# Convergence of difference analogues to the Darboux problem with functional dependence

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**Abstract**

We consider the Darboux problem with functional dependence for $z$, $D_x z$ and $D_y z$ on the right-hand side of the differential equation. We investigate a wide class of difference schemes for the differential-functional problem. In the present paper we prove convergence theorems by means of consistency and stability statements.

## 1 Introduction

Take $a$, $b > 0$ and $\alpha$, $\beta \geq 0$. Define $E = [0, a] \times [0, b]$, $E^0 = [-\alpha, a] \times [-\beta, b] \backslash (0, a] \times (0, b]$, and $B = [-\alpha, 0] \times [-\beta, 0]$ Given a function $z : E^0 \cup E \to R$ and a point $(x, y) \in E$, we define the functional $z_{(x,y)} : B \to R$ by $z_{(x,y)}(\xi, \eta) = z(x+\xi, y+\eta)$ for $(\xi, \eta) \in B$. Suppose that we are given a function $f : \Omega := E \times X_0 \times X_1 \times X_2 \to R$, where $X_0, X_1, X_2$ are some subsets of the set of all functions from $B$ to $R$. Take a differentiable function $\phi : E^0 \to R$. Consider the Darboux problem

$$D_{xy} z(x, y) = f\Big(x, y, z_{(x,y)}, (D_x z)_{(x,y)}, (D_y z)_{(x,y)}\Big), \tag{1}$$

$$z(x, y) = \phi(x, y), \qquad (x, y) \in E^0. \tag{2}$$

We will assume that there exists a classical solution to problem (1), (2), i.e. a continuous function $v : E^0 \cup E \to R$ which satisfies (2) on $E^0$, is of class $C^2$ on $E$, and satisfies (1) on the set $E$.

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Fix a constant $c \geq 1$. Define the set of acceptable steps

$$I_d = \left\{ (h,k) \in (0,a] \times (0,b] \;\middle|\; \frac{\alpha}{h}, \frac{\beta}{k} \in \mathcal{N}_0, \quad k\frac{1}{c} \leq h \leq k\,c \right\},$$

where the symbol $\mathcal{N}_0$ denotes all natural numbers including 0. We write $x_i = ih$ and $y_j = jk$. Take $Z_{hk} = \Big\{(x_i, y_j) \mid\; i,j \in \mathcal{Z}\Big\}$. Denote by $E^0_{hk}$ the set of all $(x_i, y_j) \in E^0 \cap Z_{hk}$. Denote $E^+_{hk} = (0,a] \times (0,b] \cap Z_{hk}$. Define

$$\begin{aligned} E_{hk} &= \Big\{(x_i, y_j) \in Z_{hk} \,|\, (x_{i+1}, y_{j+1}) \in E^+_{hk}\Big\} \\ \tilde{E}_{hk} &= (E^0 \cup E) \cap Z_{hk}, \qquad B_{hk} = B \cap Z_{hk}. \end{aligned}$$

If $z : \tilde{E}_{hk} \to R$, we denote $z^{(i,j)} = z(x_i, y_j)$. Let $\mathcal{F}(X,Y)$ be the set of all functions from a set $X$ into $Y$.

We will need some difference operators $\delta_1, \delta_2, \delta_{12}$ which correspond to the derivatives $D_x, D_y, D_{xy}$, respectively. We define these operators as follows

$$\begin{aligned} \delta_1 z^{(i,j)} &= \frac{z^{(i+1,j)} - z^{(i,j)}}{h}, \quad (x_i, y_j), (x_{i+1}, y_j) \in \tilde{E}_{hk}, \\ \delta_2 z^{(i,j)} &= \frac{z^{(i,j+1)} - z^{(i,j)}}{k}, \quad (x_i, y_j), (x_i, y_{j+1}) \in \tilde{E}_{hk}, \\ \delta_{12} z^{(i,j)} &= \frac{z^{(i+1,j+1)} - z^{(i+1,j)} - z^{(i,j+1)} + z^{(i,j)}}{hk}, \quad (x_i, y_j), (x_{i+1}, y_{j+1}) \in \tilde{E}_{hk}, \end{aligned}$$

for $z \in \mathcal{F}(\tilde{E}_{hk}, R)$.

Define also a discrete counterpart of $z_{(x,y)}$. If $z \in \mathcal{F}(\tilde{E}_{hk}, R)$ and $(x_i, y_j) \in E_{hk}$, then we define the function $z_{[i,j]} \in \mathcal{F}(B_{hk}, R)$ by $z_{[i,j]}(x_\mu, y_\nu) = z^{(i+\mu, j+\nu)}$ for $(x_\mu, y_\nu) \in B_{hk}$.

Suppose that $f_{hk} : \Omega_{hk} := E_{hk} \times \mathcal{F}(B_{hk}, R^3) \to R$ and $\phi_{hk} : E^0_{hk} \to R$. We consider the difference scheme in correspondence with differential-functional problem (1), (2).

$$\delta_{12} z^{(i,j)} = f_{hk}\Big(x_i, y_j, z_{[i,j]}, (\delta_1 z)_{[i,j]}, (\delta_2 z)_{[i,j]}\Big) \qquad (x_i, y_j) \in E_{hk}, \tag{3}$$

$$z^{(i,j)} = \phi^{(i,j)}_{hk}, \qquad (x_i, y_j) \in E^0_{hk}. \tag{4}$$

In [KL] we investigate the Darboux problem without partial derivatives in the right-hand side. We develop there a general theory of convergence under some relatively weak assumptions of non-linear Perron-type estimates of the right-hand-side function. This theory corresponds with the existence and uniqueness theory for hyperbolic equations in this way that typical integral forms of the Darboux problem and their basic properties are reflected on the ground of difference schemes by also very natural inverse summation formulas from which one can deduce a priori bounds for all discrete solutions and their errors, that is: the differences between the solutions to the difference scheme and the solution to differential problem restricted to the mesh. In the above mentioned item of the literature as well as in [L2] and [L3], we can find a standard way of dealing with convergence theorems, namely:

if the difference scheme is consistent with the differential or differential-functional problem (at least on a class of solutions which are sufficiently regular) and if it is stable (not too sensitive with respect to perturbations of the right-hand sides and the initial data), then the solutions to difference scheme converge to the solution of the differential problem provided it exists and it is unique. The paper [L3] is devouted to a class of finite difference approximations to parabolic problems and the convergence is obtained by some realization of discrete maximum principle. In [L2] we consider a strongly coupled hyperbolic system of first-order equations whose difference analogues are proved to converge due to a recurrence comparison formula for a properly transformed error equation. Considering difference schemes for the Darboux problem with functional dependence, we begin our analysis of error equations by means of an inverse formula, which is very similar to getting an integral fixed point equation for differential-functional problem (1), (2).

## 2 Main examples

We illustrate in the present section how to specify the above difference operators and how to produce a new right-hand side in the difference scheme on the ground of the function $f$. Finally, we give two very common types of functional dependence which could be easily specified from (1).

**Example 1.** Suppose that we are given three interpolation operators $I_0, I_1, I_2 : \mathcal{F}(B_{h,k}, R) \to C(B, R)$. We can define the discrete counterpart of the function $f$ in the following way

$$f_{hk}\Big(x_i, y_j, w_0, w_1, w_2\Big) = f\Big(x_i, y_j, I_0 w_0, I_1 w_1, I_2 w_2\Big) \tag{5}$$

for $(x_i, y_j) \in B_{hk}$ and $w_\nu \in \mathcal{F}(B_{hk}, R)$, $(\nu = 0,1,2)$. If there is no functional dependence, i.e. if $f(x, y, w_0, w_1, w_2) = \tilde{f}(x, y, w_0(0,0), w_1(0,0), w_2(0,0))$ for some function $\tilde{f} : E \times R^3 \to R$ and for all $(x, y, w_0, w_1, w_2) \in \Omega$, then we can put simply $(I_\nu w_\nu)(x, y) = w_\nu(0,0)$ in formula (5), and we have the difference scheme

$$\delta_{12} z^{(i,j)} = \tilde{f}\Big(x_i, y_j, z^{(i,j)}, \delta_1 z^{(i,j)}, \delta_2 z^{(i,j)}\Big). \tag{6}$$

It is seen that in that case the function $f_{hk}$ coincides with the function $f$ restricted to the mesh.

**Example 2.** Suppose that we are given the same operators $I_0, I_1, I_2$ as in Example 1. Take another interpolation operator $\tilde{I} : C(B, R) \to C(B'(h, k), R)$, where

$$B'(h,k) = \Big\{(x+\xi, y+\eta)\Big|\ (x,y) \in B,\ (\xi,\eta) \in [0, h/2] \times [0, k/2]\Big\}.$$

Define the function $F_{h,k} : \Omega \to R$ by

$$\begin{aligned} F_{hk}(P) = f(P) + & \\ \frac{h}{2} D_x f(P) + \frac{k}{2} D_y f(P) + D_{w_0} f(P)\Big((\tilde{I} w_0)_{(h/2,k/2)} - w_0\Big) + & \\ D_{w_1} f(P)\Big((\tilde{I} w_1)_{(h/2,k/2)} - w_1\Big) + D_{w_2} f(P)\Big((\tilde{I} w_2)_{(h/2,k/2)} - w_2\Big) & \end{aligned} \tag{7}$$

for $P = (x, y, w_0, w_1, w_2) \in \Omega$. Now, instead of formula (5), we write

$$f_{hk}\Big(x_i, y_j, w_0, w_1, w_2\Big) = F_{hk}\Big(x_i, y_j, I_0 w_0, I_1 w_1, I_2 w_2\Big) \tag{8}$$

for $(x_i, y_j) \in E_{hk}$ and $w_\nu \in \mathcal{F}(B_{hk}, R)$, $(\nu = 0, 1, 2)$. It is seen that the function $F_{hk}(x, y, \dots)$ approximates the value $f(x+h/2, y+k/2, \dots)$ for it is derived from the Taylor formula of second order, possible when the function $f$ is sufficiently regular. This difference scheme, as such, is to approximate the solution to the differential-functional problem much better than the scheme in Example 1.

**Example 3.** The interpolation operators $I_0, I_1, I_2$ in Examples 1 and 2 should provide a relevant approximation for sufficiently regular functions. One such example is the spline interpolation: we define $I_0 = I_1 = I_2$ by

$$\begin{aligned}(I_\nu w_\nu)(s,t) = \\ & w_\nu^{(i,j)} \left(1 - \frac{s - x_i}{h}\right)\left(1 - \frac{t - y_j}{k}\right) + w_\nu^{(i+1,j+1)} \frac{s - x_i}{h} \frac{t - y_j}{k} + \\ & w_\nu^{(i+1,j)} \frac{s - x_i}{h}\left(1 - \frac{t - y_j}{k}\right) + w_\nu^{(i,j+1)}\left(1 - \frac{s - x_i}{h}\right)\frac{t - y_j}{k}\end{aligned} \tag{9}$$

for $\nu = 0, 1, 2$ and for $(s,t) \in B$, $(x_i, y_j) \in B_{hk}$ such that $x_i \le s \le x_{i+1}$ and $y_j \le t \le y_{j+1}$.

**Example 4.** Besides typical examples such as the Darboux problems for equations without functional dependence such as

$$D_{xy}z(x,y) = F(x, y, z(x,y), D_x z(x,y), D_y z(x,y)),$$

we can find in equation (1) some equations with deviations and with the Volterra-type integral dependence. Suppose that $F : E \times R^3 \to R$, $\omega_0, \omega_1, \omega_2 : E \to E_0 \cup E$ and $G_0, G_1, G_2 : B \times R \to R$. Assume that these functions are continuous and that the functions $\omega_\nu$ for $\nu = 0, 1, 2$ satisfy the condition

$$(x - \alpha, y - \beta) \le \omega_\nu(x,y) \le (x,y) \text{ for } (x,y) \in E.$$

Consider the equations

$$D_{xy}z(x,y) = F\Big(x, y, z(\omega_0(x,y)), D_x z(\omega_1(x,y)), D_y z(\omega_2(x,y))\Big) \tag{10}$$

and

$$\begin{aligned}D_{xy}z(x,y) = \\ & F\left(x, y, \int_B G_0(s, t, z(s+x, t+y))\, dt\, ds,\right. \\ & \left.\int_B G_1(s, t, D_x z(s+x, t+y))\, dt\, ds, \int_B G_2(s, t, D_y z(s+x, t+y))\, dt\, ds\right).\end{aligned} \tag{11}$$

The first (deviated) equation can be specified from equation (1) when we substitute

$$\begin{aligned}f(x, y, w_0, w_1, w_2) = \\ & F\Big(x, y, w_0(\omega_0(x,y) - (x,y)), w_1(\omega_1(x,y) - (x,y)), w_2(\omega_2(x,y) - (x,y))\Big)\end{aligned}$$

for $(x, y, w_0, w_1, w_2) \in \Omega$. Indeed, we have then the equality

$$\begin{aligned} & f\Big(x, y, z_{(x,y)}, (D_x z)_{(x,y)}, (D_y z)_{(x,y)}\Big) = \\ & \quad F\Big(x, y, z((\omega_0(x, y) - (x, y)) + (x, y)), \\ & \qquad D_x z((\omega_1(x, y) - (x, y)) + (x, y)), D_y z((\omega_2(x, y) - (x, y)) + (x, y))\Big) \end{aligned}$$

The differential-integral equation can be obtained from equation (1) when we define the function $f$ on $\Omega$ by

$$\begin{aligned} & f(x, y, w_0, w_1, w_2) = \\ & \quad F\left(x, y, \int_B G_0(s, t, w_0(s, t))\, dt\, ds,\right. \\ & \qquad \left. \int_B G_1(s, t, w_1(s, t))\, dt\, ds, \int_B G_2(s, t, w_2(s, t))\, dt\, ds\right) \end{aligned}$$

The explanation is similar to that in the former case. Changing $w_\nu$ into its discrete counterpart $I_\nu w_\nu$ (cf. (5)) leads to particular quadratures for the above integrals.

## 3 Notations and assumptions

Define the discrete operators $\mathcal{L}_0 : \mathcal{F}(\tilde{E}_{hk}, R) \to \mathcal{F}(E^+_{hk}, R)$ and

$$\mathcal{L}_1 : \mathcal{F}(\tilde{E}_{hk}, R) \to \mathcal{F}((0, k) + E_{hk}, R),\ \ \mathcal{L}_2 : \mathcal{F}(\tilde{E}_{hk}, R) \to \mathcal{F}((h, 0) + E_{hk}, R).$$

Given a function $z \in \mathcal{F}(\tilde{E}_{hk}, R)$, we define

$$\begin{aligned} \mathcal{L}_0 z^{(i,j)} &= z^{(i,0)} + z^{(0,j)} - z^{(0,0)} + \\ &\quad hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} f_{hk}\Big(x_\mu, y_\nu, z_{[\mu,\nu]}, (\delta_1 z)_{[\mu,\nu]}, (\delta_2 z)_{[\mu,\nu]}\Big), \\ \mathcal{L}_1 z^{(i,j)} &= \delta_1 z^{(i,0)} + k \sum_{\nu=0}^{j-1} f_{hk}\Big(x_i, y_\nu, z_{[i,\nu]}, (\delta_1 z)_{[i,\nu]}, (\delta_2 z)_{[i,\nu]}\Big), \\ \mathcal{L}_2 z^{(i,j)} &= \delta_2 z^{(0,j)} + h \sum_{\mu=0}^{i-1} f_{hk}\Big(x_\mu, y_j, z_{[\mu,j]}, (\delta_1 z)_{[\mu,j]}, (\delta_2 z)_{[\mu,j]}\Big), \end{aligned} \tag{12}$$

for $(x_i, y_j) \in E^+_{hk}$, $(0, k) + E_{hk}$, $(h, 0) + E_{hk}$, respectively. Observe that if the function $z \in \mathcal{F}(\tilde{E}_{hk}, R)$ is a solution to equation (3), then we have

$$\begin{aligned} z^{(i,j)} &= \mathcal{L}_0 z^{(i,j)} && \Big((x_i, y_j) \in E^+_{hk}\Big), \\ \delta_1 z^{(i,j)} &= \mathcal{L}_1 z^{(i,j)} && \Big((x_i, y_j) \in (0, k) + E_{hk}\Big), \\ \delta_2 z^{(i,j)} &= \mathcal{L}_2 z^{(i,j)} && \Big((x_i, y_j) \in (h, 0) + E_{hk}\Big). \end{aligned} \tag{13}$$

A function $\gamma : I_d \to R_+$ is said to be of the class $\Gamma_0$ if $\lim_{(h,k)\to(0,0)} \gamma(h, k) = 0$. We introduce assumptions which will guarantee consistency and stability of the difference scheme.

**Assumption 1** *Suppose that there is a function $v \in C(E^0 \cup E, R)$ which satisfies (1), (2) and the function $v_{|_{E^0}} = \phi$ is of class $C^2$, the function $v_{|_E}$ is of class $C^3$. Of the function $v$ satisfying these conditions we will say that it is of class $C^{2,3}$.*

**Assumption 2** *Suppose that there is $\gamma_0[w] \in \Gamma_0$ such that*

$$\Big| f_{hk}\Big(x_i, y_j, (w_0)_{|_{B_{hk}}}, (w_1)_{|_{B_{hk}}}, (w_2)_{|_{B_{hk}}}\Big) - f(x_i, y_j, w_0, w_1, w_2)\Big| \le \gamma_0[w](h,k)$$

*for $(x_i, y_j, w_0, w_1, w_2) \in \Omega$, where $w = (w_0, w_1, w_2)$.*

**Assumption 3** *Suppose that there are constants $L_0, L_1, L_2 \in R_+$ (independent of $(h,k)$) such that*

$$\begin{gathered} |f_{hk}(x_i, y_j, w_0, w_1, w_2) - f_{hk}(x_i, y_j, \bar{w}_0, \bar{w}_1, \bar{w}_2)| \le \\ L_0\|w_0 - \bar{w}_0\| + L_1\|w_1 - \bar{w}_1\| + L_2\|w_2 - \bar{w}_2\| \end{gathered}$$

*for $(x_i, y_j, w_0, w_1, w_2), (x_i, y_j, \bar{w}_0, \bar{w}_1, \bar{w}_2) \in \Omega_{hk}$.*

If $z \in \mathcal{F}(\tilde{E}_{hk}, R)$, then we will denote by $\xi_{hk}^{(i,j)}[z]$ $((x_i, y_j) \in E_{hk})$ the following residual expression

$$\xi_{hk}^{(i,j)}[z] = \delta_{12} z^{(i,j)} - f_{hk}\Big(x_i, y_j, z_{[i,j]}, (\delta_1 z)_{[i,j]}, (\delta_2 z)_{[i,j]}\Big). \tag{14}$$

## 4 Lemmas on consistency and stability

We start this section with a lemma on the consistency of the difference scheme with the differential-functional problem.

**Lemma 1** *Suppose that Assumptions 1, 2 and 3 are satisfied. Then the function $\gamma(h,k) = \max_{i,j} |\xi_{hk}^{(i,j)}[v]|$ is of class $\Gamma_0$.*

*Proof.* Since the function $v$ is of class $C^{2,3}$ we can expand $\delta_{12} v^{(i,j)}$ in the Taylor power series with the error of the third order. Then we obtain the estimate

$$|\delta_{12} v^{(i,j)} - D_{xy} v(x_i, y_j)| \le \|(h,k)\| \sum_{\mu,\nu \ge 0;\, \mu+\nu \le 3} \|D_x^\mu D_y^\nu v\| \tag{15}$$

for $(x_i, y_j) \in E_{h,k}$. Moreover, we can get

$$\begin{aligned} &\|(\delta_1 (v_{|_{\tilde{E}_{hk}}})_{[i,j]} - (D_x v)_{|_{\tilde{E}_{hk}}})_{[i,j]}\| = \\ &\qquad \max_{(x_\mu, y_\nu) \in B_{hk}} |\delta_1 v^{(i+\mu, j+\nu)} - D_x v(x_{i+\mu}, y_{j+\nu}| \le h\|D_{xx} v\|, \\ &\|(\delta_2 (v_{|_{\tilde{E}_{hk}}})_{[i,j]} - (D_y v)_{|_{\tilde{E}_{hk}}})_{[i,j]}\| = \\ &\qquad \max_{(x_\mu, y_\nu) \in B_{hk}} |\delta_2 v^{(i+\mu, j+\nu)} - D_y v(x_{i+\mu}, y_{j+\nu}| \le k\|D_{yy} v\|. \end{aligned} \tag{16}$$

Consequently, we get

$$\begin{gathered} |\xi_{hk}^{(i,j)}[v]| \le \|(h,k)\| \sum_{\mu,\nu \ge 0;\, \mu+\nu \le 3} \|D_x^\mu D_y^\nu v\| + \\ \gamma_0[v, \delta_1 v, \delta_2 v](h,k) + L_1 h\|D_{xx} v\| + L_2 k\|D_{yy} v\| \end{gathered} \tag{17}$$

for $(x_i, y_j) \in E_{hk}$, where $\gamma_0[v, \delta_1 v, \delta_2 v] \in \Gamma_0$ is taken out of Assumption 2. This completes the proof.

We formulate a lemma on stability of the difference scheme.

**Lemma 2** *Suppose that Assumption 3 is satisfied. Take $z, \bar{z} \in \mathcal{F}(\tilde{E}_{hk}, R)$; the function $z$ satisfying (3), (4), and the function $\bar{z}$ satisfying the inequalities*

$$
\begin{aligned}
|\xi_{hk}^{(i,j)}[\bar{z}]| &\leq \bar{\gamma}(h,k) \quad \text{for} \ \ (x_i, y_j) \in E_{hk}, \\
|z^{(i,j)} - \bar{z}^{(i,j)}| &\leq \bar{\gamma}_0(h,k) \quad \text{for} \ \ (x_i, y_j) \in E_{hk}^0, \\
|\delta_1 z^{(i,j)} - \delta_1 \bar{z}^{(i,j)}| &\leq \bar{\gamma}_1(h,k) \quad \text{for} \ \ (x_i, y_j), (x_{i+1}, y_j) \in E_{hk}^0, \\
|\delta_2 z^{(i,j)} - \delta_2 \bar{z}^{(i,j)}| &\leq \bar{\gamma}_2(h,k) \quad \text{for} \ \ (x_i, y_j), (x_i, y_{j+1}) \in E_{hk}^0,
\end{aligned}
\tag{18}
$$

*where $\bar{\gamma}, \bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2 \in \Gamma_0$. Then we have*

$$
\begin{aligned}
|z^{(i,j)} - \bar{z}^{(i,j)}| &\leq W_0(x_i, y_j) \quad \text{for} \ \ (x_i, y_j) \in \tilde{E}_{hk}, \\
|\delta_1 z^{(i,j)} - \delta_1 \bar{z}^{(i,j)}| &\leq W_1(x_i, y_j) \quad \text{for} \ \ (x_i, y_j), (x_{i+1}, y_j) \in \tilde{E}_{hk}, \\
|\delta_2 z^{(i,j)} - \delta_2 \bar{z}^{(i,j)}| &\leq W_2(x_i, y_j) \quad \text{for} \ \ (x_i, y_j), (x_i, y_{j+1}) \in \tilde{E}_{hk},
\end{aligned}
$$

*where the functions $W_0, W_1, W_2 : E_0 \cup E \to R$ are defined by*

$$
\begin{aligned}
W_0(x,y) &= \begin{cases} 3\bar{\gamma}_0(h,k) + xy\bar{\gamma}(h,k) + \int_0^x \int_0^y \tilde{W}(s,t)\,dt\,ds & \text{on } E, \\ W_0(\max\{x,0\}, \max\{y,0\}) & \text{on } E_0, \end{cases} \\
W_1(x,y) &= \begin{cases} \bar{\gamma}_1(h,k) + y\bar{\gamma}(h,k) + \int_0^y \tilde{W}(x,t)\,dt & \text{on } E, \\ W_1(\max\{x,0\}, \max\{y,0\}) & \text{on } E_0, \end{cases} \\
W_2(x,y) &= \begin{cases} \bar{\gamma}_2(h,k) + x\bar{\gamma}(h,k) + \int_0^x \tilde{W}(s,y)\,ds & \text{on } E, \\ W_2(\max\{x,0\}, \max\{y,0\}) & \text{on } E_0, \end{cases}
\end{aligned}
\tag{19}
$$

*and the function $\tilde{W} : E \to R_+$ is a unique solution to the Darboux problem*

$$
\begin{aligned}
&D_{xy} z(x,y) = L_0 z(x,y) + L_1 D_x z(x,y) + L_2 D_y z(x,y), \\
&\begin{cases} z(0,y) = \Big(3L_0\bar{\gamma}_0 + L_1\bar{\gamma}_1 + L_2\bar{\gamma}_2 + \bar{\gamma}(yL_1 + 1)\Big)\frac{e^{yL_1}-1}{L_1}, & \text{for } y \in [0,b], \\ z(x,0) = \Big(3L_0\bar{\gamma}_0 + L_1\bar{\gamma}_1 + L_2\bar{\gamma}_2 + \bar{\gamma}(xL_2 + 1)\Big)\frac{e^{xL_2}-1}{L_2}, & \text{for } x \in [0,a], \end{cases}
\end{aligned}
\tag{20}
$$

*where $\bar{\gamma}_\nu = \bar{\gamma}_\nu(h,k)$ for $\nu = 0, 1, 2$, and $\bar{\gamma} = \bar{\gamma}(h,k)$.*

*Proof.* In view of formula (12) we deduce for $(x_i, y_j) \in E^+_{hk}$ the estimate

$$
\begin{aligned}
&|z^{(i,j)} - \bar{z}^{(i,j)}| \leq \\
&\quad |\mathcal{L}_0 z^{(i,j)} - \mathcal{L}_0 \bar{z}^{(i,j)}| + hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} |\xi_{hk}^{(\mu,\nu)}[\bar{z}]| \leq \\
&\quad 3\bar{\gamma}_0(h,k) + x_i y_j \bar{\gamma}(h,k) + \\
&\quad hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \Big( L_0 \|(z-\bar{z})_{[\mu,\nu]}\| + L_1 \|(\delta_1(z-\bar{z}))_{[\mu,\nu]}\| + L_2 \|(\delta_2(z-\bar{z}))_{[\mu,\nu]}\| \Big) \leq \\
&\quad 3\bar{\gamma}_0(h,k) + x_i y_j \bar{\gamma}(h,k) + \\
&\quad hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \Big( L_0 W_0(x_\mu, y_\nu) + L_1 W_1(x_\mu, y_\nu) + L_2 W_2(x_\mu, y_\nu) \Big) \leq \\
&\quad 3\bar{\gamma}_0(h,k) + x_i y_j \bar{\gamma}(h,k) + \\
&\quad \int_0^{x_i} \int_0^{y_j} \Big( L_0 W_0(s,t) + L_1 W_1(s,t) + L_2 W_2(s,t) \Big)\, dt\, ds \leq \\
&\quad W_0(x_i . y_j)
\end{aligned}
$$

Now, we take $(x_i, y_j) \in (0,k) + E_{hk}$ and derive the estimate

$$
\begin{aligned}
&|\delta_1 z^{(i,j)} - \delta_1 \bar{z}^{(i,j)}| \leq \\
&\quad |\mathcal{L}_1 z^{(i,j)} - \mathcal{L}_1 \bar{z}^{(i,j)}| + k \sum_{\nu=0}^{j-1} |\xi_{hk}^{(i,\nu)}[\bar{z}]| \leq \\
&\quad \bar{\gamma}_1(h,k) + y_j \bar{\gamma}(h,k) + \\
&\quad k \sum_{\nu=0}^{j-1} \Big( L_0 \|(z-\bar{z})_{[i,\nu]}\| + L_1 \|(\delta_1(z-\bar{z}))_{[i,\nu]}\| + L_2 \|(\delta_2(z-\bar{z}))_{[i,\nu]}\| \Big) \leq \\
&\quad \bar{\gamma}_1(h,k) + y_j \bar{\gamma}(h,k) + \\
&\quad k \sum_{\nu=0}^{j-1} \Big( L_0 W_0(x_i, y_\nu) + L_1 W_1(x_i, y_\nu) + L_2 W_2(x_i, y_\nu) \Big) \leq \\
&\quad \bar{\gamma}_1(h,k) + y_j \bar{\gamma}(h,k) + \\
&\quad \int_0^{y_j} \Big( L_0 W_0(x_i,t) + L_1 W_1(x_i,t) + L_2 W_2(x_i,t) \Big)\, dt \leq \\
&\quad W_1(x_i . y_j).
\end{aligned}
$$

Taking $(x_i, y_j) \in (h, 0) + E_{hk}$, we derive the estimate

$$
\begin{aligned}
&|\delta_2 z^{(i,j)} - \delta_2 \bar{z}^{(i,j)}| \leq \\
&\quad |\mathcal{L}_2 z^{(i,j)} - \mathcal{L}_2 \bar{z}^{(i,j)}| + k \sum_{\mu=0}^{i-1} |\xi_{hk}^{(\mu,j)}[\bar{z}]| \leq \\
&\quad \bar{\gamma}_2(h,k) + x_i \bar{\gamma}(h,k) + \\
&\quad h \sum_{\mu=0}^{i-1} \Big( L_0 \|(z - \bar{z})_{[\mu,j]}\| + L_1 \|(\delta_1(z - \bar{z}))_{[\mu,j]}\| + L_2 \|(\delta_2(z - \bar{z}))_{[\mu,j]}\| \Big) \leq \\
&\quad \bar{\gamma}_2(h,k) + x_i \bar{\gamma}(h,k) + \\
&\quad k \sum_{\mu=0}^{i-1} \Big( L_0 W_0(x_\mu, y_j) + L_1 W_1(x_\mu, y_j) + L_2 W_2(x_\mu, y_j) \Big) \leq \\
&\quad \bar{\gamma}_2(h,k) + x_i \bar{\gamma}(h,k) + \\
&\quad \int_0^{x_i} \Big( L_0 W_0(s, y_j) + L_1 W_1(s, y_j) + L_2 W_2(s, y_j)\, ds \Big) \leq \\
&\quad W_2(x_i . y_j).
\end{aligned}
$$

These estimates establish the assertion of our lemma, which finishes the proof.

## 5 The main result - convergence theorem

Our convergence result is based on consistency and stability. The main theorem will be followed by some efficient error estimates.

**Theorem 1** *Suppose that Assumptions 1, 2 and 3 are satisfied. Assume that the function $z \in \mathcal{F}(\tilde{E}_{hk}, R)$ is a solution to problem (3), (4) satisfying within $E_{hk}^0$ the inequalities*

$$
\begin{aligned}
|\phi_{hk}^{(i,j)} - \phi(x_i, y_j)| &\leq \bar{\gamma}_0(h,k) \quad \textit{for} \ \ (x_i, y_j) \in E_{hk}^0, \\
|\delta_1 \phi_{hk}^{(i,j)} - \delta_1 \bar{\phi}^{(i,j)}| &\leq \bar{\gamma}_1(h,k) \quad \textit{for} \ \ (x_i, y_j), (x_{i+1}, y_j) \in E_{hk}^0, \\
|\delta_2 \phi_{hk}^{(i,j)} - \delta_2 \bar{\phi}^{(i,j)}| &\leq \bar{\gamma}_2(h,k) \quad \textit{for} \ \ (x_i, y_j), (x_i, y_{j+1}) \in E_{hk}^0,
\end{aligned} \tag{21}
$$

*where $\bar{\gamma}, \bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2 \in \Gamma_0$. Then we have*

$$
|v^{(i,j)} - z^{(i,j)}|,\ |D_x v(x_i, y_j) - \delta_1 \bar{z}^{(i,j)}|,\ |D_y v(x_i, y_j) - \delta_2 \bar{z}^{(i,j)}| \to 0
$$

*as $(h,k) \to (0,0)$.*

*Proof.* It follows from Lemma 1 that the function $\gamma(h,k) = \max_{i,j} |\xi_{hk}^{(i,j)}[v]|$ is of class $\Gamma_0$. Define $\bar{\gamma} \in \Gamma_0$ as the right-hand side of inequality (17):

$$
\begin{aligned}
\bar{\gamma}(h,k) = \|(h,k)\| &\sum_{\mu,\nu \geq 0;\, \mu+\nu \leq 3} \|D_x^\mu D_y^\nu v\| + \\
&\gamma_0[v, \delta_1 v, \delta_2 v](h,k) + L_1 h \|D_{xx} v\| + L_2 k \|D_{yy} v\|.
\end{aligned} \tag{22}
$$

If we put $\bar{z} = v_{|_{\tilde{E}_{hk}}}$, then formulas (17), (22), (21) yield (18). It follows from Lemma 2 that

$$\begin{array}{rcll} |z^{(i,j)} - v^{(i,j)}| & \leq & W_0(x_i, y_j) & \text{for } \ (x_i, y_j) \in \tilde{E}_{hk}, \\ |\delta_1 z^{(i,j)} - \delta_1 v^{(i,j)}| & \leq & W_1(x_i, y_j) & \text{for } \ (x_i, y_j), (x_{i+1}, y_j) \in \tilde{E}_{hk}, \\ |\delta_2 z^{(i,j)} - \delta_2 v^{(i,j)}| & \leq & W_2(x_i, y_j) & \text{for } \ (x_i, y_j), (x_i, y_{j+1}) \in \tilde{E}_{hk}, \end{array}$$

where the functions $W_0, W_1, W_2 : E_0 \cup E \to R$ are defined by (19) with $\tilde{W}$ satisfying (20). Because of the continuous dependence on the initial data, we claim that the function $\tilde{W}$, and consequently the functions $W_0, W_1, W_2$ tend to 0 as $(h, k) \to 0$. Finally, we derive

$$\begin{aligned} & |D_x v(x_i, y_j) - \delta_1 z^{(i,j)}| \leq \\ & \quad |D_x v(x_i, y_j) - \delta_1 v^{(i,j)}| + |\delta_1 v^{(i,j)} - \delta_1 z^{(i,j)}| \leq h\|D_{xx} v\| + W_1(x_i, y_j) \\ & \quad \text{for } (x_i, y_j), (x_{i+1}, y_j) \in \tilde{E}_{hk}, \\ & |D_y v(x_i, y_j) - \delta_2 z^{(i,j)}| \leq \\ & \quad |D_y v(x_i, y_j) - \delta_2 v^{(i,j)}| + |\delta_2 v^{(i,j)} - \delta_2 z^{(i,j)}| \leq k\|D_{yy} v\| + W_2(x_i, y_j) \\ & \quad \text{for } (x_i, y_j), (x_i, y_{j+1}) \in \tilde{E}_{hk}. \end{aligned} \tag{23}$$

This completes the proof.

In order to give some explicite error estimates we will majorate it by means of the following lemma the function $\tilde{W}$ satisfying equation (20).

**Lemma 3** *Suppose that a function $z : E \to R$ satisfies the equation*

$$\begin{cases} D_{xy} z(x, y) = L_0 z(x, y) + L D_x z(x, y) + L D_y z(x, y), & \\ z(x, 0) = C(1 + x)(e^{Lx} - 1)/L, & \textit{for } x \in [0, a], \\ z(0, y) = C(1 + y)(e^{Ly} - 1)/L, & \textit{for } y \in [0, b], \end{cases} \tag{24}$$

*for some $C, L \in R_+$. Then we have*

$$z(x, y) \leq \frac{C}{L} e^{L(x+y)} \sum_{\nu=0}^{\infty} \frac{\big((L_0 + L^2)xy\big)^{\nu}}{(\nu!)^2} \leq \frac{C}{L} e^{L(x+y) + (L_0 + L^2)xy} \tag{25}$$

*for $(x, y) \in E$.*

*Proof.* Define the function $\tilde{z} : E \to R$ as $\tilde{z}(x, y) = e^{-L(x+y)} z(x, y)$ for $(x, y) \in E$. It is clear that the function $\tilde{z}$ satisfies the equation

$$D_{xy} \tilde{z}(x, y) = e^{-L(x+y)} z(x, y)(L_0 + L^2) = \tilde{z}(x, y)(L_0 + L^2) \tag{26}$$

and the initial condition

$$\begin{cases} \tilde{z}(x, 0) = C(1 + x)(1 - e^{-Lx})/L \leq C(1 + x)/L, & \text{for } x \in [0, a], \\ \tilde{z}(0, y) = C(1 + y)(1 - e^{-Ly})/L \leq C(1 + y)/L, & \text{for } y \in [0, b]. \end{cases} \tag{27}$$

If we solve the comparison problem with respect to problem (26), (27), then we obtain the estimate

$$\tilde{z}(x,y) \le \frac{C}{L}\sum_{\nu=0}^{\infty}\frac{\Big((L_0+L^2)xy\Big)^{\nu}}{(\nu!)^2}. \tag{28}$$

The remaining part of the proof is trivial.

**Corollary 1** *Suppose that the assumptions of Theorem 1 are satisfied. Then we have*

$$\begin{aligned}
&|v^{(i,j)} - z^{(i,j)}| \le \qquad (29)\\
&\qquad 3\bar{\gamma}_0(h,k) + x_i y_j \bar{\gamma}(h,k) + x_i y_j \,\frac{C(h,k)}{L}\, e^{(x_i+y_j)L + x_i y_j (L_0+L^2)},\\
&\qquad \text{for } (x_i,y_j) \in E^{+}_{hk},\\
&|D_x v(x_i,y_j) - \delta_1 z^{(i,j)}| \le \qquad (30)\\
&\qquad h\|D_{xx}v\| + \bar{\gamma}_1(h,k) + y_j\bar{\gamma}(h,k) + y_j\,\frac{C(h,k)}{L}\, e^{(x_i+y_j)L + x_i y_j (L_0+L^2)},\\
&\qquad \text{for } (x_i,y_j),(x_{i+1},y_j) \in E_{hk},\\
&|D_y v(x_i,y_j) - \delta_2 z^{(i,j)}| \le \qquad (31)\\
&\qquad k\|D_{yy}v\| + \bar{\gamma}_2(h,k) + x_i\bar{\gamma}(h,k) + x_i\,\frac{C(h,k)}{L}\, e^{(x_i+y_j)L + x_i y_j (L_0+L^2)},\\
&\qquad \text{for } (x_i,y_j),(x_i,y_{j+1}) \in E_{hk},
\end{aligned}$$

*where*

$$\begin{aligned}
L &= L_1 + L_2,\\
C(h,k) &= 3L_0\bar{\gamma}_0(h,k) + L_1\bar{\gamma}_1(h,k) + L_2\bar{\gamma}_2(h,k) + \bar{\gamma}(h,k)(L+1).
\end{aligned}$$

*Proof.* From Lemma 3 we obtain the estimate

$$\tilde{W}(x,y) \le \frac{C}{L}\, e^{L(x+y)+(L_0+L^2)xy} \tag{32}$$

with $C = C(h,k)$. Observe that

$$\begin{aligned}
\int_0^x\int_0^y e^{L(s+t)+(L_0+L^2)st}\,dt\,ds &\le xy e^{L(x+y)+(L_0+L^2)xy},\\
\int_0^y e^{L(x+t)+(L_0+L^2)xt}\,dt\,ds &\le y e^{L(x+y)+(L_0+L^2)xy},\\
\int_0^x e^{L(s+y)+(L_0+L^2)sy}\,dt\,ds &\le x e^{L(x+y)+(L_0+L^2)xy}.
\end{aligned}$$

Our assertion follows from the estimates in the proof of Theorem 1. This completes the proof.

Note that Corollary 1 yields some effective error estimates in dependence on the perturbations of the data and on the a priori bounds of classical solutions to the Darboux problem. Applying the more subtle estimate from Lemma 3 and the exact values of the integrals, we can obtain a more accurate error estimate.

## 6 An existence theorem

We give in Example 4 two generic kinds of functional dependence (deviated and integral) not only because of the awareness of a noticeable imbalance between the deviated and the integral dependence, namely: the Lipschitz condition holds for itegral functionals with somehow regular kernels, whereas any non-trivial deviations affect this property. The same behaviour has been observed while dealing with difference analogues of differential-functional problems. We will quote a theorem from [L1] which includes much more difficult type of equations with delays. First, we quote the assumptions concerning the case in question.

**A**$[f]$**.** Suppose that $f : \Omega_{CL} := E \times X_0 \times X_1 \times X_2 \to R$, where $X_0 = C_L(B, R)$ (the class of continuous functions satisfying the Lipschitz condition), $X_1 = C_{0+L}(B, R)$ (the class of continuous functions satisfying the Lipschitz condition with respect to the second variable) and $X_2 = C_{L+0}(B, R)$ (the class of continuous functions satisfying the Lipschitz condition with respect to the first variable). Assume that the function $f$ is continuous on $\Omega_C$ and there are $L, L_0, L_1, L_2 \in R_+$ such that

$$\begin{aligned}&|f(x, y, w_0, w_1, w_2) - f(\bar{x}, \bar{y}, \bar{w}_0, \bar{w}_1, \bar{w}_2)| \leq \\ &\qquad L\|(x - \bar{x}, y - \bar{y})\| + L_0\|w_0 - \bar{w}_0\|_L + L_1\|w_1 - \bar{w}_1\|_{0+L} + L_2\|w_2 - \bar{w}_2\|_{L+0}\end{aligned}$$

for all $(w_0, w_1, w_2), (\bar{w}_0, \bar{w}_1, \bar{w}_2) \in X_0 \times X_1 \times X_2$ and $(x, y) \in E$, where the norms $\|\cdot\|_L$, $\|\cdot\|_{0+L}$, $\|\cdot\|_{L+0}$ in functional spaces $X_0, X_1, X_2$ are defined by

$$\begin{aligned}\|w_0\|_L &= \|w_0\| + \sup_{(x,y)\neq(\bar{x},\bar{y})} \frac{|w_0(x, y) - w_0(\bar{x}, \bar{y})|}{\|(x - \bar{x}, y - \bar{y})\|},\\ \|w_1\|_{0+L} &= \|w_1\| + \sup_{(x,y)\neq(x,\bar{y})} \frac{|w_1(x, y) - w_1(x, \bar{y})|}{|y - \bar{y}|},\\ \|w_2\|_{L+0} &= \|w_2\| + \sup_{(x,y)\neq(\bar{x},y)} \frac{|w_2(x, y) - w_2(\bar{x}, y)|}{|x - \bar{x}|}\end{aligned}$$

for $w_\nu \in X_\nu$ $(\nu = 0, 1, 2)$.

**A**$[\phi]$**.** Suppose that $\phi : E_0 \to R$ is differentiable and $D_x\phi \in C_{0+L}(E_0, R)$, $D_y\phi \in C_{L+0}(E_0, R)$.

Observe the fact of losing the global character of existence, which results in demanding that the Lipschitz constants $L_1, L_2$ be sufficiently small.

**A**$[C_\nu]$**.** Suppose that there are $\theta \in (0, 1)$ and $C_f \in R_+$ such that

$$\theta = L_0(ab + a + b) + L_1(1 + b) + L_2(1 + a) \qquad \text{and} \qquad \|f(\cdot, \cdot, 0, 0, 0)\| \leq C_f.$$

If the assumption A$[C_\nu]$ holds, we can define a few positive constants: $C$, $C_0$, $C_1$, $C_2$ by

$$\begin{aligned}C &= \frac{C_f + 3L_0\|\phi\| + \|\phi\|_L(2L_0 + L_1 + L_2)}{1 - \theta}, \qquad (33)\\ C_0 &= (ab + a + b)C + 3\|\phi\| + 2\|\phi\|_L,\\ C_1 &= \|\phi\|_L + (1 + b)C, \qquad C_2 = \|\phi\|_L + (1 + a)C.\end{aligned}$$

Define the set $\mathcal{X}_L[C_0, C_1, C_2]$ by the fomula

$$\mathcal{X}_L[C_0, C_1, C_2] = \\ \{(z_0, z_1, z_2) \in \mathcal{X} \mid \|z_0\|_L \le C_0,\ \|z_1\|_{0+L} \le C_1,\ \|z_2\|_{L+0} \le C_2\},$$

where

$$\begin{aligned} \|z_0\|_L &= \sup_{(x,y)\in E} \|(z_0)_{(x,y)}\|_L, \\ \|z_1\|_{0+L} &= \sup_{(x,y)\in E} \|(z_1)_{(x,y)}\|_{0+L}, \\ \|z_2\|_{L+0} &= \sup_{(x,y)\in E} \|(z_2)_{(x,y)}\|_{L+0}. \end{aligned}$$

We cite after [L1] the existence theorem for differential-functional problem (1), (2).

**Theorem 2** *Suppose that the assumptions A[$\phi$], A[$f$] and A[$C_\nu$] are satisfied. Then there is a unique solution $z = (z_0, z_1, z_2)$ to a natural integral equivalent of problem (1), (2) in the class $\mathcal{X}_L[C_0, C_1, C_2]$. Moreover, we have $z_1 = D_x z_0$ and $z_2 = D_y z_0$ on $E_0 \cup E$, and the function $z_0$ is a classical solution to problem (1), (2).*

Existence results can be found also in [By], [Cz], [LLV]. We shall formulate assumptions which reflect the character of these sufficient conditions for existence and uniqueness. The ideas and methods used in the proofs of stability and convergence statements can be found to be parallel to that of existence and uniqueness.

## 7 Other stability, consistency and convergence results

Define the discrete norms

$$\begin{aligned} \|w\|_L &= \|w\| + \|\delta_1 w\| + \|\delta_2 w\|, \\ \|w\|_{L+0} &= \|w\| + \|\delta_1 w\|, \quad \|w\|_{0+L} = \|w\| + \|\delta_2 w\| \end{aligned} \tag{34}$$

for $w \in \mathcal{F}(X, R)$, where $X$ stands either for $\tilde{E}_{hk}$ or $B_{hk}$ in dependence on the context. The difference operators in the above definition are discrete counterparts of the Lipschitz constants.

**Assumption 4** *Suppose that there are constants $L_0, L_1, L_2 \in R_+$ (independent of $(h, k)$) such that*

$$|f_{hk}(x_i, y_j, w_0, w_1, w_2) - f_{hk}(x_i, y_j, \bar{w}_0, \bar{w}_1, \bar{w}_2)| \le \\ L_0\|w_0 - \bar{w}_0\|_L + L_1\|w_1 - \bar{w}_1\|_{0+L} + L_2\|w_2 - \bar{w}_2\|_{L+0}$$

*for $(x_i, y_j, w_0, w_1, w_2), (x_i, y_j, \bar{w}_0, \bar{w}_1, \bar{w}_2) \in \Omega_{hk}$.*

Note that what Assumption 4 states of the function $f_{hk}$ is very close to A[$f$], compare also Example 1 and 2.

We start the main body of this section with a lemma on the consistency of the difference scheme with the differential-functional problem.

**Lemma 4** *Suppose that Assumptions 1, 2 and 3 are satisfied. Then the function* $\gamma(h,k) = \max_{i,j} |\xi_{hk}^{(i,j)}[v]|$ *is of class* $\Gamma_0$.

We omit the proof as it is similar to that of Lemma 1. We formulate a lemma on stabiliy of the difference scheme.

**Lemma 5** *Suppose that Assumption 4 is satisfied. Take* $z, \bar{z} \in \mathcal{F}(\tilde{E}_{hk}, R)$*; the function* $z$ *satisfying (3), (4), and the function* $\bar{z}$ *satisfying the inequalities*

$$\begin{aligned} |\xi_{hk}^{(i,j)}[\bar{z}]| &\leq \bar{\gamma}(h,k) \quad \textit{for} \;\; (x_i, y_j) \in E_{hk}, \\ |z^{(i,j)} - \bar{z}^{(i,j)}| &\leq \bar{\gamma}_0(h,k) \quad \textit{for} \;\; (x_i, y_j) \in E_{hk}^0, \\ |\delta_1 z^{(i,j)} - \delta_1 \bar{z}^{(i,j)}| &\leq \bar{\gamma}_1(h,k) \quad \textit{for} \;\; (x_i, y_j), (x_{i+1}, y_j) \in E_{hk}^0, \\ |\delta_2 z^{(i,j)} - \delta_2 \bar{z}^{(i,j)}| &\leq \bar{\gamma}_2(h,k) \quad \textit{for} \;\; (x_i, y_j), (x_i, y_{j+1}) \in E_{hk}^0, \\ |\delta_{12} z^{(i,j)} - \delta_{12} \bar{z}^{(i,j)}| &\leq \bar{\gamma}_{12}(h,k) \quad \textit{for} \;\; (x_i, y_j), (x_i, y_{j+1}), (x_{i+1}, y_j) \in E_{hk}^0, \end{aligned} \tag{35}$$

*where* $\bar{\gamma}, \bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_{12} \in \Gamma_0$*. Then we have*

$$\begin{aligned} |z^{(i,j)} - \bar{z}^{(i,j)}| &\leq C_0(h,k) \quad \textit{for} \;\; (x_i, y_j) \in \tilde{E}_{hk}, \\ |\delta_1 z^{(i,j)} - \delta_1 \bar{z}^{(i,j)}| &\leq C_1(h,k) \quad \textit{for} \;\; (x_i, y_j), (x_{i+1}, y_j) \in \tilde{E}_{hk}, \\ |\delta_2 z^{(i,j)} - \delta_2 \bar{z}^{(i,j)}| &\leq C_2(h,k) \quad \textit{for} \;\; (x_i, y_j), (x_i, y_{j+1}) \in \tilde{E}_{hk}, \\ |\delta_{12} z^{(i,j)} - \delta_{12} \bar{z}^{(i,j)}| &\leq C_{12}(h,k) \quad \textit{for} \;\; (x_i, y_j), (x_{i+1}, y_{j+1}) \in \tilde{E}_{hk}, \end{aligned} \tag{36}$$

*where*

$$\begin{aligned} C_0(h,k) &= 3\bar{\gamma}_0(h,k) + ab\bar{\gamma}(h,k) + abC(h,k), \\ C_1(h,k) &= \bar{\gamma}_1(h,k) + b\bar{\gamma}(h,k) + bC(h.k), \\ C_2(h,k) &= \bar{\gamma}_2(h,k) + a\bar{\gamma}(h,k) + aC(h,k), \\ C_{12}(h,k) &= \bar{\gamma}(h,k) + C(h,k) \end{aligned} \tag{37}$$

*with*

$$C(h,k) = \frac{3\bar{\gamma}_0(h,k)L_0 + \bar{\gamma}_1(h,k)(L_0 + L_1) + \bar{\gamma}_2(h,k)(L_0 + L_2) + \bar{\gamma}(h,k)\theta}{1 - \theta}, \tag{38}$$

*and* $C_0(h,k)$, $C_1(h,k)$, $C_2(h,k)$, $C_{12}(h,k) \to 0$ *as* $h$, $k \to 0$.

*Proof.* The estimates are obvious on $E_{hk}^0$. If we define the error $z - \bar{z}$, then there is no question of the explicite solvability of the error equation. The only matter is how to estabish its relevant estimate. In view of formula (12) and the above remark on recurrence solvability we deduce for $(x_i, y_j) \in E^+_{hk}$ the estimate

$$
\begin{aligned}
&|z^{(i,j)} - \bar{z}^{(i,j)}| \le \\
&\quad |\mathcal{L}_0 z^{(i,j)} - \mathcal{L}_0 \bar{z}^{(i,j)}| + hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} |\xi_{hk}^{(\mu,\nu)}[\bar{z}]| \le \\
&\quad 3\bar{\gamma}_0(h,k) + x_i y_j \bar{\gamma}(h,k) + hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \Big( L_0 \|(z-\bar{z})_{[\mu,\nu]}\|_L + \\
&\quad L_1 \|(\delta_1(z-\bar{z}))_{[\mu,\nu]}\|_{L+0} + L_2 \|(\delta_2(z-\bar{z}))_{[\mu,\nu]}\|_{0+L} \Big) \le \\
&\quad 3\bar{\gamma}_0(h,k) + x_i y_j \bar{\gamma}(h,k) + hkij \Big( L_0 (C_0(h,k) + C_1(h,k) + C_2(h,k)) + \\
&\quad L_1 (C_1(h,k) + C_{12}(h,k)) + L_2 (C_2(h,k) + C_{12}(h,k)) \Big) \le C_0(h,k).
\end{aligned}
$$

Now, we take $(x_i, y_j) \in (0,k) + E_{hk}$ and derive the estimate

$$
\begin{aligned}
&|\delta_1 z^{(i,j)} - \delta_1 \bar{z}^{(i,j)}| \le \\
&\quad |\mathcal{L}_1 z^{(i,j)} - \mathcal{L}_1 \bar{z}^{(i,j)}| + k \sum_{\nu=0}^{j-1} |\xi_{hk}^{(i,\nu)}[\bar{z}]| \le \\
&\quad \bar{\gamma}_1(h,k) + y_j \bar{\gamma}(h,k) + k\, j \Big( L_0 (C_0(h,k) + C_1(h,k) + C_2(h,k)) + \\
&\quad L_1 (C_1(h,k) + C_{12}(h,k)) + L_2 (C_2(h,k) + C_{12}(h,k)) \Big) \le C_1(h,k).
\end{aligned}
$$

Taking $(x_i, y_j) \in (h,0) + E_{hk}$, we derive the estimate

$$
\begin{aligned}
&|\delta_2 z^{(i,j)} - \delta_2 \bar{z}^{(i,j)}| \le \\
&\quad |\mathcal{L}_2 z^{(i,j)} - \mathcal{L}_2 \bar{z}^{(i,j)}| + k \sum_{\mu=0}^{i-1} |\xi_{hk}^{(\mu,j)}[\bar{z}]| \le \\
&\quad \bar{\gamma}_2(h,k) + x_i \bar{\gamma}(h,k) + h\, i \Big( L_0 (C_0(h,k) + C_1(h,k) + C_2(h,k)) + \\
&\quad L_1 (C_1(h,k) + C_{12}(h,k)) + L_2 (C_2(h,k) + C_{12}(h,k)) \Big) \le C_2(h,k).
\end{aligned}
$$

Finally, taking $(x_i, y_j) \in E_{hk}$ such that $(x_{i+1}, y_j), (x_i, y_{j+1}) \in E_{hk}$ , we get

$$
\begin{aligned}
&|\delta_{12} z^{(i,j)} - \delta_{12} \bar{z}^{(i,j)}| \le \\
&\quad |f_{hk}\Big(x_i, y_j, z_{[i,j]}, (\delta_1 z)_{[i,j]}, (\delta_2 z)_{[i,j]}\Big) - \\
&\quad f_{hk}\Big(x_i, y_j, \bar{z}_{[i,j]}, (\delta_1 \bar{z})_{[i,j]}, (\delta_2 \bar{z})_{[i,j]}\Big)| + k \sum_{\mu=0}^{i-1} |\xi_{hk}^{(\mu,j)}[\bar{z}]| \le \\
&\quad \bar{\gamma}(h,k) + \Big( L_0 (C_0(h,k) + C_1(h,k) + C_2(h,k)) + \\
&\quad L_1 (C_1(h,k) + C_{12}(h,k)) + L_2 (C_2(h,k) + C_{12}(h,k)) \Big) \le C_{12}(h,k).
\end{aligned}
$$

These estimates establish the assertion of our lemma, which finishes the proof.

**Theorem 3** *Suppose that Assumptions 1, 2, 4 and A[$C_\nu$] are satisfied. Assume that the function $z \in \mathcal{F}(\tilde{E}_{hk}, R)$ is a solution to problem (3), (4) satisfying within $E^0_{hk}$ the inequalities*

$$\begin{aligned} |\phi^{(i,j)}_{hk} - \phi(x_i, y_j)| &\leq \bar{\gamma}_0(h,k) \quad \textit{for} \ \ (x_i, y_j) \in E^0_{hk}, \\ |\delta_1\phi^{(i,j)}_{hk} - \delta_1\bar{\phi}^{(i,j)}| &\leq \bar{\gamma}_1(h,k) \quad \textit{for} \ \ (x_i, y_j), (x_{i+1}, y_j) \in E^0_{hk}, \\ |\delta_2\phi^{(i,j)}_{hk} - \delta_2\bar{\phi}^{(i,j)}| &\leq \bar{\gamma}_2(h,k) \quad \textit{for} \ \ (x_i, y_j), (x_i, y_{j+1}) \in E^0_{hk}, \\ |\delta_{12}\phi^{(i,j)}_{hk} - \delta_{12}\bar{\phi}^{(i,j)}| &\leq \bar{\gamma}_{12}(h,k) \quad \textit{for} \ \ (x_i, y_j), (x_i, y_{j+1}), (x_{i+1}, y_j) \in E^0_{hk}, \end{aligned} \tag{39}$$

*where $\bar{\gamma}, \bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2 \in \Gamma_0$. Then we have*

$$\begin{aligned} |v^{(i,j)} - z^{(i,j)}| &\leq C_0(h,k) \to 0, \\ |D_x v(x_i, y_j) - \delta_1\bar{z}^{(i,j)}| &\leq h\|D_{xx}v\| + C_1(h,k) \to 0, \\ |D_y v(x_i, y_j) - \delta_2\bar{z}^{(i,j)}| &\leq k\|D_{yy}v\| + C_2(h,k) \to 0, \\ |D_{xy} v(x_i, y_j) - \delta_{12}\bar{z}^{(i,j)}| &\leq \|(h,k)\| \sum_{\mu,\nu\geq 0;\, \mu+\nu\leq 3} \|D^\mu_x D^\nu_y v\| + C_{12}(h,k) \to 0. \end{aligned} \tag{40}$$

*Proof.* It follows from Lemma 1 that the function $\gamma(h,k) = \max_{i,j} |\xi^{(i,j)}_{hk}[v]|$ is of class $\Gamma_0$. Define $\bar{\gamma}{\in}\Gamma_0$ as the right-hand side of inequality (17). Assertion (40) is obtained by means of Lemma 5 with $\bar{\gamma}$, $\bar{\gamma}_0$, $\bar{\gamma}_1$, $\bar{\gamma}_2$, $\bar{\gamma}_{12}$ satisfying (35). This completes the proof.

**Remark.** The last inequality in (35) and (36) seem unnatural and inconvenient, but these constraints are contained by themselves just in the definitions of $\|\delta_1 z\|_{0+L} = \|\delta_1 z\| + \|\delta_{12} z\|$ and $\|\delta_2 z\|_{L+0} = \|\delta_2 z\| + \|\delta_{12} z\|$. Our error estimates are local, which is due to Assumption 4 and A[$C_\nu$]. Some parts of our assumptions (for instance on the boundedness of $f$) are not applied in their explicite forms. They are hidden somehow in the regularity of the solutions to the Darboux problem.

## 8 Numerical examples

We illustrate the results of our numerical experiments performed by PC IBM 486. Three differential equations whose share solution is

$$u(t,x) \,=\, 1 \,+\, t\,x^2 \,-\, x\,t^3 \tag{41}$$

are considered in $E = [0, 0.5] \times [0, 0.5]$. We introduce the usual mesh with $h = k = 0.005$ and show some computed values at the main diagonal of the square $E$.

**Numerical example 1.** We compute approximate solutions of the following non-linear equation

$$D_{tx}u(t,x) = u(t,x) + \sin\left(D_t u\,(t,x) + D_x u\,(t,x)\right) \,+ f_1(t,x), \tag{42}$$

where the function $f_1 : E \to R$ is defined as follows

$$f_1(t,x) = -1 - 3t^2 + 2x + t^3 x - t x^2 + \sin\left(t^3 - 2tx + 3t^2 x - x^2\right).$$

The following table contains the diagonal values of $U_{enh}(x_i, x_i)$ and $U(x_i, x_i)$, the solutions of enhanced and usual difference schemes, and their errors $err_{enh}$ and $err$, respectively.

| $x_i$ | $U_{enh}(x_i, x_i)$ | $err_{enh}$ | $U(x_i, x_i)$ | $err$ |
|---|---|---|---|---|
| 0.05 | 1.00011878 | 0.00000003 | 1.00010687 | −0.00001188 |
| 0.10 | 1.00090013 | 0.00000013 | 1.00085493 | −0.00004507 |
| 0.15 | 1.00286904 | 0.00000029 | 1.00277278 | −0.00009597 |
| 0.20 | 1.00640053 | 0.00000053 | 1.00623895 | −0.00016105 |
| 0.25 | 1.01171959 | 0.00000084 | 1.01148191 | −0.00023684 |
| 0.30 | 1.01890122 | 0.00000122 | 1.01858020 | −0.00031980 |
| 0.35 | 1.02787044 | 0.00000169 | 1.02746249 | −0.00040626 |
| 0.40 | 1.03840224 | 0.00000224 | 1.03790770 | −0.00049230 |
| 0.45 | 1.05012163 | 0.00000288 | 1.04954522 | −0.00057353 |
| 0.50 | 1.06250361 | 0.00000361 | 1.06185512 | −0.00064488 |

In the usual scheme we take $U[i, j] \approx u(t_i, x_j)$ and progressive difference operators instead of $D_t u(\dots)$ and $D_x u(\dots)$. The enhancement requires some modifications, namely: $f_1(t_i, x_j)$ is replaced by $f_1\left(t_{i+1/2}, x_{j+1/2}\right)$, and

$$
\begin{aligned}
\frac{U[i,j] + U[i+1,j] + U[i,j+1] + v}{4} &\approx u\left(t_{i+1/2}, x_{j+1/2}\right),\\
\frac{1}{2}\left(\frac{U[i+1,j] - U[i,j]}{h} + \frac{v - U[i,j+1]}{h}\right) &\approx D_t u\left(t_{i+1/2}, x_{j+1/2}\right),\\
\frac{1}{2}\left(\frac{U[i,j+1] - U[i,j]}{h} + \frac{v - U[i+1,j]}{h}\right) &\approx D_x u\left(t_{i+1/2}, x_{j+1/2}\right),
\end{aligned}
$$

where $v$ is an approximate value of $U[i+1, j+1]$ obtained in a certain number of iterations. In fact, this is an explicit scheme which is close to a second-order implicit scheme. The above table shows how much the enhanced scheme improves the approximation.

**Numerical example 2.** We consider a differential equation with simple delays $t/2$ and $x/2$. Of course, it is no need to give initial data in a 'thick' set $E_0$, because these delays act within the set $E$.

$$
\begin{aligned}
D_{tx} u(t,x) = &-u\left(t, \frac{x}{2}\right) + 4\,u\left(\frac{t}{2}, x\right)\\
&+ \frac{21}{8}\,u(t,x) - 7\,t\,D_t u\left(\frac{t}{2}, \frac{x}{2}\right) + D_x u\left(\frac{t}{2}, \frac{x}{2}\right) + f_2(t,x),
\end{aligned} \tag{43}
$$

where $f_2(t, x)$ is given by the formula

$$f_2(t,x) = -5.625 - 3t^2 + 0.125\,t^3 + 2x - 0.5\,tx - 2.625\,tx^2.$$

We obtain the following table

| $x_i$ | $U(x_i, x_i)$ | $err$ |
|---|---|---|
| 0.05 | 1.00010746 | −0.00001129 |
| 0.10 | 1.00085979 | −0.00004021 |
| 0.15 | 1.00278952 | −0.00007923 |
| 0.20 | 1.00627974 | −0.00012026 |
| 0.25 | 1.01156452 | −0.00015423 |
| 0.30 | 1.01872942 | −0.00017058 |
| 0.35 | 1.02771199 | −0.00015676 |
| 0.40 | 1.03830232 | −0.00009768 |
| 0.45 | 1.05014368 | 0.00002493 |
| 0.50 | 1.06273311 | 0.00023311 |

The above table shows the discrete values and the adequate errors at every tenth diagonal knot of our mesh. Concerning the points between two knots, we derive functions as mean value of these from two or four natural neighbouring knots, which corresponds to applying the linear spline interpolation. In order to get a significant decrease in error, similarly as in the former numerical example, one can use the concept of enhancement.

**Numerical example 3.** Finally, we consider a kind of the Voterra dependence represented by an integral over the set $[t-1, t] \times [x-1, x]$.

$$D_{tx}u(t,x) = D_t u\,(t,x) + D_x u\,(t,x) \,+\, 24\int_{t-1}^{t}\int_{x-1}^{x} u(s,y)\,dy\,ds \,+\, f_3(t,x), \qquad (44)$$

where $f_3(t,x)$ is defined by the formula

$$f_3(t,x) = -17 - 20\,t + 15t^2 - 11t^3 - 16\,x + 48tx - 33t^2x + \\ 24\,t^3x + 11\,x^2 - 24\,tx^2.$$

In this example, we need some initial data in $E_0 := [-1, 0.5] \times [-1, 0.5]\backslash E$. In fact, our initial data are given by (41). The meaning of the following table is clear.

| $x_i$ | $U(x_i, x_i)$ | $err$ |
|---|---|---|
| 0.05 | 1.00012740 | 0.00000865 |
| 0.10 | 1.00107177 | 0.00017177 |
| 0.15 | 1.00367755 | 0.00080880 |
| 0.20 | 1.00878061 | 0.00238061 |
| 0.25 | 1.01719461 | 0.00547586 |
| 0.30 | 1.02969647 | 0.01079647 |
| 0.35 | 1.04701205 | 0.01914330 |
| 0.40 | 1.06980335 | 0.03140335 |
| 0.45 | 1.09865893 | 0.04854018 |
| 0.50 | 1.13408995 | 0.07158995 |

What is worth mentioning is that the integrals over subsets of $E_0$ are accurate, whereas the remaining parts of these integrals are replaced by summation formulas from $(0,0)$ to $(i,j)$, which reflects a sort of approximation by means of piecewise constant functions. The necessity of dealing with numerous sums results in a noticeable decrease in accuracy and speed of computations. In particular, the above table contains reliable values only in $[0,0.3)\times[0,0.3)$. Presumably, applying other quadratures and averaging operators would lessen the error.

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