Finite Dimensional Hopf Algebras Coacting on Coalgebras

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Introduction

Let H be a finite dimensional Hopf algebra and let C be a left H-comodule coalgebra. In [2], a Morita-Takeuchi context arising from a left H-comodule coalgebra has been constructed. Utilizing that Morita- Takeuchi context we may characterize the Hopf-Galois coactions on coalgebras, and use it to prove the duality theorem for crossed coproducts. In this note, we show that the Morita-Takeuchi context constructed in [2] is generated by the left comodule ${}_{C\rtimes H}C$, where $C \rtimes H$ is the smash coproduct coalgebra of C by H. As a consquence, we obtain that the coaction of Hopf algebra H on C is Galois if and only if ${}_{C\rtimes H}C$ is a cogenerator. This dualizes the corresponding result in [1]. Another functorial description of Galois coactions is in Theorem 2.8, which is the dualization of the weak structure theorem in [4].

In Section 3, we define the cotrace map for an H^*-coextension C/R. There are various descriptions of the cotrace map being injective. For instance, the comodule ${}_{C\rtimes H}C$ is an injective comodule; the canonical map G in the Morita-Takeuchi context is injective; the cohom functor $h_{C\rtimes H-}(C,-)$ is equivalent to the cotensor functor $C\square_{C\rtimes H}-$ cf.Theorem 3.5.

1 Preliminaries

Throughout k is a fixed field. All coalgebras, algebras, vector spaces and unadorned $\otimes$, Hom, etc, are over k. C, D always denote coalgebras and H is a Hopf algebra. We refer to [9] for detail on coalgebras and comodules. We adapt the usual sigma notation for the comultiplications of coalgebras, and adapt the following sigma notation

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for a (left) C-comodule structure map ρ_X of X:

$$\rho_X(x) = \sum x_{(-1)} \otimes x_{(0)}.$$

For a left H-comodule M, we use the following sigma notation to denote the comodule structure map ρ_M of M:

$$\rho_M(m) = \sum m_{<-1>} \otimes m_{<0>}.$$

Let $\mathbf{M}^C$ (or ${}^C\mathbf{M}$) denote the category of right (or left) C-comodules. If $\alpha : C \longrightarrow D$ is a coalgebra map, then any left C-comodule X may be treated as a left D-comodule in a natural way:

$$(\alpha \otimes 1)\rho : X \longrightarrow C \otimes X \longrightarrow D \otimes X.$$

A $(C - D)$-bicomodule is a left C-comodule and a right D-comodule X, denoted by ${}_CX_D$, such that the C-comodule structure map $\rho_C : \ X \longrightarrow C \otimes X$ is right D-colinear (or a D-comodule map).

For a right C-comodule M and a left C-comodule N, the contensor product $M\square_C N$ is the kernel of

$$\rho_M \otimes 1 - 1 \otimes \rho_N : \ M \otimes N \rightrightarrows M \otimes C \otimes N.$$

The functors $M\square_C-$ and $-\square_C N$ are left exact and preserve direct sums. If ${}_CX_D$ and ${}_DY_E$ are bicomodules, then $X\square_D Y$ is a $(C - E)$-bicomodule with comodule structures induced by those of X and Y.

We recall from [10] the definition of a cohom functor and some of its basic properties. A comodule ${}_CX$ is quasi-finite if $\mathrm{Com}_{C-}(Y, X)$ is finite dimensional for any finite dimensional comodule ${}_CY$. A comodule ${}_CX$ is finitely cogenerated if it is isomorphic to a subcomodule of $C \otimes W$ for some finite dimensional space W. A finitely cogenerated comodule is quasi-finite. But the coverse is not true. A comodule $X \in {}^C\mathbf{M}$ is said to be a cogenerator if for any comodule $M \in {}^C\mathbf{M}$ there is a space W such that $M \hookrightarrow X \otimes W$ as comodules. The following lemma relates the existence of the cohom functor to quasi-finiteness:

Basic Lemma[10]: Let ${}_CX_D$ be a bicomodule. Then ${}_CX$ is quasi-finite if and only if the functor $X\square_D- : \ {}^D\mathbf{M} \longrightarrow {}^C\mathbf{M}$ has a left adjoint functor, denoted by $h_{C-}(X, -)$. That is, for comodules ${}_CY$ and ${}_DW$,

$$\mathrm{Com}_{D-}(h_{C-}(X, Y), W) \simeq \mathrm{Com}_{C-}(Y, X\square_D W) \qquad (\#)$$

Where,

$$h_{C-}(X, Y) = \varinjlim_{\mu} \mathrm{Com}_{C-}(Y_\mu, X)^* \simeq \varinjlim_{\mu} (Y_\mu^* \square_C X)^*$$

is a left D-comodule, $\{Y_\mu\}$ is the directed family of finite dimensional subcomodules of ${}_CY$ such that $Y = \bigcup_\mu Y_\mu$. In particular, if $C = X$, $D = k$, then $h_{C-}(C, -)$ is nothing else but the forgetful functor $U : \ {}^C\mathbf{M} \longrightarrow \mathbf{M}$, here $\mathbf{M}$ is the k-module category; if $C = D$, $X = C$, $h_{C-}(C, -)$ is the identity functor from ${}^C\mathbf{M}$ to ${}^C\mathbf{M}$. Let θ denote the canonical C-colinear map $Y \longrightarrow X\square_D h_{C-}(X, Y)$ which corresponds to the identity map $h_{C-}(X, Y) \longrightarrow h_{C-}(X, Y)$ in (#). Similarly, there is a right version of the basic lemma for a quasi-finite comodule X_D.

Assume that ${}_CX$ is a quasi-finite comodule. Consider a bicomodule ${}_CX_k$. Then $e_{C-}(X) = h_{C-}(X,X)$ is a coalgebra, called the co-endomorphism coalgebra of X. The comultiplication of $e_{C-}(X)$ corresponds to $(\theta \otimes 1)\theta : \ X \longrightarrow X \otimes e_{C-}(X) \otimes e_{C-}(X)$ in (#), and the counit of $e_{C-}(X)$ corresponds to the identity map 1_X. Also X is a $C - e_{C-}(X)$-bicomodule with right comodule structure map θ, the canonical map $X \longrightarrow X \otimes h_{C-}(X,X)$.

A Morita-Takeuchi (M-T) context $(C, D, {}_CP_D, {}_DQ_C, f, g)$ consists of coalgebras C, D, bicomodules ${}_CP_D, {}_DQ_C$, and bicolinear maps $f : \ C \longrightarrow P\Box_D Q$ and $g : \ D \longrightarrow Q\Box_C P$ satisfying the following commutative diagrams:

$$\begin{tikzcd} P \arrow[r, "\sim"] \arrow[d, "\sim"] & P\Box_D D \arrow[d, "1\Box g"] \\ C\Box_C P \arrow[r, "f\Box 1"] & P\Box_D Q\Box_C P \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, "\sim"] \arrow[d, "\sim"] & Q\Box_C C \arrow[d, "1\Box f"] \\ D\Box_D Q \arrow[r, "g\Box 1"] & Q\Box_C P\Box_D Q \end{tikzcd}$$

The context is said to be *strict* if both f and g are injective (equivalently, isomorphic). In this case we say that C is M-T equivalent to D, denoted by $C \sim D$.

Let H be a Hopf algebra, C a coalgebra. C is said to be a right H-module coalgebra if
i). C is a right H-module,
ii). $\Delta(c \leftharpoonup h) = \sum c_{(1)} \leftharpoonup h_{(1)} \otimes c_{(2)} \leftharpoonup h_{(2)}, \ c \in C, h \in H,$
iii). $\varepsilon(c \leftharpoonup h) = \varepsilon(c)\varepsilon(h)$.

Dually, a coalgebra C is called a left H-comodule coalgebra if
i). C is a left H-comodule,
ii). $\sum c_{<-1>} \otimes \Delta(c_{<0>}) = \sum c_{(1)<-1>}c_{(2)<-1>} \otimes c_{(1)<0>} \otimes c_{(2)<0>}$,
iii). $\sum \varepsilon(c_{<0>})c_{<-1>} = \varepsilon(c)1_H$.

If H is a finite dimensional Hopf algebra, a coalgebra C is a right H-module coalgebra if and only if C is a left H^*-comodule coalgebra. On the other hand, for any Hopf algebra H and right H-module coalgebra C, the convolution algebra C^* is a left H-module algebra with H-module structure induced by transposition.

Let C be a right H-module coalgebra, H a Hopf algebra. Denote by H^+ the augmentation ideal kerε which is a Hopf ideal. Then $CH^+ = C \leftharpoonup H^+$ is a coideal of C, and C/CH^+ is a coalgebra with a trivial right H-module structure. Let R be the quotient coalgebra C/CH^+. It is not hard to check that R^* is the invariant subalgebra of the left H-module algebra C^*. Dual to the terminology of 'H-extension', we call C/R an *H-coextension.* View C as a left and right R-comodule. There is a canonical linear map

$$\beta : \ C \otimes H \longrightarrow C\Box_R C, \ c \otimes h \mapsto \sum c_{(1)}\Box c_{(2)} \leftharpoonup h.$$

If β is bijective, then C/R is said to be an H-Galois coextension cf.[7] (sometimes it is called H-cogalois cf.[3] [8]).

Let C be a left H-comodule coalgebra. We may form a smash coproduct coalgebra $C \rtimes H$ which has counit $\varepsilon_C \rtimes \varepsilon_H$ and comultiplication as follows:

$$\Delta(c \rtimes h) = \sum (c_{(1)} \rtimes c_{(2)<-1>}h_{(1)}) \otimes (c_{(2)<0>} \rtimes h_{(2)}).$$

If H is finite dimensional, C^* is a left H^*-module algebra. We have the usual smash product algebra $C^*\#H^*$. It is easy to see that $C^*\#H^*$ is exactly the convolution algebra $(C \rtimes H)^*$.

Now let H be a finite dimensional Hopf algebra, C a left H-comodule coalgebra. We recall from [2] the M-T context arising from a left H-comodule coalgebra C. Let R be the quotient coalgebra C/CH^{*+}. Then C may be viewed as a left or a right R-comodule in a natural way. There is a canonical left $C \rtimes H$-coaction on C given by

$$\rho^l(c) = \sum (c_{(1)} \rtimes c_{(2)<-1>}) \otimes c_{(2)<0>} \tag{1}$$

This coaction is compatible with the right R-coaction on C, and makes C into a $(C \rtimes H - R)$-bicomodule.

Let T be a left integral of H^* and λ be the distinguished group-like element cf.[6] of H which satisfies:

$$Th^* =< h^*, \lambda > T, \ \forall h^* \in H^*.$$

There is a right coaction of $C \rtimes H$ on C as follows:

$$\rho^r(c) = \sum c_{(1)<0>} \otimes (c_{(2)<0>} \rtimes S^{-1}(c_{(1)<-1>}c_{(2)<-1>})\lambda) \tag{2}$$

With the above right $C \rtimes H$-coaction and the natural left R-coaction C becomes an $(R - C \rtimes H)$-bicomodule. The Morita-Takeuchi context arising from C is

$$(C \rtimes H, R, \ {}_{C\rtimes H}C_R, \ {}_RC_{C\rtimes H}, F, G) \tag{3}$$

where the bicolinear maps F, G are given by
$F: \ C \rtimes H \longrightarrow C\Box_R C, \ c \rtimes h \mapsto \sum c_{(1)}\Box c_{(2)<0>} < T, c_{(2)<-1>}h >$, and
$G: \ R \longrightarrow C\Box_{C\rtimes H} C, \ \overline{c} \mapsto \sum c_{(1)<0>}\Box c_{(2)<0>} < T, c_{(1)<-1>}c_{(2)<-1>} >$.

In [2] we use the above M-T context to show the duality theorem for crossed coproducts. Moreover, the bicolinear map F in (3) can be used to describe the Galois coextension, that is, C/R is H^*-Galois if and only if F is injective cf.[2, Th.1.2].

2 The Hopf comodule category

Let H be a Hopf algebra. If C is a left H-comodule coalgebra, we have the smash coproduct coalgebra $C \rtimes H$. Denote by ${}^{C\rtimes H}\mathbf{M}$ the category of left $C \rtimes H$-comodules and morphisms.

Lemma 2.1. A comodule M is in ${}^{C\rtimes H}\mathbf{M}$ if and only if M is a left C-comodule and a left H-comodule satisfying the compatibility condition: $\forall m \in M$,

$$\sum m_{<0>(-1)} \otimes m_{<-1>} \otimes m_{<0>(0)} = \sum m_{<-1><0>} \otimes m_{(-1)<-1>}m_{(0)<-1>} \otimes m_{(0)<0>} \tag{4}$$

Proof. Straightforward. ∎

A left C-comodule M is called a Hopf comodule if it is a left H-comodule and satisfies the compatibility condition (4). Write ${}^{(C,H)}\mathbf{M}$ for the category of Hopf comodules and morphisms. Lemma 2.1 states that ${}^{C\rtimes H}\mathbf{M} \sim {}^{(C,H)}\mathbf{M}$. A left C-comodule M is said to be a Hopf bimodule if M is a right H-module and satisfies the compatibility condition:

$$\rho(m \leftharpoonup h) = \sum m_{(-1)} \leftharpoonup h_{(1)} \otimes m_{(0)} \leftharpoonup h_{(2)}, \ m \in M, h \in H. \tag{5}$$

The category of Hopf bimodules and morphisms is denoted by ${}^{C}\mathbf{M}_{H}$. If H is finite dimensional, then we have that ${}^{(C,H)}\mathbf{M} \sim {}^{C}\mathbf{M}_{H^*}$. In the sequel, H is a finite dimensional Hopf algebra, C is a left H-comodule coalgebra. We identify ${}^{(C,H)}\mathbf{M}$, ${}^{C\rtimes H}\mathbf{M}$ with ${}^{C}\mathbf{M}_{H^*}$. Let H^{*+} be the augmentation ideal $ker(\varepsilon_{H^*} : H^* \longrightarrow k)$. Let R be the quotient coalgebra C/CH^{*+}. To a Hopf comodule $M \in {}^{(C,H)}\mathbf{M}$ we associate an R-comodule $\overline{M} = M/MH^{*+}$. The functor $\overline{(-)} : {}^{(C,H)}\mathbf{M} \longrightarrow {}^{R}\mathbf{M}$ has a right adjoint functor $C\square_R- : {}^{R}\mathbf{M} \longrightarrow {}^{(C,H)}\mathbf{M}$ cf.[7]. On the other hand, C is a $(C \rtimes H, R)$-bicomodule, and as a left $C \rtimes H$-comodule is quasi-finite. So the cohom functor $h_{C\rtimes H-}(C,-) : {}^{(C,H)}\mathbf{M} = {}^{C\rtimes H}\mathbf{M} \longrightarrow {}^{R}\mathbf{M}$ exists and it is a left adjoint functor of the functor $C\square_R-$. By the uniqueness of adjointness, $h_{C\rtimes H-}(C,-)$ is equivalent to $\overline{(-)}$. Let η be the natural (isomorphic) transformation from $\overline{(-)}$ to $h_{C\rtimes H-}(C-)$. For a Hopf comodule $M \in {}^{(C,H)}\mathbf{M}$, we have the following commutative diagram:

$$\begin{tikzcd} M \arrow[rr, "\theta_M"] \arrow[dr, "\nu_M"'] & & C\square_R h_{C\rtimes H-}(C,M) \\ & C\square\overline{M} \arrow[ur, "1\otimes\eta_M"'] & \end{tikzcd} \tag{6}$$

where θ is the canonical (adjoint) map mentioned in Section 1 and ν_M is the adjoint map:

$$M \longrightarrow C\square_R\overline{M} : \ m \mapsto \sum m_{(-1)} \otimes \overline{m_{(0)}}.$$

In the sequel, $\square$ means the cotensor product over R.
Lemma 2.2. Let M be a Hopf comodules. The following sequence is exact:

$$0 \longrightarrow M \leftharpoonup H^{*+} \longrightarrow M \overset{(\epsilon\otimes 1)\theta_M}{\longrightarrow} h_{C\rtimes H-}(C,M) \longrightarrow 0.$$

Proof. Follows from the foregoing commutative diagram (6). ∎

We need the following preparation to show Proposition 2.4. It is well-known that a finite dimensional Hopf algebra is a Frobenius algebra. Let Θ be the Frobenius isomorphism:

$${}_H H_{H^*} \longrightarrow {}_H H^*_{H^*},$$

where the actions are canonical, i.e,

$$h \leftharpoonup p = \sum < p, h_{(1)} > h_{(2)}, \ h \rightharpoonup p = \sum p_{(1)} < p_{(2)}, h >, \ h \in H, p \in H^*.$$

Θ^{-1} makes H a right H^*-free module with basis $t = \Theta^{-1}(\epsilon)$, which is a left integral of H. Let T be $S^*(\Theta(1))$, where S^* is the antipode of H^*. Then T is a left integral of H^* cf.[5, 6]. Define a map

$$\widetilde{T} : H \longrightarrow H, \ h \mapsto h \leftharpoonup T = \sum < T, h_{(1)} > h_{(2)} =< T, h > \lambda,$$

where λ is the distinguished group-like element of H satifying

$$Tp = T < p, \lambda >, \ \forall p \in H^*.$$

In fact, $\widetilde{T}$ is a map onto 1-dimensional subspace $k\lambda$ of H because $< T, t >= 1$ cf.[6].

Lemma 2.3. Let H be a finite dimensional Hopf algebra and let $\widetilde{T}, \lambda$ be as above. The following sequence is exact:

$$0 \longrightarrow H \leftharpoondown H^{*+} \longrightarrow H \xrightarrow{\widetilde{T}} k\lambda \longrightarrow 0.$$

Proof. It is enough to show that $H \leftharpoondown H^{*+}$ is the kernel of $\widetilde{T}$. The inclusion $H \leftharpoondown H^{*+} \subseteq ker\widetilde{T}$ is easily seen. We show the anti-inclusion. For $h \in H$, there is some $p \in H^*$ such that $h = t \leftharpoondown p$. If $\widetilde{T}(h) = 0$, then $0 = \widetilde{T}(t \leftharpoondown p) = t \leftharpoondown pT$. Since t is the basis of H, we have that $pT = 0$. But T is a left integral of H^*. It follows that $< p, 1 >= 0$, i.e, $p \in H^{*+}$. So we have that $ker\widetilde{T} \subseteq H \leftharpoondown H^{*+}$. ■

Proposition 2.4. Let C be a left H-comodule coalgebra, R the quotient coalgebra C/CH^{*+}. Then
1). $\eta_C : R \longrightarrow h_{C\bowtie H-}(C, C) = e_{C\bowtie H-}(C)$ is a coalgebra isomorphism.
2). $C \simeq h_{C\bowtie H-}(C, C \bowtie H)$ as $(R, C \bowtie H)$-bicomodules.

Proof. 1). It is clear that η_C is a left R-colinear isomorphism. It remains to check that η_C is a coalgebra map. Note that the adjoint map $\theta_C : C \longrightarrow C\Box e_{C\bowtie H-}(C)$ makes C into an $e_{C\bowtie H-}(C)$-comodule cf.[10]. That is, $(1 \otimes \Delta_e)\theta_C = (\theta_C \otimes 1)\theta_C$, where Δ_e is the comultiplication of $e_{C\bowtie H-}(C)$. It follows from the diagram (6) that $\theta_C = (1 \otimes \eta_C)\nu_C$. The above two equalities arrive at the identity for $c \in C$:

$$\sum c_{(1)}\Box\Delta_e\eta_C(\overline{c_{(2)}}) = \sum c_{(1)}\Box\eta_C(\overline{c_{(2)}})\Box\eta_C(\overline{c_{(3)}}).$$

This implies that η_C is a coalgebra map.

2). Let M be $C \bowtie H$ in the diagram (6).
Then $\eta_{C\bowtie H} : \overline{C \bowtie H} \longrightarrow h_{C\bowtie H-}(C, C \bowtie H)$ is an R-colinear isomorphism. We have to show that $\eta_{C\bowtie H}$ is right $C \bowtie H$-colinear and $\overline{C \bowtie H} \simeq C$ as $(R - C \bowtie H)$-bicomodules. Observe that the canonical adjoint map

$$\theta_{C\bowtie H} : \ C \bowtie H \longrightarrow C\Box h_{C\bowtie H-}(C, C \bowtie H)$$

is a $C \bowtie H$-bicolinear map. It follows that the map $\eta_{C\bowtie H} = (\epsilon \otimes 1)\theta_{C\bowtie H}$ is an $(R, C \bowtie H)$-bicolinear map. To show that $\overline{C \bowtie H} \simeq C$ as $(R, C \bowtie H)$-bicomodules, we define a map ψ as follows:

$$\psi : C \bowtie H \longrightarrow C \otimes k\lambda : \ c \bowtie h \mapsto \sum c_{<0>} \otimes < T, c_{<-1>}h > \lambda.$$

It is clear that ψ is a left R-colinear. Moreover, ψ is a right $C \bowtie H$-colinear map. In fact, for $c \bowtie h \in C \bowtie H$, we have

$$\begin{aligned}
& \rho_C(\psi(c \bowtie h)) \\
= \ & \sum c_{<0>(1)} \otimes [c_{<0>(2)} \bowtie S^{-1}(c_{<-1>}) < T, c_{<-2>}h > \lambda] \\
= \ & \sum c_{<0>(1)} \otimes [c_{<0>(2)} \bowtie S^{-1}(c_{<-1>})c_{<-2>}h_{(2)} < T, c_{<-3>}h_{(1)} >] \\
= \ & \sum c_{<0>(1)} \otimes [c_{<0>(2)} \bowtie h_{(2)} < T, c_{<-1>}h_{(1)} >] \\
= \ & \sum c_{(1)<0>} < T, c_{(1)<-1>}c_{(2)<-1>}h_{(1)} > \otimes c_{(2)<0>} \bowtie h_{(2)} \\
= \ & \sum \psi(c_{(1)} \bowtie c_{(2)<-1>}h_{(1)}) \otimes c_{(2)<0>} \bowtie h_{(2)} \\
= \ & (\psi \otimes 1)\Delta(c \bowtie h).
\end{aligned}$$

Now ψ is surjective because:

$$\psi(\sum c_{<0>} \bowtie S^{-1}(c_{<-1>}) < T, c_{<-2>}h > \lambda) = c\otimes < T, t > \lambda = c \otimes \lambda, c \in C.$$

Let $(C \bowtie H)^+$ be $(C \bowtie H) \leftharpoonup H^{*+}$, where the right H^*-module structure of $C \bowtie H$ is given by

$$(c \bowtie h) \leftharpoonup p = \sum c_{<0>} \bowtie c_{<-1>}h_{(2)} < T, c_{<-2>}h_{(1)} >, \quad p \in H^*, c \bowtie h \in C \bowtie H.$$

We show that $ker\psi = (C \bowtie H)^+$. The inclusion $(C \bowtie H)^+ \subseteq ker\psi$ is clear. To show the other inclusion, we need to show that $C \bowtie H$ is a free H^*-module. Let $C \otimes H$ be the free H^*-module with H^*-structure stemming from H. Define a map

$$\zeta :\ C \bowtie H \longrightarrow C \otimes H,\ c \bowtie h \mapsto \sum c_{<0>} \otimes c_{<-1>}h.$$

For $p \in H^*$, we have:

$$\begin{aligned}
\zeta((c \bowtie h) \leftharpoonup p) &= \sum \zeta(c_{<0>} \bowtie h_{(2)} < p, c_{<-1>}h_{(1)} >) \\
&= \sum c_{<0>} \otimes c_{<-1>}h_{(2)} < p, c_{<-2>}h_{(1)} > \\
&= \sum c_{<0>} \otimes (c_{<-1>}h_{(2)}) \leftharpoonup p \\
&= \sum \zeta(c \bowtie h) \leftharpoonup p.
\end{aligned}$$

It is obvious that ζ is an isomorphism. It follows from the fact that $C \otimes H$ is a free H^*-module that $C \bowtie H$ is H^*-free. Now if $x = \sum c_i \bowtie h_i \in ker\psi$, then

$$\begin{aligned}
\psi(x) &= \sum c_{i<0>}\otimes < T, c_{i<-1>}h > \lambda \\
&= \sum c_{i<0>} \otimes c_{i<-1>}h_{(2)} < T, c_{i<-2>}h_{(1)} > \\
&= 0
\end{aligned}$$

This means that $x \leftharpoonup T = 0$ in $C \bowtie H$. Let $\{x_i\}$ be a basis of the free H^*-module $C \bowtie H$. Suppose that $x = \sum x_i \leftharpoonup p_i$. That $0 = x \leftharpoonup T = \sum x_i \leftharpoonup p_iT$ implies that $p_iT = 0, \forall i$. It follows that $p_i \in H^{*+}$ for all i, and hence $x \in (C \bowtie H) \leftharpoonup H^{*+}$. Therefore $\overline{C \bowtie H} \simeq C \otimes k\lambda \simeq C$. ∎

Theorem 2.5. The Morita-Takeuchi context $(C \bowtie H, R, C, C, F, G)$ in (3) is generated by the comodule ${}_{C\bowtie H}C$.

Proof. A M-T context generated by a quasi-finite comodule was constructed by Takeuchi in [10]. The M-T context generated by the quasi-finite comodule ${}_{C\bowtie H}C$ is

$$(C \bowtie H, e_{C\bowtie H-}(C),\ {}_{C\bowtie H}C_{e_{C\bowtie H-}(C)},\ h_{C\bowtie H-}(C, C \bowtie H), f, g),$$

where, f is the canonical map $\theta_{C\bowtie H} : C \bowtie H \longrightarrow C\Box h_{C\bowtie H-}(C, C \bowtie H)$, and g is the composite map:

$$e_{C\bowtie H-}(C) \longrightarrow h_{C\bowtie H-}(C, C \bowtie H\Box_{C\bowtie H}C) \longrightarrow h_{C\bowtie H-}(C, C \bowtie H)\Box_{C\bowtie H}C.$$

By Proposition 2.4, we have that $R \cong e_{C\bowtie H-}(C)$ and $h_{C\bowtie H-}(C, C \bowtie H) \simeq {}_RC_{C\bowtie H}$. It remains to be shown that the following two diagrams are commutative.

$$\begin{tikzcd}
C \bowtie H \arrow[rr, "f"] \arrow[dr, "F"'] & & C\Box h_{C\bowtie H-}(C, C \bowtie H) \arrow[dl, "1\Box\mu"] \\
& C\Box C &
\end{tikzcd} \tag{7}$$

and

$$\begin{tikzcd} R \arrow[rr, "g"] \arrow[dr, "G"'] & & h_{C\bowtie H-}(C, C\bowtie H)\square_{C\bowtie H}C \arrow[dl, "\mu\square 1"] \\ & C\square_{C\bowtie H}C & \end{tikzcd} \tag{8}$$

where μ is the composite isomorphism

$$h_{C\bowtie H-}(C, C\bowtie H) \xrightarrow{\eta^{-1}_{C\bowtie H}} \overline{C\bowtie H} \xrightarrow{\overline{\psi}} C,$$

and $\overline{\psi}$ is induced by the map ψ in the proof of Proposition 2.4. To show the diagram (7), it is enough to verify that the following diagram commutes because we have the commutative diagram (6).

$$\begin{tikzcd} C\bowtie H \arrow[rr, "\nu_{C\bowtie H}"] \arrow[dr, "F"'] & & C\square\overline{C\bowtie H} \arrow[dl, "1\square\overline{\psi}"] \\ & C\square C & \end{tikzcd} \tag{9}$$

In fact, for $c\bowtie h\in C\bowtie H$,

$$\begin{aligned} (1\square\overline{\psi})f(c\bowtie h) &= \sum c_{(1)}\square\overline{\psi}(\overline{c_{(2)}\bowtie h}) \\ &= \sum c_{(1)}\square c_{(2)<0>} < T, c_{(2)<-1>}h > \\ &= F(c\bowtie h) \end{aligned}$$

Now we establish the diagram (8). Note that we have a relation between f and g expressed by commutativity of the following diagram:

$$\begin{tikzcd} C \arrow[r, "\sim"] \arrow[d, "\sim"'] & C\bowtie H\square_{C\bowtie H}C \arrow[d, "f\square 1"] \\ C\square R \arrow[r, "1\square g"] & C\square h_{C\bowtie H-}(C, C\bowtie H)\square_{C\bowtie H}C \end{tikzcd}$$

Explicitly, for $c\in C$, we have the identity:

$$\sum c_{(1)}\square g(\overline{c_{(2)}}) = \sum f(c_{(1)}\bowtie c_{(2)<-1>})\square c_{(2)<0>}.$$

This implies that the map g is determined by f, i.e,

$$g(\overline{c}) = \sum(\epsilon\otimes 1)f(c_{(1)}\bowtie c_{(2)<-1>})\square c_{(2)<0>}, \forall\overline{c}\in R.$$

Now we compute

$$\begin{aligned} (\mu\otimes 1)g(\overline{c}) &= \sum(\mu\otimes 1)[(\epsilon\otimes 1)f(c_{(1)}\bowtie c_{(2)<-1>})\square c_{(2)<0>}] \\ &= \sum(\epsilon\otimes 1\otimes 1)(1\otimes\mu\otimes 1)[f(c_{(1)}\bowtie c_{(2)<-1>})\square c_{(2)<0>}] \\ &= \sum(\epsilon\otimes 1\otimes 1)[F(c_{(1)}\bowtie c_{(2)<-1>})\square c_{(2)<0>}] \\ &= \sum(\epsilon\otimes 1\otimes 1)[c_{(1)}\square c_{(2)<0>} < T, c_{(2)<-1>}c_{(3)<-1>} > \square c_{(3)<0>}] \\ &= \sum c_{(1)<0>}\square c_{(2)<0>} < T, c_{(1)<-1>}c_{(2)<-1>} > \\ &= G(\overline{c}) \end{aligned}$$

where we omitted the subscript $C\bowtie H$ and R of the cotensor product, and we use the commutativity of diagram (7) in the third equality. The proof is complete. ■

Now we can prove:

Corollary 2.6. Let C/R be an H^*-coextension. Then C/R is H^*-Galois if and only if ${}_{C\bowtie H}C$ is a cogenerator.

Proof. It follows from [2, Th.1.2] that C/R is H^*-Galois if and only if the canonical map F is injective. Since The above M-T context is generated by comodule ${}_{C\bowtie H}C$, F is injective if and only if ${}_{C\bowtie H}C$ is a cogenerator cf.[10, 3.2]. ∎

Note that the kernel of the canonical map F is a subcoalgebra of the smash coproduct $C \bowtie H$. If $C \bowtie H$ is a simple coalgebra, then F is injective, and hence ${}_{C\bowtie H}C$ is a cogenerator.

Corollary 2.7. If C/R is an H^*-Galois coextension, then the functor $C\square_R-$ is equivalent to the cohom functor $h_{R-}(C,-)$.

Proof. Let $S = C\square_R-$, $T = C\square_{C\bowtie H}-$. Then the bicolinear maps F and G may be identified with the natural transformations $F:\ I \longrightarrow ST$ and $G:\ I \longrightarrow TS$ cf.[10, 2.4]. If C/R is H^*-Galois then F is an isomorphism, and then the pair $(F^{-1}\ : ST \longrightarrow I, G:\ I \longrightarrow TS)$ yields an adjoint relation $S \dashv T$, i.e, S is a left adjoint functor of T. On the other hand, $h_{R-}(C,-)$ is a left adjoint functor of T because ${}_RC$ is quasi-finite cf.[2, 1.3]. By the uniquess of adjointness the statement holds. ∎

The above result is dual to [11, Th.3.2]. If we call $C\square_R-$ the induction functor and call $h_{R-}(C,)$ the coinduction functor, then induction functor and coinduction functors coincides when the coextension is Galois. To end this section, we give a dualization of the so-called weak structure theorem for Hopf modules in [4].

Theorem 2.8. Let C/R be an H^*-coextension. Then C/R is H^*-Galois if and only if the canonical map $\nu_M:\ M \longrightarrow C\square\overline{M}$ is an isomorphism for every $C \bowtie H$-comodule M.

Proof. Let $M = C \bowtie H$. Then the composite map

$$C \bowtie H \overset{\nu_{C\bowtie H}}{\longrightarrow} C\square\overline{C \bowtie H} \overset{1\square\overline{\psi}}{\longrightarrow} C\square C$$

is exactly the canonical map F in the M-T context. If $\nu_{C\bowtie H}$ is an isomorphism, then F is injective and C/R is H^*-Galois by [2, Th.1.2].

Conversely, suppose that C/R is H^*-Galois. We dualize the diagram in [4, 2.13]. Let β' be the Galois isomorphism:

$$C \otimes H^* \longrightarrow C\square C,\ c \otimes p \mapsto \sum c_{(1)} \leftharpoondown p\square c_{(2)}.$$

Given a $C \bowtie H$-comodule M, β' induces an isomorphism

$$\beta_M:\ M \otimes H^* \longrightarrow C\square M,\ m \otimes p \mapsto \sum m_{(-1)} \leftharpoondown p \otimes m_{(0)}.$$

Denote by δ the following composite isomorphism:

$$M \otimes H^* \otimes H^* \overset{\sigma}{\longrightarrow} M \otimes H^* \otimes H^* \overset{\beta_M\otimes 1}{\longrightarrow} C\square M \otimes H^*$$

where $\sigma(m \otimes p \otimes q) = \sum m \otimes p_{(1)}q \otimes p_{(2)}$. Now it is straightfroward to verify that the following diagram is commutative:

$$\begin{tikzcd}
C\Box(M \otimes H^*) \arrow[r, shift left, "1\otimes\leftharpoonup"] \arrow[r, shift right, "1\otimes 1\otimes\epsilon"'] & C\Box M \arrow[r] & C\Box M \arrow[r] & 0 \\
M \otimes H^* \otimes H^* \arrow[u, "\delta"] \arrow[r, shift left, "\leftharpoonup\otimes 1"] \arrow[r, shift right, "1\otimes\leftharpoonup"'] & M \otimes H^* \arrow[u, "\beta"] \arrow[r, "\leftharpoonup"] & M \arrow[u, "\nu_M"] \arrow[r] & 0
\end{tikzcd}$$

where the uper sequence is exact since C as an R-comodule is coflat (or equivalently injective), and the bottom one is exact because: if $\sum m_i p = 0$ in M, then

$$(\leftharpoonup \otimes 1 - 1\otimes \leftharpoonup)(\sum m_i \leftharpoonup p_{i(1)} \otimes S^*(p_{i(2)}) \otimes p_{i(3)} = \sum m_i \otimes p_i.$$

As β_M and δ are isomorphisms, ν_M is an isomorphism too. ■

3 The cotrace map

Throughtout this section H is a finite dimensional Hopf algebra, and C is a left H-comodule coalgebra. Let T be the left integral of H^* as in the previous section. We define a map from $R = C/C \leftharpoonup H^{*+}$ to C by passage to the quotient:

$$\widetilde{T}: \ R \longrightarrow C, \ \overline{c} \mapsto \sum c_{<0>} < T, c_{<-1>} > .$$

If $c = x \leftharpoonup p, x \in C, p \in H^*$, then $\widetilde{T}(\overline{c}) = \epsilon(p)\widetilde{T}(\overline{x})$. This means that $\widetilde{T}$ is well-defined. It is clear that $\widetilde{T}$ is both left and right R-colinear. The map $\widetilde{T}$ is called the *cotrace map* of C. Let G be the canonical map $R \longrightarrow C\Box_{C\rtimes H}C$ in the M-T context (3). Let D be the image $\widetilde{T}(R)$ of $\widetilde{T}$. One may easily calculate that the following diagram is commutative:

$$\begin{tikzcd}
R \arrow[rr, "G"] \arrow[dr, "\widetilde{T}"'] & & C\Box_{C\rtimes H}C \\
& D \arrow[ur, "\Delta"'] &
\end{tikzcd} \tag{10}$$

Note that in general the comultiplication map Δ can not extend from D to C. Since Δ is injective G is injective if and only if $\widetilde{T}$ is injective.

Proposition 3.1. Let C/R be an H^*-coextension. The following are equivalent:
1). The cotrace map $\widetilde{T}$ is injective.
2). The canonical map G is injective.
3). ${}_{C\rtimes H}C$ (or $C_{C\rtimes H}$) is an injective comodule.
4). The functor $\overline{(-)}$ is exact.
If one of the above conditions holds, then R as a (left or right) R-comodule is a direct summand of C.

Proof. It is sufficient to show that 2) $\Longleftrightarrow$ 3) and this follows from Theorem 2.5 and [10, 3.2]. If $\widetilde{T}$ is injective, then $\widetilde{T}$ splits because R as an R-comodule is injective. ■

Corollary 3.2. Let C/R be an H^*-coextension. If $\widetilde{T}$ is injective, then for any R-comodule N the adjoint map

$$\partial_N: \ \overline{C\Box N} \longrightarrow N, \overline{\sum c_i \Box n_i} \mapsto \sum \epsilon(c_i) n_i$$

is an isomorphism.

Proof. Let ∂ be the canonical map cf.[10, 1.13]

$$h_{C\bowtie H-}(C, C\Box N) \longrightarrow h_{C\bowtie H-}(C, C)\Box N.$$

We have the following commutative diagram:

$$\begin{tikzcd} h_{C\bowtie H-}(C, C\Box N) \arrow[r, "\partial"] & h_{C\bowtie H-}(C, C)\Box N \\ \overline{C\Box N} \arrow[u, "\eta_{C\Box N}"] \arrow[r, "\partial_N"] & N = R\Box N \arrow[u, "\eta_C \Box 1"'] \end{tikzcd}$$

Since ${}_{C\bowtie H}C$ is injective, ∂ is an isomorphism cf. [10, 1.14]. It follows that ∂_N is an isomorphism. ■

Corollary 3.3. Let C/R be an H^*-coextension. The following are equivalent:
1). C/R is H^*-Galois and the cotrace map is injective.
2). $C\Box-$ defines an M-T equivalence between ${}^R\mathbf{M}$ and ${}^{C\bowtie H}\mathbf{M}$.

If R is cocommutative, then the cotrace map is injective when C/R is H^*-Galois cf.[11]. In this case condition 1) in Cor.3.3 may be weakened. In [7], Schneider showed that 2) of Cor.3.3 is equivalent to C/R being Galois and the existence of a 'total integral', i.e, an augmental H^*-linear map from C to H^*. In fact, we have:
Proposition 3.4. Let C/R be an H^*-coextension. The following are equivalent:
1). $\widetilde{T}$ is injective.
2). There exists an H^*-linear map $\phi: \ C \longrightarrow H^*$ such that $\epsilon_{H^*}\phi = \epsilon_C$.

Proof. Suppose that $\widetilde{T}$ is injective. Let π be the section of $\widetilde{T}$ such that $\pi\widetilde{T} = 1_R$. Define a map ϕ as follows:

$$\phi \ C \longrightarrow H^*, \ c \mapsto \sum \epsilon\pi(c \leftharpoonup T_{(2)}) S^{*-1}(T_{(1)}),$$

where T is the left integral of H^* as before. ϕ is augmental because

$$\epsilon\phi(c) = \epsilon\pi(c \leftharpoonup T) = \epsilon\pi\widetilde{T}(c) = \epsilon(c), \forall c \in C.$$

Observe that we have the identity:

$$\sum pT_{(2)} \otimes S^{*-1}(T_{(1)}) = \sum T_{(2)} \otimes S^{*-1}(T_{(1)})p, \ \forall p \in H^* \tag{11}$$

This yields

$$\begin{aligned} \phi(c \leftharpoonup p) &= \sum \epsilon\pi(c \leftharpoonup pT_{(2)}) S^{*-1}(T_{(1)}) \\ &= \sum \epsilon\pi(cT_{(2)}) S^{*-1}(T_{(1)})p \\ &= \phi(c)p \end{aligned}$$

and hence ϕ is H^*-linear.

Conversely, if there is augmental H^*-linear map $\phi:\ C \longrightarrow H^*$, we define a map π as follows:

$$\pi:\ C \longrightarrow R,\ c \mapsto \sum \overline{c_{(1)}} < \phi(c_{(2)}), t >,$$

where t is the left integral of H in the previous section. Note that $< T, t >= 1$. We have

$$\begin{aligned}
\pi\widetilde{T}(\overline{c}) &= \sum\sum \pi(c_{<0>} < T, c_{<-1>} >)\\
&= \sum \overline{c_{<0>(1)}} < \phi(c_{<0>(2)}), t >< T, c_{<-1>} >\\
&= \sum \overline{c_{(1)<0>}} < \phi(c_{(2)<0>}), t >< T, c_{(1)<-1>}c_{(2)<-1>} >\\
&= \sum \overline{c_{(1)} \leftharpoondown T_{(1)}} < \phi(c_{(2)} \leftharpoondown T_{(2)}), t >\\
&= \sum \overline{c_{(1)}} < \phi(c_{(2)})T, t >\\
&= \sum \overline{c_{(1)}\epsilon\phi(c_{(2)})} < T, t >\\
&= \overline{c}
\end{aligned}$$

We have shown that $\widetilde{T}$ is injective. ∎

To end this section we give a functorial characterization of the cotrace map which is dual to [11, Th.2.1].

Theorem 3.5. Let C/R be an H^*-coextension. The cotrace map is injective if and only if the functor $\overline{(-)}$ (cohom functor) is equivalent to the the functor $C\Box_{C\rtimes H}-$ (cotensor functor) via the natural transformation

$$\tau_M:\ \overline{M} \longrightarrow C\Box_{C\rtimes H}M,\ \overline{m} \mapsto \sum m_{(-1)} \leftharpoondown T_{(1)}\Box m_{(0)} \leftharpoondown T_{(2)} = \rho(m \leftharpoondown T).$$

Proof. Suppose that τ_M is an isomorphism for any left $C \rtimes H$- comodule M. Let $M = C$. Then τ_C is exactly the canonical map G, and hence the cotrace map is injective by Proposition 3.1.

Conversely, suppose that $\widetilde{T}$ is injective. For a left $C \rtimes H$-comodule M, we first verify that τ_M is well-defined. To show $\rho_C(m \leftharpoondown T) \in C\Box_{C\rtimes H}M$, it is equivalent to show that $\rho_C(m \leftharpoondown T) \in C\Box_C M$ and

$$\sum p \rightharpoonup (m_{(-1)} \leftharpoondown T_{(1)}) \otimes m_{(0)} \leftharpoondown T_{(2)} = \sum m_{(-1)} \leftharpoondown T_{(1)} \otimes m_{(0)} \leftharpoondown T_{(2)}p$$

for any $p \in H^*$, where

$$p \rightharpoonup c = c \leftharpoondown S^{*-1}(p^\lambda) = \sum c \leftharpoondown S^{*-1}(p_{(1)}) < p_{(2)}, \lambda > \qquad (*)$$

and λ is the group-like element of H mentioned in Section 1. That $\rho_C(m \leftharpoondown T)$ is in $C\Box_C M$ is clear. The equation $(*)$ holds if the following equation holds.

$$\sum T_{(1)}S^{*-1}(p^\lambda) \otimes T_{(2)} = \sum T_{(1)} \otimes T_{(2)}p,\ p \in H^*$$

This is true because

$$\begin{aligned}
\sum T_{(1)}S^{*-1}(p^\lambda) \otimes T_{(2)} &= \sum T_{(1)} < p_{(2)}, \lambda > S^{*-1}(p_{(1)}) \otimes T_{(2)}\\
&= \sum T_{(1)}p_{(2)}S^{*-1}(p_{(1)}) \otimes p_{(3)}\\
&= \sum T_{(1)} \otimes T_{(2)}p.
\end{aligned}$$

It is clear that τ_M is R-colinear. To show that τ_M is an isomorphism, we define a map as follows:

$$\xi_M : \ C\square_{C\rtimes H}M \longrightarrow \overline{M}, \ \xi_M(c \rtimes m) = \epsilon\pi(c)\overline{m},$$

where the map $\pi : \ C \longrightarrow R$ is the section of the cotrace map $\widetilde{T}$. For simplicity, we write $c\square m$ for an element $\sum c_i\square m_i \in C\square_{C\rtimes H}M$. $c\square m$ has to satisfy the following identity in $C \otimes C \rtimes H \otimes M$:

$$\begin{aligned}\sum c_{<0>(1)} \otimes c_{<0>(2)} \rtimes S^{-1}(c_{<-1>})\lambda) \otimes m &= c \otimes \rho_{C\rtimes H}(m)\\ &= \sum c \otimes m_{(-1)} \rtimes m_{(0)<-1>} \otimes m_{(0)<0>}\end{aligned}$$

This yields the equation;

$$\sum c_{<0>(1)} \otimes c_{<0>(2)} < p, S^{-1}(c_{<-1>})\lambda > \otimes m = \sum c \otimes m_{(-1)} \otimes m_{(0)} \leftharpoonup p.$$

Now we have

$$\begin{aligned}&\sum c \otimes m_{(-1)} \leftharpoonup T_{(1)} \otimes m_{(0)} \leftharpoonup T_{(2)}\\ =\ &\sum c_{<0>(1)} \otimes c_{<0>(2)} \leftharpoonup T_{(1)} < T_{(2)}, S^{-1}(c_{<-1>})\lambda > \otimes m\\ =\ &\sum c_{(1)<0>} \otimes c_{(2)<0>} < T, c_{(2)<-1>}S^{-1}(c_{(1)<-1>}c_{(2)<-2>})\lambda > \otimes m\\ =\ &\sum c_{(1)<0>} \otimes c_{(2)} < T, S^{-1}(c_{(1)<-1>})\lambda > \otimes m\\ =\ &\sum c_{(1)<0>} < T, c_{(1)<-1>} > \otimes c_{(2)} \otimes m\\ =\ &\sum c_{(1)} \leftharpoonup T \otimes c_{(2)} \otimes m\end{aligned}$$

where we use the identity $< T, S^{-1}(h)\lambda > = < T, h >$ cf.[2]. It follows from the above equation that we have

$$\begin{aligned}\xi\tau_M(c\square m) &= \sum \epsilon\pi(c)\rho_C(m \leftharpoonup T)\\ &= \sum \epsilon\pi(c)m_{(-1)} \leftharpoonup T_{(1)} \otimes m_{(0)} \leftharpoonup T_{(2)}\\ &= \sum \epsilon\pi(c_{(1)} \leftharpoonup T)c_{(2)} \otimes m\\ &= \sum \epsilon(c_{(1)})c_{(2)} \otimes m\\ &= c \otimes m\end{aligned}$$

that is, $\xi\tau_M = I$. To show that $\tau_M\xi = I$ is easy. ■

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