On the Stability of Cauchy Additive Mappings

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Abstract

It is well-known that the concept of Hyers-Ulam-Rassias stability originated by Th. M. Rassias (Proc. Amer. Math. Soc. 72(1978), 297-300) and the concept of Ulam-Gavruta-Rassias stability by J. M. Rassias (J. Funct. Anal. U.S.A. 46(1982), 126-130; Bull. Sc. Math. 108 (1984), 445-446; J. Approx. Th. 57 (1989), 268-273) and P. Gavruta ("An answer to a question of John M. Rassias concerning the stability of Cauchy equation", in: Advances in Equations and Inequalities, in: Hadronic Math. Ser. (1999), 67-71). In this paper we give results concerning these two stabilities.

1 Introduction

The stability problem of functional equations originated from a question of S. Ulam[21] concerning the stability of group homomorphism: *Let* $(G_1, \circ)$ *be a group and* $(G_2, *)$ *a metric group with a metric* $d(\cdot, \cdot)$. *Given* $\epsilon > 0$, *does there exist a* $\delta > 0$ *such that if* $f : G_1 \to G_2$ *satisfies*

$$d(f(x \circ y), f(x) * f(y)) \leq \delta, \quad \text{for all } x, y \in G_1,$$

then there exists a homomorphism $h : G_1 \to G_2$ *with*

$$d(f(x), h(x)) \leq \epsilon, \text{ for all } x \in G_1?$$

D. H. Hyers[5] gave a first affirmative answer to the question of Ulam, for Banach spaces:

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Let $f : E \to E'$ be a mapping, where E and E' are Banach spaces, such that

$$\| f(x+y) - f(x) - f(y) \|_{E'} \le \epsilon,$$

for all $x, y \in E$ and for some ϵ. Then there exists a unique additive mapping $L : E \to E'$ such that

$$\| f(x) - L(x) \| \le \epsilon.$$

In 1978, Th. M. Rassias[17] proved the following generalization of Hyers[5]:

Proposition 1.1. *Let $f : E \to E'$ be a mapping, where E is a real normed space and E' is a Banach space. Assume that there exist $\epsilon > 0$ such that*

$$\| f(x+y) - f(x) - f(y) \| \le \epsilon(\| x \|^p + \| y \|^p), \tag{1.1}$$

for all $x, y \in E$, where $p \in [0, 1)$. Then there exists a unique additive mapping $L : E \to E'$ such that

$$\| f(x) - L(x) \| \le \frac{2\epsilon}{2 - 2^p} \| x \|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$.

In 1991, Z. Gajda[3] gave an affirmative answer to Th. M. Rassias' question whether his theorem can be extended for values of p greater than one.

However it was shown by Z. Gajda[3] and Th. M. Rassias and P. Semrl[18] that one can not prove a theorem similar to [17].

The inequality (1.1) that was introduced for the first time by Th. M. Rassias[17] provided a lot of influence in the development of a generalization of the Hyers-Ulam concept. This new concept of stability is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations (see the book of D. H. Hyers, G. Isac and Th. M. Rassias[6]).

In 1982-1989, J. M. Rassias([14], [15], [16]) proved the following generalization of Hyers[5]:

Proposition 1.2. *Let $f : E \to E'$ be a mapping, where E is a real normed space and E' is a Banach space. Assume that there exists a $\theta > 0$ such that*

$$\| f(x+y) - [f(x) + f(y)] \| \le \theta \| x \|^p \| y \|^q, \tag{1.3}$$

for all $x, y \in E$, where $r = p + q \neq 1$. Then there exists a unique additive mapping $L : E \to E'$ such that

$$\| f(x) - L(x) \| \le \frac{\theta}{|2^r - 2|} \| x \|^r, \tag{1.4}$$

for all $x \in E$.

However, the case $r = 1$ in inequality (1.3) is singular. A counter-example has been given by P. Gavruta[4]. The above-mentioned stability involving a product of different powers of norms is called Ulam-Gavruta-Rassias stability by B. Bouikhalene, E. Elqorachi and M. A. Sibaha[20], as well as by K. Ravi and M. Arunkumar[19], P. Nakmahachalasint[9], and B. Bouikhalene and E. Elqorachi[1].

More generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings can be find in [2], [7], [8], [10], [11] and [13].

C. Park, Y. Cho and M. Han[12] proved that a mapping satisfying one of the following inequalities,

$$\begin{aligned}\| f(x)+f(y)+f(z) \| &\leq \| 2f(\frac{x+y+z}{2}) \|,\\ \| f(x)+f(y)+f(z) \| &\leq \| f(x+y+z) \|,\\ \| f(x)+f(y)+2f(z) \| &\leq \| 2f(\frac{x+y}{2}+z) \|,\end{aligned}$$

is a Cauchy additive mapping and they gave some stability of these mappings. In this paper, we give improved results concerning these mappings.

2 Hyers-Ulam-Rassias Stability

In this paper we note that X is a normed vector space and Y is a Banach space. It was shown in [12] that a mapping $f : X \to Y$ satisfying the inequality

$$\| f(x)+f(y)+f(z) \|_Y \leq \| 2f(\frac{x+y+z}{2}) \|_Y$$

is Cauchy additive. Now we prove the Hyers-Ulam-Rassias stability of these mappings in Banach spaces.

Theorem 2.1. *Let $r > 1$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\| f(x)+f(y)+f(z) \|_Y \leq \| 2f(\frac{x+y+z}{2}) \|_Y + \epsilon(\| x \|_X^r + \| y \|_X^r + \| z \|_X^r), \tag{2.1}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that

$$\| f(x) - L(x) \|_Y \leq \frac{6+2^r}{2^r-2}\epsilon \| x \|_X^r. \tag{2.2}$$

Proof. From (2.1) with $x = y = z = 0$, we get $\| 3f(0) \|_Y \leq \| 2f(0) \|_Y$ which implies $\| f(0) \|_Y = 0$ and $f(0) = 0$. Also, by letting $y = x$, $z = -2x$ in (2.1) we get

$$\| 2f(x) + f(-2x) \|_Y \leq (2+2^r)\epsilon \| x \|_X^r,$$

for all $x \in X$. So, we get

$$\| 2f(\frac{x}{2}) + f(-x) \|_Y \leq \frac{2+2^r}{2^r}\epsilon \| x \|_X^r. \tag{2.3}$$

Next, by letting $y = -x$ and $z = 0$ in (2.1) we get

$$\| f(x) + f(-x) \|_Y \leq 2\epsilon \| x \|_X^r. \tag{2.4}$$

Hence, we have due to (2.3) and (2.4) that

$$\begin{aligned}
&\| 2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}) \|_Y \le \sum_{j=l}^{m-1} \| 2^j f(\frac{x}{2^j}) - 2^{j+1} f(\frac{x}{2^{j+1}}) \|_Y \\
&\le \sum_{j=l}^{m-1} \| 2^j f(\frac{x}{2^j}) + 2^{j+1} f(\frac{-x}{2^{j+1}}) - 2^{j+1} f(\frac{-x}{2^{j+1}}) - 2^{j+1} f(\frac{x}{2^{j+1}}) \|_Y \\
&\le \sum_{j=l}^{m-1} \left[\| 2^j f(\frac{x}{2^j}) + 2^{j+1} f(\frac{-x}{2^{j+1}}) \|_Y + \| 2^{j+1} f(\frac{-x}{2^{j+1}}) + 2^{j+1} f(\frac{x}{2^{j+1}}) \|_Y \right] \\
&\le \frac{6+2^r}{2^r} \epsilon \| x \|_X^r \sum_{j=l}^{m-1} \left(\frac{2}{2^r} \right)^j,
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So we can define the mapping $L : X \to Y$ by $L(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$, for all $x \in X$.

Moreover, by letting $l = 0$ and passing the limit $m \to \infty$, we get (2.2).

Next, we claim that $L(x)$ is a Cauchy additive mapping. First of all, we get by (2.4) that

$$\begin{aligned}
\| L(x) + L(-x) \|_Y &\le \lim_{n\to\infty} 2^n \| f(\frac{x}{2^n}) + f(-\frac{x}{2^n}) \|_Y \le \lim_{n\to\infty} 2^{n+1} \epsilon \| \frac{x}{2^n} \|_X^r \\
&= \lim_{n\to\infty} \frac{2^{n+1}\epsilon}{2^{nr}} \| x \|_X^r = 0,
\end{aligned}$$

for $r > 1$. So we have $L(-x) = -L(x)$.

Therefore we get by the definition of $L(x)$ and (2.1) that

$$\begin{aligned}
&\| L(x) + L(y) - L(x+y) \|_Y = \| L(x) + L(y) + L(-x-y) \|_Y \\
&= \lim_{n\to\infty} 2^n \| f(\frac{x}{2^n}) + f(\frac{y}{2^n}) + f(\frac{-x-y}{2^n}) \|_Y \\
&\le \lim_{n\to\infty} \left(\frac{2}{2^r} \right)^n \epsilon \; [\; \| x \|_X^r + \| y \|_X^r + \| x+y \|_X^r] = 0,
\end{aligned}$$

for all $x, y \in X$. So the function $L : X \to Y$ is Cauchy additive.

Now, to prove uniqueness of the function $L(x)$, let us assume that $T : X \to Y$ be another Cauchy additive mapping satisfying (2.2).

Then we obtain

$$\begin{aligned}
&\| L(x) - T(x) \|_Y = \lim_{n\to\infty} 2^n \| L(\frac{x}{2^n}) - T(\frac{x}{2^n}) \|_Y \\
&\le \lim_{n\to\infty} 2^n [\| L(\frac{x}{2^n}) - f(\frac{x}{2^n}) \|_Y + \| T(\frac{x}{2^n}) - f(\frac{x}{2^n}) \|_Y] \\
&\le \lim_{n\to\infty} \left(\frac{2}{2^r} \right)^n \left(\frac{12 + 2^{r+1}}{2^r - 2} \right) \epsilon \; \| x \|_X^r = 0,
\end{aligned}$$

for all $x \in X$. So we can conclude that $L(x) = T(x)$ for all $x \in X$. This proves the uniqueness of L. Thus the mapping $L : X \to Y$ is a unique Cauchy additive mapping satisfying (2.2). ∎

Theorem 2.2. *Let $r < 1$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\| f(x) + f(y) + f(z) \|_Y \leq \| 2f(\frac{x+y+z}{2}) \|_Y + \epsilon(\| x \|_X^r + \| y \|_X^r + \| z \|_X^r), \quad (2.5)$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that

$$\| f(x) - L(x) \|_Y \leq \frac{2 + 3 \cdot 2^r}{2 - 2^r} \epsilon \| x \|^r. \quad (2.6)$$

Proof. From (2.5) with $y = x$ and $z = -2x$, we get

$$\| f(x) + \frac{1}{2} f(-2x) \|_Y \leq \frac{2 + 2^r}{2} \epsilon \| x \|_X^r. \quad (2.7)$$

Hence, we have by (2.4) and (2.7)

$$\begin{aligned}
\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \|_Y &= \sum_{j=l}^{m-1} \| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \|_Y \\
\leq \sum_{j=l}^{m-1} & \left[\| \frac{1}{2^j} f(2^j x) + \frac{1}{2^{j+1}} f(-2^{j+1} x) \|_Y + \frac{1}{2^{j+1}} \| f(-2^{j+1} x) + f(2^{j+1} x) \|_Y \right] \\
\leq \sum_{j=l}^{m-1} & \left[\left(\frac{2 + 2^r}{2^{j+1}} \right) \epsilon \| 2^j x \|_X^r + \frac{2\epsilon}{2^{j+1}} \| 2^{j+1} x \|_X^r \right] \\
\leq \sum_{j=l}^{m-1} & \left(\frac{2 + 3 \cdot 2^r}{2} \right) \left(\frac{2^r}{2} \right)^j \epsilon \| x \|_X^r,
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$.

Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So we can define the mapping $L : X \to Y$ by $L(x) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$, for all $x \in X$.

Moreover, by letting $l = 0$ and passing the limit $m \to \infty$, we get

$$\| f(x) - L(x) \|_Y \leq \frac{2 + 3 \cdot 2^r}{2 - 2^r} \epsilon \| x \|_X^r.$$

The rest is similar to the proof of Theorem 2.1. ∎

It was shown in [12] that a mapping $f : X \to Y$ satisfying the inequality

$$\| f(x) + f(y) + f(z) \|_Y \leq \| f(x + y + z) \|_Y$$

is Cauchy additive. Now we prove the Hyers-Ulam-Rassias stability of these mappings in Banach spaces.

Theorem 2.3. *Let $r > 1$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\| f(x) + f(y) + f(z) \|_Y \leq \| f(x + y + z) \|_Y + \epsilon(\| x \|_X^r + \| y \|_X^r + \| z \|_X^r), \quad (2.8)$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that

$$\| f(x) - L(x) \|_Y \leq \frac{6 + 2^r}{2^r - 2} \epsilon \| x \|_X^r. \quad (2.9)$$

Proof. One can easily check that $\| f(0) \|_Y = 0$ which implies $f(0) = 0$. Also, by letting $y = x$ and $z = -2x$ in (2.9), we get

$$\| 2f(x) + f(-2x) \|_Y \le (2 + 2^r)\epsilon \| x \|_X^r, \tag{2.10}$$

for all $x \in X$. So we have

$$\| 2f(\frac{x}{2}) + f(-x) \|_Y \le \frac{2 + 2^r}{2^r}\epsilon \| x \|_X^r.$$

Next, by letting $y = -x$ and $z = 0$ in (2.9), we get

$$\| f(x) + f(-x) \|_Y \le 2\epsilon \| x \|_X^r.$$

The rest is similar to the proof of Theorem 2.1. ∎

Theorem 2.4. *Let $r < 1$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying (2.8). Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that*

$$\| f(x) - L(x) \|_Y \le \frac{2 + 3 \cdot 2^r}{2 - 2^r} \epsilon \| x \|_X^r, \quad \textit{for all } x \in X. \tag{2.11}$$

Proof. Since we get from (2.10),

$$\| 2f(x) + f(-2x) \|_Y \le (2 + 2^r)\epsilon \| x \|_X^r,$$

for all $x \in X$, we obtain

$$\| f(x) + \frac{1}{2}f(-2x) \|_Y \le \frac{2 + 2^r}{2}\epsilon \| x \|_X^r,$$

So by defining $L(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$, we get (2.11). The rest is similar to the proof of Theorem 2.2. ∎

It was shown in [12] that a mapping $f : X \to Y$ satisfying the inequality

$$\| f(x) + f(y) + 2f(z) \|_Y \le \| 2f(\frac{x + y}{2} + z) \|_Y$$

is Cauchy additive. Now we prove the Hyers-Ulam-Rassias stability of these mappings in Banach spaces.

Theorem 2.5. *Let $r > 1$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\| f(x) + f(y) + 2f(z) \|_Y \le \| 2f(\frac{x + y}{2} + z) \|_Y + \epsilon(\| x \|_X^r + \| y \|_X^r + \| z \|_X^r), \tag{2.12}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that

$$\| f(x) - L(x) \|_Y \le \frac{5 + 2^r}{2^r - 2}\epsilon \| x \|_X^r. \tag{2.13}$$

Proof. From (2.12) with $x = y = z = 0$, we get $f(0) = 0$. Also, by letting $x = 2x$, $y = 0$ and $z = -x$ in (2.12), we get

$$\| f(2x) + 2f(-x) \|_Y \leq (1 + 2^r)\epsilon \| x \|_X^r. \tag{2.14}$$

Next, by letting $y = -x$ and $z = 0$ in (2.14), we have

$$\| f(x) + f(-x) \|_Y \leq 2\epsilon\| x \|_X^r. \tag{2.15}$$

By a similar method to the proof of Theorem 2.1, we can define $L(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$. Now we claim that the mapping $L(x)$ is Cauchy additive. Due to (2.12) and (2.14), we obtain

$$\begin{aligned}
&\| L(x) + L(y) - L(x+y) \|_Y \\
&= \lim_{n\to\infty} 2^n \| f(\frac{x}{2^n}) + f(\frac{y}{2^n}) - f(\frac{x+y}{2^n}) \|_Y \\
&\leq \lim_{n\to\infty} 2^n \left[\| f(\frac{x}{2^n}) + f(\frac{y}{2^n}) + 2f(\frac{-x-y}{2^{n+1}}) \|_Y + \| 2f(\frac{-x-y}{2^{n+1}}) + f(\frac{x+y}{2^n}) \|_Y\right] \\
&\leq \lim_{n\to\infty} \left(\frac{2}{2^r}\right)^n \epsilon \left[\| x \|_X^r + \| y \|_X^r + \frac{2+2^r}{2^r} \| x+y \|_X^r\right] = 0,
\end{aligned}$$

for $r > 1$. The rest is similar to the proof of Theorem 2.1. ∎

Theorem 2.6. *Let $r < 1$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying the inequality (2.12). Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that*

$$\| f(x) - L(x) \|_Y \leq \frac{1 + 3 \cdot 2^r}{2 - 2^r}\epsilon\| x \|^r, \quad \textit{for all } x \in X. \tag{2.16}$$

Proof. In this case, we define $L(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$. Then, due to (2.12) and (2.14), we obtain

$$\begin{aligned}
&\| L(x) + L(y) - L(x+y) \|_Y \\
&= \lim_{n\to\infty} \frac{1}{2^n} \| f(2^n x) + f(2^n y) - f(2^n(x+y)) \|_Y \\
&\leq \lim_{n\to\infty} \frac{1}{2^n} \| f(2^n x) + f(2^n y) + 2f(\frac{2^n(-x-y)}{2}) \|_Y + \\
&+ \lim_{n\to\infty} \| 2f(\frac{2^n(-x-y)}{2}) + f(2^n(x+y)) \|_Y \\
&\leq \lim_{n\to\infty} \left(\frac{2^r}{2}\right)^n \epsilon \left[\| x \|_X^r + \| y \|_X^r + \frac{2+2^r}{2^r} \| x+y \|_X^r\right] = 0,
\end{aligned}$$

for $r < 1$. The rest is similar to the proof of Theorem 2.2. ∎

3 Ulam-Gavruta-Rassias Stability

In this section, we will give results concerning Ulam-Gavruta-Rassias stability.

Theorem 3.1. *Let $r > \frac{1}{3}$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\| f(x) + f(y) + f(z) \|_Y \le \| 2f(\frac{x+y+z}{2}) \|_Y + \epsilon(\| x \|_X^r \cdot \| y \|_X^r \cdot \| z \|_X^r), \quad (3.1)$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that

$$\| f(x) - L(x) \|_Y \le \frac{2^r}{2^{3r} - 2} \, \epsilon \, \| x \|^{3r}. \quad (3.2)$$

Proof. From (3.1) with $x = y = z = 0$, we get $\| f(0) \|_Y = 0$ which implies $f(0) = 0$. Also, by letting $y = x$ and $z = -2x$ in (3.1), we get

$$\| 2f(x) + f(-2x) \|_Y \le 2^r \epsilon \, \| x \|^{3r}.$$

So, we obtain

$$\| 2f(\frac{x}{2}) + f(-x) \|_Y \le \frac{\epsilon}{2^{2r}} \| x \|^{3r}. \quad (3.3)$$

Next, by letting $y = -x$ and $z = 0$ in (3.1), we get

$$\| f(x) + f(-x) \|_Y = 0 \quad (3.4)$$

which implies $-f(x) = f(-x)$. Hence, we have

$$\begin{aligned} \| 2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}) \|_Y &\le \sum_{j=l}^{m-1} \| 2^j f(\frac{x}{2^j}) - 2^{j+1} f(\frac{x}{2^{j+1}}) \|_Y \\ &\le \sum_{j=l}^{m-1} \| 2^j f(\frac{x}{2^j}) + 2^{j+1} f(\frac{-x}{2^{j+1}}) \|_Y \le \sum_{j=l}^{m-1} \frac{2^j}{2^{2r}} \epsilon \, \| \frac{x}{2^j} \|_X^{3r} \\ &\le \sum_{j=l}^{m-1} \frac{\epsilon}{2^{2r}} \| x \|^{3r} \left(\frac{2}{2^{3r}}\right)^j, \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$, if $r > \frac{1}{3}$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So we can define the mapping $L : X \to Y$ by $L(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$, for all $x \in X$.

Moreover, by letting $l = 0$ and passing the limit $m \to \infty$, we get (3.2).

Next, we note from (3.4)

$$\| L(x) + L(-x) \|_Y = \lim_{n\to\infty} 2^n \| f(\frac{x}{2^n}) + f(-\frac{x}{2^n}) \|_Y = 0$$

which implies $L(-x) = -L(x)$. The rest is similar to the proof of Theorem 2.1. ∎

Theorem 3.2. *Let $r < \frac{1}{3}$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying the inequality (3.1). Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that*

$$\| f(x) - L(x) \|_Y \le \frac{2^r}{2 - 2^{3r}} \, \epsilon \, \| x \|_X^{3r}. \quad (3.5)$$

Proof. From (3.1) with $y = x$, $z = -2x$, we get

$$\| f(x) + \frac{1}{2} f(-2x) \|_Y \le 2^{r-1} \epsilon \| x \|_X^{3r}. \tag{3.6}$$

Hence, we get by (3.4) and (3.6) that

$$\begin{aligned} \| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \|_Y &\le \sum_{j=l}^{m-1} \| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \|_Y \\ \le \sum_{j=l}^{m-1} \left[\| \frac{1}{2^j} f(2^j x) + \frac{1}{2^{j+1}} f(-2^{j+1} x) \|_Y \right] &\le \sum_{j=l}^{m-1} \frac{2^r}{2^{j+1}} \epsilon \| 2^j x \|_X^{3r} \\ \le \sum_{j=l}^{m-1} 2^{r-1} \left(\frac{2^{3r}}{2} \right)^j \epsilon \| x \|_X^{3r}, \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$.

Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So we can define the mapping $L : X \to Y$ by $L(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$, for all $x \in X$.

Moreover, by letting $l = 0$ and passing the limit $m \to \infty$, we get (3.5). The rest is similar to the proof of Theorem 2.1. ■

Theorem 3.3. *Let $r > \frac{1}{3}$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\| f(x) + f(y) + f(z) \|_Y \le \| f(x + y + z) \|_Y + \epsilon(\| x \|_X^r \cdot \| y \|_X^r \cdot \| z \|_X^r), \tag{3.7}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that

$$\| f(x) - L(x) \|_Y \le \frac{2^r}{2^{3r} - 2} \epsilon \| x \|_X^{3r}. \tag{3.8}$$

Proof. One can easily check $\| 3f(0) \|_Y \le \| f(0) \|_Y$ which implies $\| f(0) \|_Y = 0 = f(0)$. Also, by letting $y = x$ and $z = -2x$ in (3.8) we get

$$\| 2f(x) + f(-2x) \|_Y \le 2^r \epsilon \| x \|_X^{3r}, \quad \text{for} \quad \text{all } x \in X, \tag{3.9}$$

which implies by replacing x as $\frac{x}{2}$ that

$$\| 2f(\frac{x}{2}) + f(-x) \|_Y \le \frac{1}{2^{2r}} \epsilon \| x \|_X^{3r}.$$

Next, by letting $y = -x$ and $z = 0$, we have $f(-x) = -f(x)$. The rest is similar to the proof of Theorem 3.1. ■

Theorem 3.4. *Let $r < \frac{1}{3}$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying the inequality (3.7). Then there exists a unique Cauchy additive mapping $L : X \to Y$ such that*

$$\| f(x) - L(x) \|_Y \le \frac{2^r}{2 - 2^{3r}} \epsilon \| x \|_X^{3r}, \quad \textit{for all } x \in X. \tag{3.10}$$

Proof. we get from (3.9) that

$$\| f(x) + \frac{1}{2} f(-2x) \|_{Y} \leq 2^{r-1} \epsilon \| x \|_X^{3r}.$$

The rest is similar to the proof of Theorem 3.2. ∎

Theorem 3.5. *Let $r > \frac{1}{3}$ and ϵ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\| f(x) + f(y) + 2f(z) \|_Y \leq \| 2f(\frac{x+y}{2} + z) \|_{Y} + \epsilon(\| x \|_X^r \cdot \| y \|_X^r \cdot \| z \|_X^r), \quad (3.11)$$

for all $x, y, z \in X$. Then the mapping $f : X \to Y$ is a Cauchy additive mapping.

Proof. One can easily get $f(0) = 0$ by letting $x = y = z = 0$ in (3.11). Also, by letting $x = 2x$, $z = -x$ and $y = 0$ in (3.11), we get

$$\| f(2x) + 2f(-x) \|_Y = 0. \quad (3.12)$$

Next, by letting $y = -x$ and $z = 0$ in (3.11), we get

$$\| f(x) + f(-x) \|_Y = 0, \quad f(-x) = -f(x). \quad (3.13)$$

Thus, by (3.12) and (3.13) we obtain

$$f(2x) = 2f(x), \quad f(x) = 2f(\frac{x}{2}), \quad f(x) = 2^n f(\frac{x}{2^n}), \quad (3.14)$$

for all $n \in N$ and $x \in X$. Since $f(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ we obtain by (3.11),(3.12)

$$\begin{aligned}
&\| f(x) + f(y) - f(x+y) \|_Y = \lim_{n\to\infty} 2^n \| f(\frac{x}{2^n}) + f(\frac{y}{2^n}) - f(\frac{x+y}{2^n}) \|_{Y} \\
&\leq \lim_{n\to\infty} 2^n \left[\| f(\frac{x}{2^n}) + f(\frac{y}{2^n}) + 2f(\frac{-x-y}{2^{n+1}}) \|_{Y} + \| 2f(\frac{-x-y}{2^{n+1}}) + f(\frac{x+y}{2^n}) \|_{Y} \right] \\
&\leq \lim_{n\to\infty} \left(\frac{2}{2^{3r}}\right)^n \epsilon \left(\| x \|_X^r \cdot \| y \|_X^r \cdot \frac{\| x+y \|_X^r}{2^r} \right) = 0,
\end{aligned}$$

for $r > \frac{1}{3}$. Thus $f(x+y) = f(x) + f(y)$. ∎

Theorem 3.6. *Let $r < \frac{1}{3}$ and $f : X \to Y$ be a mapping satisfying (3.11). Then the mapping $f : X \to Y$ is a Cauchy additive mapping.*

Proof. By a similar method to the proof of Theorem 3.5, we get

$$\| f(2x) + 2f(-x) \|_Y = 0, \quad f(-x) = -\frac{1}{2} f(2x)$$

and

$$\| f(x) + f(-x) \|_Y = 0, \quad f(-x) = -f(x).$$

Thus we obtain

$$f(x) = \frac{1}{2} f(2x) = \frac{1}{2^2} f(2^2 x) = ... = \frac{1}{2^n} f(2^n x)...,$$

for all $n \in N$ and $x \in X$. So we have $f(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$. Hence, by a similar method to the proof of Theorem 3.5, we obtain

$$\| f(x) + f(y) - f(x+y) \|_Y = \lim_{n\to\infty} \frac{1}{2^n} \| f(2^n x) + f(2^n y) - f(2^n(x+y)) \|_Y$$
$$\leq \lim_{n\to\infty} \left(\frac{2^{3r}}{2}\right)^n \left[\| x \|_X^r \cdot \| y \|_X^r \cdot \frac{\| x+y \|_X^r}{2^r} \right] = 0,$$

for $r < \frac{1}{3}$. Therefore $f(x+y) = f(x) + f(y)$. ■

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