paper, we introduce and study an Arens multiplication on $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ for certain foundation semigroups $\mathcal{S}$ with identity. As the main result, we prove that $M_a(\mathcal{S})$ or $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ is Arens regular if and only if $\mathcal{S}$ is finite. Our work improves an interesting result of Singh [15] and Young [20] for locally compact groups $\mathcal{G}$ to a more general setting of locally compact semigroups; these results are based on the previous investigation concerning $L_0^\infty(\mathcal{G})$ by Lau and Pym [9] and Isik, Pym, and Ülger [6].

2 The C*-algebra $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$

The locally compact semigroup $\mathcal{S}$ is said to be *compactly cancellative* if the sets $C^{-1}D$ and CD^{-1} are compact subsets of $\mathcal{S}$ for all compact subsets C and D of $\mathcal{S}$, where

$$\begin{aligned} C^{-1}D &= \{x \in \mathcal{S} : cx = d \text{ for some } c \in C, d \in D\}, \\ CD^{-1} &= \{x \in \mathcal{S} : c = xd \text{ for some } c \in C, d \in D\}. \end{aligned}$$

Given any $\mu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$, the complex-valued functions $g \circ \mu$ and $\mu \circ g$ are defined on $\mathcal{S}$ by

$$(g \circ \mu)(x) = \langle \mu, {}_x g \rangle$$

and

$$(\mu \circ g)(x) = \langle \mu, g_x \rangle$$

for all $x \in \mathcal{S}$, where the function ${}_x g$ and g_x are defined on $\mathcal{S}$ by

$${}_x g(y) = g(xy) \quad \text{and} \quad g_x(y) = g(yx)$$

for all $x, y \in \mathcal{S}$. The weak continuity of the mappings

$$x \mapsto \delta_x * \mu \quad \text{and} \quad x \mapsto \mu * \delta_x$$

from $\mathcal{S}$ into $M_a(\mathcal{S})$ together with that

$$(g \circ \mu)(x) = \langle \delta_x * \mu, g \rangle$$

and

$$(\mu \circ g)(x) = \langle \mu * \delta_x, g \rangle$$

for all $x \in \mathcal{S}$ imply that $g \circ \mu$ and $\mu \circ g$ are continuous. Also,

$$\|g \circ \mu\|_\infty \leq \|g\|_\infty \, \|\mu\|$$

and

$$\|\mu \circ g\|_\infty \leq \|g\|_\infty \, \|\mu\|.$$

So, if we denote by $C_b(\mathcal{S})$ the Banach space of all bounded continuous complex-valued functions on $\mathcal{S}$, then

$$L^\infty(\mathcal{S}, M_a(\mathcal{S})) \circ M_a(\mathcal{S}) \subseteq C_b(\mathcal{S})$$

and

$$M_a(\mathcal{S}) \circ L^\infty(\mathcal{S}, M_a(\mathcal{S})) \subseteq C_b(\mathcal{S}).$$

Let X be a closed subspace of $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with

$$X \circ M_a(\mathcal{S}) \subseteq X$$

and

$$M_a(\mathcal{S}) \circ X \subseteq X.$$

Then as easily checked X can be considered as a Banach $M_a(\mathcal{S})$-bimodule. In fact, X equipped with the map $(\mu, g) \mapsto g \circ \mu$ from $M_a(\mathcal{S}) \times X$ into X is a Banach left $M_a(\mathcal{S})$-module; also, X equipped with the map $(g, \mu) \mapsto \mu \circ g$ from $X \times M_a(\mathcal{S})$ into X is a Banach right $M_a(\mathcal{S})$-module; moreover,

$$(\mu \circ g) \circ \nu = \mu \circ (g \circ \nu)$$

for all $\mu, \nu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$, and so X is Banach $M_a(\mathcal{S})$-bimodule. In particular, $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ and $C_b(\mathcal{S})$ are Banach $M_a(\mathcal{S})$-bimodules.

A bounded net $(\varrho_\iota)_{\iota \in I}$ in $M_a(\mathcal{S})$ is said to be a *bounded right approximate identity for X* whenever

$$\|\varrho_\iota \circ g - g\|_\infty \to 0$$

for all $g \in X$. A *bounded left approximate identity for X* is defined similarly; by a *bounded approximate identity for X*, we shall mean a bounded left and right approximate identity for X.

Proposition 2.1. *Let $\mathcal{S}$ be a compactly cancellative locally compact semigroup. Then*

$$L_0^\infty(\mathcal{S}, M_a(\mathcal{S})) \circ M_a(\mathcal{S}) \subseteq C_0(\mathcal{S})$$

and

$$M_a(\mathcal{S}) \circ L_0^\infty(\mathcal{S}, M_a(\mathcal{S})) \subseteq C_0(\mathcal{S}).$$

In particular, $C_0(\mathcal{S})$ and $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ are Banach $M_a(\mathcal{S})$-bimodules.

Proof. Let $\mu \in M_a(\mathcal{S})$ and $g \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$. As we have already seen $\mu \circ g \in C_b(\mathcal{S})$. To prove that $\mu \circ g$ vanishes at infinity, without loss of generality, we may assume that g and μ have compact support E and D, respectively. Then

$$\operatorname{supp}(\mu \circ g) \subseteq D^{-1}E.$$

In particular, $\operatorname{supp}(\mu \circ g)$ is compact, and so

$$\mu \circ g \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S})).$$

Therefore

$$M_a(\mathcal{S}) \circ L_0^\infty(\mathcal{S}, M_a(\mathcal{S})) \subseteq C_0(\mathcal{S}).$$

The other inclusion follows similarly. ∎

A function $f \in C_b(\mathcal{S})$ is called *left uniformly continuous* if the map $x \mapsto {}_xf$ from $\mathcal{S}$ into $C_b(\mathcal{S})$ is norm continuous. A *right uniformly continuous* is defined similarly; by a *uniformly continuous* function, we shall mean a left and right uniformly continuous function. The Banach space of all left (resp. right) uniformly continuous functions on $\mathcal{S}$ is denoted by $LUC(\mathcal{S})$ (resp. $RUC(\mathcal{S})$); we also set

$$UC(\mathcal{S}) := LUC(\mathcal{S}) \cap RUC(\mathcal{S}).$$

Let us recall that a *bounded right approximate identity* in the Banach algebra $M_a(\mathcal{S})$ is a bounded net $(\varrho_\iota)_{\iota\in I} \subseteq M_a(\mathcal{S})$ such that

$$\|\mu * \varrho_\iota - \mu\| \to 0$$

for all $\mu \in M_a(\mathcal{S})$. A *bounded left approximate identity* in $M_a(\mathcal{S})$ is defined similarly; also, a *bounded approximate identity* in $M_a(\mathcal{S})$ is a bounded left and right approximate identity in $M_a(\mathcal{S})$.

Proposition 2.2. *Let $\mathcal{S}$ be a foundation semigroup with identity. Then $LUC(\mathcal{S})$ is a Banach $M_a(\mathcal{S})$-bimodule, and every bounded right approximate identity in $M_a(\mathcal{S})$ is a bounded right approximate identity for $LUC(\mathcal{S})$.*

Proof. First, recall that for each $\mu \in M_a(\mathcal{S})$, the hypothesis implies that the map $x \mapsto \mu * \delta_x$ from $\mathcal{S}$ into $M_a(\mathcal{S})$ is norm continuous; see Dzinotyweyi [5], Theorem 5.6.1. Next, note that

$${}_x(\mu \circ g) = (\mu * \delta_x) \circ g$$

for all $x \in \mathcal{S}$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. It follows that

$$\mu \circ g \in LUC(\mathcal{S})$$

for all $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$; that is,

$$M_a(\mathcal{S}) \circ L^\infty(\mathcal{S}, M_a(\mathcal{S})) \subseteq LUC(\mathcal{S}).$$

In particular,

$$M_a(\mathcal{S}) \circ LUC(\mathcal{S}) \subseteq LUC(\mathcal{S}),$$

and hence $LUC(\mathcal{S})$ is a Banach right $M_a(\mathcal{S})$-module.

Now, suppose that $(\varrho_\iota)_{\iota\in I}$ is a right approximate identity in $M_a(\mathcal{S})$ bounded by the constant $B > 0$. Then for every $f \in LUC(\mathcal{S})$ and $\varepsilon > 0$, there is a neighbourhood U of the identity element e of $\mathcal{S}$ such that

$$\|{}_xf - f\|_\infty < \varepsilon$$

for all $x \in U$. Since $\mathcal{S}$ is foundation, there exists a probability measure ϱ in $M_a(\mathcal{S})$ with

$$\operatorname{supp}(\varrho) \subseteq U.$$

Then

$$\|\varrho \circ f - f\|_\infty \leq \varepsilon.$$

So, for each $\iota \in I$ we have

$$\varrho_\iota \circ (\varrho \circ f) = (\varrho * \varrho_\iota) \circ f$$

and therefore

$$\begin{aligned} \|\varrho_\iota \circ f - f\|_\infty &\leq \|\varrho_\iota \circ f - \varrho_\iota \circ (\varrho \circ f)\|_\infty \\ &+ \|\varrho_\iota \circ (\varrho \circ f) - \varrho \circ f\|_\infty + \|\varrho \circ f - f\|_\infty \\ &\leq B\,\|f - \varrho \circ f\|_\infty \\ &+ \|(\varrho * \varrho_\iota) \circ f - \varrho \circ f\|_\infty + \|\varrho \circ f - f\|_\infty \\ &\leq 2\,(B+1)\,\varepsilon + \|\varrho * \varrho_\iota - \varrho\|. \end{aligned}$$

This shows that

$$\|\varrho_\iota \circ f - f\|_\infty \to 0;$$

that is, $(\varrho_\iota)_{\iota \in I}$ is a bounded right approximate identity for $LUC(\mathcal{S})$.

Now, recall from Sleijpen [16], Theorem 5.16, that there is a bounded approximate identity $(\nu_\gamma)_{\gamma\in\Gamma}$ in $M_a(\mathcal{S})$. By what we have already seen, $(\nu_\gamma)_{\gamma\in\Gamma}$ is a bounded right approximate identity for $LUC(\mathcal{S})$. Thus, the Cohen factorization theorem [3], Theorem 11.10, yields that

$$M_a(\mathcal{S}) \circ LUC(\mathcal{S}) = LUC(\mathcal{S}).$$

So, if $f \in LUC(\mathcal{S})$ and $\mu \in M_a(\mathcal{S})$, then

$$\sigma \circ h = f$$

for some $\sigma \in M_a(\mathcal{S})$ and $h \in LUC(\mathcal{S})$; this yields that

$$f \circ \mu = (\sigma \circ h) \circ \mu = \sigma \circ (h \circ \mu).$$

Consequently,

$$f \circ \mu \in M_a(\mathcal{S}) \circ L^\infty(\mathcal{S}, M_a(\mathcal{S})),$$

and hence

$$f \circ \mu \in LUC(\mathcal{S}).$$

It follows that

$$LUC(\mathcal{S}) \circ M_a(\mathcal{S}) \subseteq LUC(\mathcal{S})$$

whence $LUC(\mathcal{S})$ is a Banach $M_a(\mathcal{S})$-bimodule. ■

Proposition 2.2 has the following analogue for $RUC(S)$.

Proposition 2.3. *Let $\mathcal{S}$ be a foundation semigroup with identity. Then $RUC(\mathcal{S})$ is a Banach $M_a(\mathcal{S})$-bimodule, and every bounded left approximate identity in $M_a(\mathcal{S})$ is a bounded left approximate identity for $RUC(\mathcal{S})$.*

A combination of Propositions 2.2 and 2.3 lead naturally to the following result.

Corollary 2.4. *Let $\mathcal{S}$ be a foundation semigroup with identity. Then $UC(\mathcal{S})$ is a Banach $M_a(\mathcal{S})$-bimodule, and every bounded approximate identity in $M_a(\mathcal{S})$ is a bounded approximate identity for $UC(\mathcal{S})$.*

The following remark shows that in Corollary 2.4, the space $UC(\mathcal{S})$ cannot be replaced by $LUC(\mathcal{S})$ or $RUC(\mathcal{S})$.

Remark 2.5. Let $\mathcal{S}$ be a foundation semigroup with identity.

(a) There is a bounded approximate identity for $LUC(\mathcal{S})$ only if

$$LUC(\mathcal{S}) \subseteq RUC(\mathcal{S});$$

indeed, if there is a bounded approximate identity $(\nu_\gamma)_{\gamma\in\Gamma} \subseteq M_a(\mathcal{S})$ for $LUC(\mathcal{S})$, then for each function $f \in LUC(\mathcal{S})$ we have

$$\|f \circ \nu_\gamma - f\|_\infty \to 0;$$

since

$$f \circ \nu_\gamma \in L^\infty(\mathcal{S}, M_a(\mathcal{S})) \circ M_a(\mathcal{S}) \subseteq RUC(\mathcal{S})$$

for all $\gamma \in \Gamma$, it follows that $f \in RUC(\mathcal{S})$. Thus

$$LUC(\mathcal{S}) \subseteq RUC(\mathcal{S}).$$

(b) There is a bounded approximate identity for $RUC(\mathcal{S})$ only if

$$RUC(\mathcal{S}) \subseteq LUC(\mathcal{S});$$

this follows by an argument similar to (a).

For foundation semigroups with identity Proposition 2.1 becomes particularly interesting as the following result shows.

Proposition 2.6. *Let $\mathcal{S}$ be a compactly cancellative foundation semigroup with identity. Then every bounded approximate identity in $M_a(\mathcal{S})$ is a bounded approximate identity for $C_0(\mathcal{S})$. Furthermore,*

(a) $C_0(\mathcal{S}) \circ M_a(\mathcal{S}) = L_0^\infty(\mathcal{S}, M_a(\mathcal{S})) \circ M_a(\mathcal{S}) = C_0(\mathcal{S})$.

(b) $M_a(\mathcal{S}) \circ C_0(\mathcal{S}) = M_a(\mathcal{S}) \circ L_0^\infty(\mathcal{S}, M_a(\mathcal{S})) = C_0(\mathcal{S})$.

Proof. Since $\mathcal{S}$ is compactly cancellative, it follows from Lemma 1.2 of Lau and Loy [8] that

$$C_0(\mathcal{S}) \subseteq UC(\mathcal{S}).$$

So, by Corollary 2.4, any bounded approximate identity in $M_a(\mathcal{S})$ is a bounded approximate identity for $C_0(\mathcal{S})$.

To prove (a), recall that $M_a(\mathcal{S})$ has a bounded approximate identity; see for example Sleijpen [16], Theorem 5.16. So, there is a bounded approximate identity in $M_a(\mathcal{S})$ for $C_0(\mathcal{S})$. Moreover, $C_0(\mathcal{S})$ is a Banach $M_a(\mathcal{S})$-bimodules. So, an application of the Cohen factorization theorem [3], Theorem 11.10, implies that

$$M_a(\mathcal{S}) \circ C_0(\mathcal{S}) = C_0(\mathcal{S});$$

similarly,

$$C_0(\mathcal{S}) \circ M_a(\mathcal{S}) = C_0(\mathcal{S}).$$

These equalities together with $C_0(\mathcal{S}) \subseteq L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ complete the proof. ∎

We end this section with the following example which shows that Propositions 2.6 is, in general, not valid if the hypothesis that S is foundation with identity is dropped.

Example 2.7. Let $\mathcal{S} = [0, \infty)$ be the semigroup with the operation $xy = \max\{x, y\}$ and the usual topology of the real line. Then $\mathcal{S}$ is a non-foundation locally compact semigroup with

$$M_a(\mathcal{S}) = \{0\};$$

however, $C_0(\mathcal{S})$ is the space of all continuous complex-valued functions f on $\mathcal{S}$ with

$$\lim_{x \to +\infty} |f(x)| = 0.$$

3 An Arens product on $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$

We commence this section with the following key lemma.

Lemma 3.1. *Let $\mathcal{S}$ be a locally compact semigroup and $m \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$. Then for each $\varepsilon > 0$, there is a compact subset $C \subseteq \mathcal{S}$ with*

$$|\langle m, h \rangle| \le \varepsilon \, \|h\|_\infty$$

for all $h \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with $\operatorname{supp}(h) \subseteq \mathcal{S} \setminus C$.

Proof. Since $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ is spanned by its positive elements, we can assume $m \ge 0$. Let σ denote the restriction of m to $C_0(\mathcal{S})$. Then for every $\varepsilon > 0$, there is a compact subset C of $\mathcal{S}$ such that

$$\sigma(\mathcal{S} \setminus C) < \varepsilon/2.$$

Let (C_α) be the family of compact subsets of $\mathcal{S}$ directed by upward inclusion. Then (χ_{C_α}) is a bounded approximate identity in the C^*-algebra $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Now, let n be the linear functional on $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ defined by

$$\langle n, g \rangle = \langle m, g\, \chi_{\mathcal{S} \setminus C} \rangle$$

for all $g \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Since n is a positive functional on the C^*-algebra $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$, it follows that

$$\|n\| = \lim_\alpha \langle n, \chi_{C_\alpha} \rangle.$$

So, there exists α_0 such that

$$\langle n, \chi_{C_{\alpha_0}} \rangle \ge \|n\| - \varepsilon/2.$$

Choose a function $\phi \in C_0(\mathcal{S})$ with

$$\chi_{C_{\alpha_0}} \le \phi \le 1.$$

Then

$$\begin{aligned} \langle n, \chi_{C_{\alpha_0}} \rangle &\le \langle n, \phi \rangle \\ &\le \|n|_{C_0(\mathcal{S})}\| \\ &= \sigma(\mathcal{S} \setminus C) \end{aligned}$$

which shows that $\|n\| \leq \varepsilon$. Therefore

$$\begin{aligned} |\langle m, h\rangle| &= |\langle m, h\,\chi_{\mathcal{S}\setminus C}\rangle| \\ &= |\langle n, h\rangle| \\ &\leq \varepsilon\,\|h\|_\infty \end{aligned}$$

for all $h \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with $\operatorname{supp}(h) \subseteq \mathcal{S} \setminus C$. ∎

Let $\mathcal{S}$ be a compactly cancellative locally compact semigroup. For every $m \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ and $g \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$, we denote by mg the linear functional defined on $M_a(\mathcal{S})$ by

$$\langle mg, \mu\rangle = \langle m, \mu \circ g\rangle \quad (\mu \in M_a(\mathcal{S})).$$

Proposition 3.2. *Let $\mathcal{S}$ be a compactly cancellative foundation semigroup with identity. Then $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ is a left introverted subspace of $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$; i.e.,*

$$mg \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$$

for $m \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ and $g \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$.

Proof. The map $\mu \mapsto \langle m, \mu \circ g\rangle$ is a bounded linear functional on $M_a(\mathcal{S})$, and hence $mg \in M_a(\mathcal{S})^*$. Since $\mathcal{S}$ is a foundation semigroup with identity, mg can be considered as a function in $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ such that

$$\langle mg, \mu\rangle = \langle m, \mu \circ g\rangle \qquad (\mu \in M_a(\mathcal{S})).$$

We show that $mg \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$. To this end, without loss of generality, we may assume that g has compact support E. Recall from Lemma 3.1 that for each $\varepsilon > 0$, there exists a compact subset C of $\mathcal{S}$ such that

$$|\langle m, h\rangle| \leq \varepsilon\,\|h\|_\infty$$

for all $h \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with

$$\operatorname{supp}(h) \subseteq \mathcal{S} \setminus C.$$

Then

$$\|(mg)\chi_{EC^{-1}}\|_\infty \leq \varepsilon\,\|g\|_\infty.$$

Indeed, for each $\mu \in M_a(\mathcal{S})$ with compact support D in $\mathcal{S} \setminus EC^{-1}$, we get

$$D^{-1}E \cap C = \emptyset.$$

Therefore,

$$\operatorname{supp}(\mu \circ g) \subseteq D^{-1}E \subseteq \mathcal{S} \setminus C$$

and hence

$$\begin{aligned} \left|\int_{\mathcal{S}} mg\, d\mu\right| &= |\langle mg, \mu\rangle| \\ &= |\langle m, \mu \circ g\rangle| \\ &\leq \varepsilon\,\|\mu \circ g\|_\infty \\ &\leq \varepsilon\,\|\mu\|\,\|g\|_\infty. \end{aligned}$$

Since EC^{-1} is compact in $\mathcal{S}$, it follows that $mg \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$. ■

Let $\mathcal{S}$ be as in Proposition 3.2. We endow $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ with the first Arens product "$\cdot$" defined by

$$\langle m \cdot n, g \rangle = \langle m, ng \rangle$$

for all $m, n \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ and $g \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Then $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ with this product is a Banach algebra.

For each $\mu \in M_a(\mathcal{S})$, let μ also denote the functional in $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ defined by

$$\langle \mu, g \rangle := \int_{\mathcal{S}} g \, d\mu$$

for all $g \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$). Note that this duality defines a linear isometric embedding of $M_a(\mathcal{S})$ into $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$; indeed,

$$C_0(\mathcal{S}) \subseteq L_0^\infty(\mathcal{S}, M_a(\mathcal{S})) \subseteq L^\infty(\mathcal{S}, M_a(\mathcal{S})) \subseteq M_a(\mathcal{S})^*,$$

and

$$\sup\{|\langle \mu, \varphi \rangle| : \varphi \in C_0(\mathcal{S})\} = \|\mu\| = \sup\{|\langle \mu, f \rangle| : f \in M_a(\mathcal{S})^*\}.$$

Also, observe that for any $\mu, \nu \in M_a(\mathcal{S})$ and $g \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$,

$$\mu \cdot \nu = \mu * \nu$$

and

$$\mu g = g \circ \mu.$$

Furthermore, an easy application of Goldstein's theorem shows that $M_a(\mathcal{S})$ is weak* dense in $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$.

Proposition 3.3. *Let $\mathcal{S}$ be a compactly cancellative foundation semigroup with identity. Then $M_a(\mathcal{S})$ is a two-sided closed ideal in $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$.*

Proof. Trivially $M_a(\mathcal{S})$ is a closed subspace of $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$.

Now, suppose that $\mu \in M_a(\mathcal{S})$ and $m \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$. We show that

$$\mu \cdot m \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*;$$

that

$$m \cdot \mu \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$$

is similar. Let $\nu \in M(\mathcal{S})$ be the restriction of m to $C_0(\mathcal{S})$. Since $M_a(\mathcal{S})$ is a two-sided ideal in $M(\mathcal{S})$ we have

$$\mu * \nu \in M_a(\mathcal{S}).$$

So it suffices to show that

$$\mu \cdot m = \mu * \nu.$$

To that end, note that if $g \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))$, then

$$\mu \circ g \in C_0(\mathcal{S})$$

by Proposition 2.1. Hence

$$\langle m, \mu \circ g \rangle = \langle \nu, \mu \circ g \rangle.$$

On the one hand,

$$\begin{aligned} \langle \mu \cdot m, g \rangle &= \langle \mu, mg \rangle \\ &= \langle m, \mu \circ g \rangle, \end{aligned}$$

and on the other hand,

$$\begin{aligned} \langle \nu, \mu \circ g \rangle &= \int_{\mathcal{S}} (\mu \circ g)(y)\, d\nu(y) \\ &= \int_{\mathcal{S}} \int_{\mathcal{S}} g(xy)\ d\mu(x)\ d\nu(y) \\ &= \int_{\mathcal{S}} g(t)\ d(\mu * \nu)(t) \\ &= \langle \mu * \nu, g \rangle. \end{aligned}$$

That is $\mu \cdot m = \mu * \nu$ as required. ■

Proposition 3.4. *Let $\mathcal{S}$ be a foundation semigroup with identity. Then $M_a(\mathcal{S})$ coincides with $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ if and only if S is discrete.*

Proof. The "if" part is clear. To prove the converse, suppose that

$$M_a(\mathcal{S}) = L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*.$$

Let E be an extension of δ_e from $C_0(\mathcal{S})$ to an element of $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$, where e denotes the identity element of $\mathcal{S}$. Then $E = \mu$ for some $\mu \in M_a(\mathcal{S})$. In particular,

$$\phi(e) = E(\phi) = \mu(\phi)$$

for all $\phi \in C_0(\mathcal{S})$. Thus

$$\delta_e = \mu \in M_a(\mathcal{S});$$

that is, $\mathcal{S}$ is discrete; see Baker and Baker [1], Theorem 2.8. ■

We end this section with the following result.

Proposition 3.5. *Let $\mathcal{S}$ be a compactly cancellative foundation semigroup with identity. Then $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ has a bounded approximate identity if and only if it has an identity.*

Proof. If $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ has a bounded approximate identity (u_γ), and u is a weak* cluster point of (u_γ) in $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$, we may assume that

$$u_\gamma \longrightarrow u$$

in the weak* topology. Let $m \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$. Then the weak* continuity of the map $n \mapsto n \cdot m$ on $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ shows that

$$u_\gamma \cdot m \longrightarrow u \cdot m$$

in the weak* topology of $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$. But

$$u_\gamma \cdot m \to m$$

in the norm topology of $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$. So $u \cdot m = m$.

So, for each $\mu \in M_a(\mathcal{S})$, by the weak* continuity of the map $n \mapsto \mu \cdot n$ on $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ we conclude that

$$\mu \cdot u_\gamma \to \mu \cdot u$$

in the weak* topology of $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$. This together with that (u_γ) is a bounded approximate identity for $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ imply that

$$\mu \cdot u = \mu.$$

It follows that $m \cdot u = m$ by the weak* density of $M_a(\mathcal{S})$ in $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$. ∎

4 Arens regularity of $M_a(\mathcal{S})$ and $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$

Let us recall that the first Arens product $\odot$ on the second dual of a Banach algebra $\mathfrak{A}$ is defined by

$$\langle F \odot G, \varphi \rangle = \langle F, G\,\varphi \rangle$$

for all $F, G \in \mathfrak{A}^{**}$ and $\varphi \in \mathfrak{A}^*$, where

$$\langle G\,\varphi, a \rangle = \langle G, \varphi\, a \rangle$$

and

$$\langle \varphi\, a, b \rangle = \langle \varphi, ab \rangle$$

for $a, b \in \mathfrak{A}$. Then $\mathfrak{A}^{**}$ endowed with $\odot$ is a Banach algebra. For any G in $\mathfrak{A}^{**}$, the map

$$F \mapsto F \odot G$$

is weak*-weak* continuous on $\mathfrak{A}^{**}$. For an element F in $\mathfrak{A}^{**}$, the map

$$G \mapsto F \odot G$$

is in general not weak*-weak* continuous on $\mathfrak{A}^{**}$ unless F is in $\mathfrak{A}$.

The Banach algebra $\mathfrak{A}$ is called *Arens regular* if the map $G \mapsto F \odot G$ is weak*-weak* continuous on $\mathfrak{A}^{**}$ for all $F \in \mathfrak{A}^{**}$; this is equivalent to that the set

$$\{\varphi a : a \in \mathfrak{A}, \|a\| \leq 1\}$$

is relatively weakly compact for all $\varphi \in \mathfrak{A}^*$. Let $\ell^1(\mathcal{S})$ denote the closed subalgebra of $M(\mathcal{S})$ consisting of all discrete measures.

Lemma 4.1. *Let $\mathcal{S}$ be a foundation semigroup with identity. If $\ell^1(\mathcal{S})$ or $M_a(\mathcal{S})$ is Arens regular, then $\mathcal{S}$ is discrete.*

Proof. If $\ell^1(\mathcal{S})$ is Arens regular, then $M(\mathcal{S})$ is also Arens regular; see Young [20]. Since $M_a(\mathcal{S})$ is a closed ideal in $M(\mathcal{S})$, it follows that $M_a(\mathcal{S})$ is Arens regular; see Corollary 6.3 of [4].

It is well-known that $M_a(\mathcal{S})$ has a bounded approximate identity and is the unique predual of von Neumann algebra $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$; see [13], Lemma 2.2. In particular, $M_a(\mathcal{S})$ is weakly sequentially complete.

So, by the use of Ülger's criterion for Arens regularity of weakly sequentially complete Banach algebras, the Arens regularity of $M_a(\mathcal{S})$ implies that $M_a(\mathcal{S})$ has an identity element δ; see Ülger [18], Theorem 3.3. One can easily check that δ is also an identity element of $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$. Now, apply Theorem 3.3 to conclude that

$$M_a(\mathcal{S}) = L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*.$$

In particular, $\mathcal{S}$ is discrete by Corollary 3.4. ∎

The next examples show that Lemma 4.1 is not true without the assumption that $\mathcal{S}$ is foundation with identity.

Example 4.2. (a) Let $\mathcal{S} = [0, 1]$ be equipped with the usual multiplication and the real line topology. Then

$$M_a(\mathcal{S}) = \{c\delta_0 : c \in \mathbb{C}\},$$

and thus $\mathcal{S}$ is a non-foundation semigroup with identity and $M_a(\mathcal{S})$ is regular, but $\mathcal{S}$ is not discrete.

(b) Let $\mathcal{S} = [0, 1]$ be equipped with the multiplication $x.y = 0$ for all $x, y \in \mathcal{S}$ and the real line topology. Then $\mathcal{S}$ is a foundation semigroup without identity, and $M_a(\mathcal{S})$ and $\ell^1(\mathcal{S})$ are regular, but $\mathcal{S}$ is not discrete.

A function $f \in C_b(\mathcal{S})$ is called *weakly almost periodic* if the set $\{ f_x : x \in \mathcal{S} \}$ is relatively weakly compact in $C_b(\mathcal{S})$. The set of all weakly almost periodic functions on $\mathcal{S}$ is denoted by $WAP(\mathcal{S})$.

We now are in a position to give the main result of this paper.

Theorem 4.3. *Let $\mathcal{S}$ be a compactly cancellative foundation semigroup with identity. Then the following statements are equivalent.*

(a) *$L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ is Arens regular.*
(b) *$M_a(\mathcal{S})$ is Arens regular.*
(c) *$M(\mathcal{S})$ is Arens regular.*
(d) *$\ell^1(\mathcal{S})$ is Arens regular.*
(e) *$\mathcal{S}$ is finite.*

Proof. The equivalence of (c) and (d) is well-known; see Young [20]. If (d) holds, then $\mathcal{S}$ is discrete by Lemma 4.1, and hence finitely cancellative; i.e.,

$$\{x\}^{-1}\{y\} \quad \text{and} \quad \{x\}\{y\}^{-1}$$

are finite subsets of $\mathcal{S}$ for all $x, y \in \mathcal{S}$. This shows that (e) holds; see Dzinotyiweyi [5] or Baker and Rejali [2].

Suppose that $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ is Arens regular. Since $M_a(\mathcal{S})$ is a closed ideal of $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$, it follows that $M_a(\mathcal{S})$ is also Arens regular; see Civin and Yood [4], Corollary 6.3. This shows that (a) implies (b). Clearly (e) implies (a).

To prove that (b) implies (e), suppose that $M_a(\mathcal{S})$ is Arens regular. For every $f \in C_b(\mathcal{S})$, the set

$$\mathcal{K}(f) = \{\mu \circ f : \mu \in M_a(\mathcal{S}), \|\mu\| \leq 1\}$$

is relatively weakly compact in $C_b(\mathcal{S})$. Let (ν_γ) be an approximate identity of probability measures in $M_a(\mathcal{S})$; see Sleijpen [16], Theorem 5.16 or the second author [13], Lemma 2.2. Then

$$\nu_\beta \circ f \to f$$

in the weak topology of $C_b(\mathcal{S})$ for some subnet (ν_β) of (ν_γ). Now, invoke Proposition 2.2 to conclude that

$$\nu_\beta \circ f \in LUC(\mathcal{S})$$

for all β, and hence $f \in LUC(\mathcal{S})$. So, for every $x \in \mathcal{S}$ we have

$$\begin{aligned} \|\nu_\gamma \circ f_x - f_x\|_\infty &= \sup_{y \in \mathcal{S}} |(\nu_\gamma \circ f)(yx) - f(yx)| \\ &\leq \|\nu_\gamma \circ f - f\|_\infty. \end{aligned}$$

Next, recall from Proposition 2.2 that (ν_γ) is a bounded left approximate identity for $LUC(\mathcal{S})$, and therefore

$$\|\nu_\gamma \circ f_x - f_x\|_\infty \to 0.$$

This shows that

$$\{f_x : x \in \mathcal{S}\} \subseteq \mathcal{K}(f),$$

and hence $f \in WAP(\mathcal{S})$. We thus have shown that

$$WAP(\mathcal{S}) = C_b(\mathcal{S}).$$

But this is equivalent to that $\mathcal{S}$ is compact; see Dzinotyweyi [5], Corollary 4.3.9 .

So, the result will follow if we show that $\mathcal{S}$ is discrete. To see this, recall that $M_a(\mathcal{S})$ is the predual of von Neumann algebra $L^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$, and hence $M_a(\mathcal{S})$ is weakly sequentially complete. Thus $M_a(\mathcal{S})$ has an identity; see Ülger [18], Theorem 3.3. In particular, $\mathcal{S}$ is discrete; see Baker and Baker [1], Theorem 2.8. ∎

Corollary 4.4. *Let $\mathcal{H}$ be a locally compact subsemigroup of a locally compact group $\mathcal{G}$ with positive Haar measure and with identity. the following statements are equivalent.*

(a) $L_0^\infty(\mathcal{H}, M_a(\mathcal{H}))^*$ *is Arens regular.*

(b) $M_a(\mathcal{H})$ *is Arens regular.*

(c) $M(\mathcal{H})$ *is Arens regular.*

(d) $\ell^1(\mathcal{H})$ *is Arens regular.*

(e) $\mathcal{H}$ *is finite.*

Proof. Let $\lambda_{\mathcal{H}}$ be the restriction of the Haar measure λ of $\mathcal{G}$ on $\mathcal{H}$. Then

$$M_a(\mathcal{H}) = L^1(\mathcal{H}, \lambda_{\mathcal{H}});$$

see Sleijpen [16], Theorem 4.10. This implies that $\mathcal{H}$ is foundation. So the result follows from Theorem 4.3. ∎

A special case of this result gives Theorem 7(a) in Singh [15].

Corollary 4.5. *Let $\mathcal{G}$ be a locally compact group. Then $L_0^\infty(\mathcal{G})^*$ is Arens regular if and only if $\mathcal{G}$ is finite.*

For a more general statement of Corollary 4.5 see [12]. Another special case of our main result gives the following description of Arens regularity for the group algebra $L^1(\mathcal{G})$ of a locally compact group $\mathcal{G}$; this is due to Young [20]; see also Isik, Pym and Ülger [6], Lau and Losert [7], and Neufang [14].

Corollary 4.6. *Let $\mathcal{G}$ be a locally compact group. Then $L^1(\mathcal{G})$ is Arens regular if and only if $\mathcal{G}$ is finite.*

We conclude the paper by the following examples.

Example 4.7. (a) Let $\mathcal{S} = \{0\} \cup \{1/n : n \geq 1\} \cup \{1/2 + 1/n : n \geq 1\}$ and set

$$\mathcal{B} = \{\{x\} : x \neq 0\} \cup \{\{0\} \cup \{1/n : n \geq k\} : k \geq 1\}.$$

Then $\mathcal{S}$ with $\mathcal{B}$ as a base of the topology and the operation

$$xy = \max\{x, y\}$$

defines a compactly cancellative foundation semigroup with identity. An application of Theorem 4.3 shows that $M_a(\mathcal{S})$ and $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ are not Arens regular.

(b) Let $\mathcal{H}$ be the subsemigroup $\mathbb{R}^+$ of the additive group $\mathbb{R}$ consisting of all non-negative real numbers. Then $\mathcal{H}$ with the restriction of the usual topology of the real line defines a compactly cancellative foundation semigroup with identity; indeed, $M_a(\mathcal{H})$ coincides with the usual Lebesgue space $L^1(\mathbb{R}^+)$, and $L_0^\infty(\mathcal{H}, M_a(\mathcal{H}))$ is the space $L_0^\infty(\mathbb{R}^+)$ of all measurable functions g on $\mathbb{R}^+$ such that

$$\|g\,\chi_{(x,\infty)}\|_\infty \to 0 \quad \text{as} \quad x \to +\infty.$$

In view of Corollary 4.4, $L^1(\mathbb{R}^+)$ and $L_0^\infty(\mathbb{R}^+)^*$ are not Arens regular.

Let $\mathcal{S}$ be a compactly cancellative foundation semigroup with identity. We denote by $Z_1(L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*)$ *the first topological center* of $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ consisting of all functionals $m \in L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ for which the map $n \mapsto m \cdot n$ is weak*-weak* continuous on $L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$. Note that

$$M_a(\mathcal{S}) \subseteq Z_1(L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*).$$

Let us recall from Lau and Pym [9] that for any locally compact group $\mathcal{G}$,

$$M_a(\mathcal{G}) = Z_1(L_0^\infty(\mathcal{G})^*).$$

This result leads us to conclude the paper by the following natural conjecture.

Conjecture. For every compactly cancellative foundation semigroup with identity $\mathcal{S}$,

$$M_a(\mathcal{S}) = Z_1(L_0^\infty(\mathcal{S}, M_a(\mathcal{S}))^*).$$

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References

[1] Baker, A. C., and Baker, J. W., Algebra of measures on a locally compact semigroup III, *J. London Math. Soc.* **4** (1972) 685-695.

[2] Baker, A. C., and Rejali, A., On the Arens regularity of weighted convolution algebras, *J. London Math. Soc.* **40** (1989) 535-546.

[3] Bonsall, F. F., and Duncan, J., Complete normed algebras, Springer Verlag, Berlin, 1973.

[4] Civin, P., and Yood, B., The second conjugate space of a Banach algebra as an algebra, *Pacific J. Math.* **40** (1961) 847-870.

[5] Dzinotyiweyi, H. A., The analogue of the group algebra for topological semigroups, Pitman Research Notes in Mathematics Series, London, 1984.

[6] Isik, N., Pym, J., and Ülger, A., The second dual of the group algebra of a compact group. *J. London Math. Soc.* **35** (1987) 135-148.

[7] Lau, A. T., and Losert, V., On the second conjugate algebra of $L^1(G)$ of a locally compact group, *J. London Math. Soc.* **37** (1988) 464-470.

[8] Lau, A. T., and Loy, R. J., Amenability of convolution algebras, *Math. Scand.* **79** (1996) 283-296.

[9] Lau, A. T., and Pym, J., Concerning the second dual of the group algebra of a locally compact group, *J. London Math. Soc.* **41** (1990) 445-460.

[10] Maghsoudi, S., and Nasr-Isfahani, R., Arens regularity of semigroup algebras with certain locally convex topologies, *Semigroup Forum* **75** (2007) 345-358 .

[11] Maghsoudi, S., Nasr-Isfahani, R., and Rejali, A., The dual of semigroup algebras with certain locally convex topologies, *Semigroup Forum* **73** (2006) 367-376.

[12] Maghsoudi, S., Nasr-Isfahani, R., and Rejali, A., Strong Arens irregularity of Beurling algebras with a locally convex topology, *Arch. Math. (Basel)* **86** (2006) 437-448

[13] Nasr-Isfahani, R., Factorization in some ideals of Lau algebras with applications to semigroup algebras, *Bull. Belg. Math. Soc.* **7** (2000) 429-433.

[14] Neufang, M., A unified approach to the topological centre problem for certain Banach algebras arising in abstract harmonic analysis, *Arch. Math. (Basel)* **82** (2004) 164-171.

[15] Singh, A. I., $L_0^\infty(G)^*$ as the second dual of the group algebra $L^1(G)$ with a locally convex topology, *Michigan Math. J.* **46** (1999) 143-150.

[16] Sleijpen, G. L., Convolution measure algebras on semigroups, Dissertation Katholieke Universiteit Nijmegen, Holland, 1976.

[17] Sleijpen, G. L., The dual of the space of measures with continuous translations, *Semigroup Forum* **22** (1981) 139-150.

[18] Ülger, A., Arens regularity of weakly sequentially complete Banach algebras, *Proc. Amer. Math. Soc.* **127** (1999) 3221-3227.

[19] Wong, J. C., Convolution and separate continuity, *Pacific J. Math.* **75** (1978) 601-611.

[20] Young, N. J., The irregularity of multiplication in group algebras, *Oxford Q. J. Math.* **24** (1972) 59-62.

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