Limit-point criteria for superlinear differential equations

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Abstract

For a nonlinear differential equation $x'' + a(t)f(x) = 0$, we obtain limit-point criteria by proving first stronger results which guarantee nonexistence of nontrivial bounded (uniformly continuous) L^2-solutions under milder restrictions on the coefficient $a(t)$ and nonlinearity $f(x)$.

Introduction

The limit-circle/limit-point classification originates from the celebrated paper by Weyl [18] and is related to the problem of existence of square integrable solutions of the second order linear differential equation

$$\mathcal{L}x \stackrel{\text{def}}{=} -(p(t)x')' + q(t)x = \lambda x, \qquad \lambda \in \mathbb{C}, \tag{1}$$

on the interval $I = [t_0, \infty)$. The operator $\mathcal{L}$ is of the *limit-circle type* at infinity if for a particular complex number λ_0 every solution $\varphi(t)$ of Eq. (1) satisfies

$$\int_{t_0}^{\infty} |\varphi(t)|^2 \, dt < +\infty,$$

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otherwise, $\mathcal{L}$ is said to be of the *limit-point type* at infinity. Later on, this definition has been extended to second order nonlinear differential equations and higher order nonlinear equations (see, for instance, Atkinson [1], Bartušek et al [2, 3, 4], Graef [8], Graef and Spikes [9], and the references cited there).

As it has been mentioned by Atkinson [1], if $a(t)$ is suitably small for large t, then the solutions of Eq. (5) "will be approximately linear, or constant, so that the equation will not merely be in the limit-point condition, but will have no non-trivial solution in $L^2(T,+\infty)$; this situation is related to the theory of the essential spectrum." In fact, efforts of numerous researchers have been aimed at proving stronger results establishing nonexistence of square integrable solutions.

In 1950, Aurel Wintner [19] established that the linear differential equation

$$x'' + a(t)x = 0, \tag{2}$$

where $a(t)$ is a continuous function, cannot have nontrivial solutions of class $L^2((0,\infty),\mathbb{R})$ provided that

$$\int_0^\infty t^3\,|a(t)|^2\,dt < +\infty. \tag{3}$$

Other limit-point criteria for Eq. (2) have been given by Levinson [13] and the present authors [15], and for linear differential equation

$$(p(t)x')' + q(t)x = 0$$

by Kauffman et al [12], Patula and Waltman [16], Wong [21], Wong and Zettl [22]. An analogue of Wintner's result for the nonlinear differential equation

$$x'' + a(t)x^p = 0, \qquad p \geq 1, \tag{4}$$

has been derived by Suyemoto and Waltman [17], who proved that if condition (3) is satisfied, Eq. (4) cannot have solutions of class $L^{2p}((0,\infty),\mathbb{R})$. Further extensions of these results to nonlinear differential equations

$$x'' + a(t)f(x) = 0 \tag{5}$$

and

$$x'' + f(t,x) = 0, \tag{6}$$

under assumption (3) and some additional conditions on nonlinear functions $f(x)$ and $f(t,x)$ have been obtained by Burlak [5], Detki [6], Elias [7], Grammatikopoulos and Kulenovic [10], Hallam [11], and Wong [20, 21]. We mention also the recent paper by the present authors [14], where nonexistence of nontrivial square integrable solutions of the n-th order nonlinear differential equation

$$u^{(n)} + f(t,u) = 0$$

has been established as a by-product of the estimate for the rate of decay of the L^2 norm of solutions of the perturbed equation

$$u^{(n)} + f(t,u) = b(t).$$

For the convenience of the reader, we adapt here the main result of Grammatikopoulos and Kulenovic [10].

Theorem 1 (Grammatikopoulos and Kulenovic, 1981). *Let conditions*

$$\int_{\tau}^{\infty} t^2 \left|a(t)\right|^2 dt < +\infty \tag{7}$$

and

$$\int_{\tau}^{\infty} \left(\int_{t}^{\infty} \left(\int_{s}^{\infty} \left|a(\tau)\right|^2 d\tau \right)^{1/2} ds \right)^2 dt < +\infty \tag{8}$$

be satisfied. Then Eq. (2) cannot have nontrivial L^2-solutions.

The following criterion has been established for the nonlinear differential equation (5) by Wong [20, Theorem 1].

Theorem 2 (Wong, 1967). *Suppose that the function $f(u)$ is continuous on $\mathbb{R}$ and such that*

(i) *$u \neq 0$ implies that $f(u) \neq 0$;*
(ii) *$\liminf_{|u| \to +\infty} f(u) > 0$;*
(iii) *for every L^2-solution $x(t)$ of Eq. (5)*

$$\limsup_{t \to +\infty} \left(\int_{t}^{\infty} \left|f(x(s))\right|^2 ds \right)^{-1} \left(\int_{t}^{\infty} \left|x(s)\right|^2 ds \right) > 0.$$

Then Eq. (5) cannot have a nontrivial solution $x(t)$ for which

$$\int_{\tau}^{\infty} \left|f(x(t))\right|^2 dt < +\infty.$$

An essential feature of the proofs of these and related results is the following property:

(P) any nontrivial L^2-solution $x(t)$ of Eq. (2) should satisfy

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} x'(t) = 0.$$

This implies, in particular, that $x(t)$ must be eventually bounded, as well as its derivative $x'(t)$. This type of asymptotic behavior of solutions of Eq. (2) is a direct consequence of condition (7).

In this paper, we establish nonexistence of square integrable solutions for Eq. (5) by eliminating condition (7) from the hypotheses of Theorem 1 and using only assumption (8). Furthermore, in contrast to conditions specifically tailored to exclude existence of L^2-solutions (see, for instance, Theorem 2 and other results reported in Wong [20, 21]), our restriction on $f(x)$ is just a growth assumption. Consequently, we obtain two limit-point criteria for Eq. (5) as corollaries to stronger nonexistence results.

1 Auxiliary results

In what follows, we always assume that

(A1) $x(t)$, $a(t)$, and $f(x)$ are real-valued functions;

(A2) f is continuous on $\mathbb{R}$;

(A3) for all $x \in \mathbb{R}$, f satisfies

$$|f(x)| \leq M\,|x|^p\,,$$

where $M > 0$ and $p \geq 1$ are constants.

Before stating and proving main results of this paper, we would like to comment on several important aspects of the problem. First, we note that a milder version of condition (7), that is,

$$a(t) \in L^2((T, +\infty); \mathbb{R}), \tag{9}$$

has been used by Grammatikopoulos and Kulenovic [10] to show that an L^2-solution $x(t)$ of Eq. (2) satisfies

$$\lim_{t\to+\infty} x'(t) = 0.$$

Since condition (8) yields (9), it is natural to expect boundedness of any L^2-solution $x(t)$ of Eq. (5) provided its derivative $x'(t)$ is bounded. The following proposition confirms our conjecture.

Lemma 3. *Let $x(t)$ be a continuously differentiable function that belongs to the class $L^p((T, +\infty), \mathbb{R})$ for some $p > 0$, and assume that its derivative $x'(t)$ is bounded. Then $x(t)$ is also bounded.*

Proof. Suppose, contrary to our claim, that $x(t)$ is unbounded. Since $x(t)$ is continuous, there exists an increasing sequence of real numbers $\{t_n\}_{n\geq 1}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$|x(t_n)| = n.$$

Furthermore, there exists another sequence of real numbers $\{p_n\}_{n\geq 1}$ such that $p_n \to \infty$ as $n \to \infty$, $t_n < p_n$ for all $n \geq 1$,

$$|x(p_n)| = n/2, \qquad |x(t)| \geq n/2,$$

and

$$\operatorname{sgn}\ x(t) = \operatorname{sgn}\ x(t_n)$$

for all $t \in [t_n, p_n]$.

It is not difficult to see that for $n \geq 1$ one has

$$(n/2)^p\,(p_n - t_n) \leq \int_{t_n}^{p_n} |x(t)|^p\,dt \leq \|x\|_{L^p}^p < +\infty,$$

which yields

$$\lim_{n\to+\infty} (p_n - t_n) = 0. \tag{10}$$

By the Mean Value Theorem, we conclude that

$$n/2 = |x(p_n) - x(t_n)| = |x'(\xi_n)|\,(p_n - t_n), \qquad \xi_n \in (t_n, p_n),$$

for all $n \geq 1$. If $N > 0$ is an upper bound for $|x'(t)|$, then

$$1 \leq n \leq 2N(p_n - t_n),$$

which contradicts (10). Therefore, our assumption is wrong and $x(t)$ is bounded. ■

Remark 4. *Dealing with L^2-solutions of Eq. (5), we may restrict ourselves only to those satisfying condition*

$$\lim_{t\to+\infty} x'(t) = 0.$$

Then, by Lemma 3, an L^2-solution $x(t)$ must be bounded. Lemma 3 and the proof of Theorem 8 prompt how property (P) can be "recovered" from its second part, that is, from the boundedness of the derivative $x'(t)$.

Remark 5. *We note that Lemma 3 can be enhanced by replacing continuous differentiability of $x(t)$ and boundedness of $x'(t)$ with the uniform continuity of the function $x(t)$. The proof of this fact is very similar to that of Lemma 3 and thus is omitted. As a consequence, we can obtain corollary to our main result (Theorem 8) by replacing expression "bounded L^2-solutions" with "uniformly continuous L^2-solutions". This is another way of recovering property (P) since any continuous function $x(t)$ such that $\lim_{t\to+\infty} x(t)$ exists in $\mathbb{R}$ is* a fortiori *uniformly continuous.*

Now we restrict our attention to L^2-solutions $x(t)$ of Eq.(5) satisfying $\lim_{t\to+\infty} x(t) = 0$ and such that $x'(t)$ is uniformly continuous. The following proposition prompts one more way to recover the property (P).

Lemma 6. *Assume that a function $x(t)$ is of class $C^1([T,+\infty),\mathbb{R})$ and satisfies $\lim_{t\to+\infty} x(t) = l$, where $l \in \mathbb{R}$. Then, $\lim_{t\to+\infty} x'(t) = 0$ if and only if the derivative $x'(t)$ is uniformly continuous.*

Proof. The direct implication is obvious. To prove the converse implication, assume, for the sake of contradiction, that there exist a real number $\varepsilon_0 > 0$ and an increasing sequence of real numbers $\{t_n\}_{n\geq 1}$ such that $t_n \to +\infty$ as $n \to +\infty$ and $|x'(t_n)| \geq \varepsilon_0$. We can pick a subsequence of the sequence $\{t_n\}_{n\geq 1}$, also denoted by $\{t_n\}_{n\geq 1}$, such that $t_{n+1} - t_n > 1$ for all $n \geq 1$. Since $x'(t)$ is uniformly continuous, one can choose an $\eta = \eta(\varepsilon_0) > 0$ such that

$$|x'(t) - x'(s)| < \varepsilon_0/2$$

for all t, $s \geq T$ satisfying $|t - s| < \eta$. Select now a real number ε so that $0 < \varepsilon < \min\{1/2, \eta\}$, and define a sequence of open intervals $V_n = (t_n - \varepsilon, t_n + \varepsilon)$, $n \geq 1$. It follows from the choice of $\{t_n\}_{n\geq 1}$ and ε that $V_n \cap V_m = \emptyset$ for all $m \neq n$.

Let $t \in V_n$, and $s = t_n$, then one has $|x'(t) - x'(t_n)| < \varepsilon_0/2$, which yields

$$|x'(t)| > \varepsilon_0/2, \quad t \in V_n,\ n \geq 1. \tag{11}$$

Applying the Mean Value Theorem, we conclude that

$$x'(t_n + \varepsilon/2) - x'(t_n - \varepsilon/2) = \varepsilon x'(s_n), \tag{12}$$

where $s_n \in (t_n - \varepsilon/2, t_n + \varepsilon/2)$ for all $n \geq 1$. Since the left-hand member of (12) vanishes as $n \to +\infty$, this contradicts (11). The proof is complete. ∎

Example 7. *To show that the uniform continuity is essential for Lemma 6, consider the function $x : [0, +\infty) \to [0, +\infty)$ defined by*

$$x(t) = \begin{cases} n^{3/4}\left[\exp\left((t-n)^2(\alpha(n)-t)^2\right)-1\right], & \text{if } t \in [n, \alpha(n)], \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha(n) = n + 2n^{-1/4}$ and $n \geq 17$. A straightforward computation gives

$$\lim_{t\to+\infty} x(t) = 0,$$

but since $x'(t)$ is not uniformly continuous,

$$\lim_{n\to+\infty} x'(n + 2^{-1}n^{-1/4}) = 3/2 \neq 0.$$

2 Nonexistence of L^2 solutions

Define the function $\varphi(t)$ by

$$\varphi(t) \stackrel{\text{def}}{=} \int_t^\infty \left(\int_\tau^\infty |a(s)|^2\, ds\right)^{1/2} d\tau, \qquad t \geq T.$$

Theorem 8. *Assume that the function $\varphi(t)$ is of class $L^2((T, +\infty), \mathbb{R})$. Then Eq. (5) cannot have nontrivial bounded square integrable solutions.*

Proof. Assume that $x(t)$ is a bounded L^2-solution of Eq. (5), that is, there exists a positive constant N such that

$$|x(t)| \leq N, \qquad t \geq T.$$

Integrating Eq. (5) from t_1 to t_2, where $t_2 \geq t_1 \geq T$, and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |x'(t_2) - x'(t_1)| &\leq M \int_{t_1}^{t_2} |a(t)|\, |x(t)|^p\, dt \\ &\leq MN^{p-1} \left(\int_{t_1}^{t_2} |a(s)|^2\, ds\right)^{1/2} \left(\int_{t_1}^{\infty} |x(s)|^2\, ds\right)^{1/2} \\ &\leq MN^{p-1} \|x\|_{L^2} \left(\int_{t_1}^{t_2} |a(s)|^2\, ds\right)^{1/2}. \end{aligned}$$

It follows from the latter inequality that

$$\lim_{t\to+\infty} x'(t) \text{ exists in } \mathbb{R}.$$

Furthermore, using the l'Hôpital's rule, we conclude that

$$\lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} \frac{x(t)}{t} = 0.$$

Thus, we have

$$x'(t) = \int_t^{\infty} a(s) f(x(s)) ds, \qquad t \geq T. \tag{13}$$

Integration of (13) from t_1 to t_2, where $t_2 \geq t_1 \geq T$, and application of the Cauchy-Schwarz inequality yield

$$\begin{aligned} |x(t_2) - x(t_1)| &\leq M N^{p-1} \int_{t_1}^{t_2} \left(\int_t^{\infty} |a(s)| \, |x(s)| \, ds \right) dt \\ &\leq M N^{p-1} \|x\|_{L^2} \int_{t_1}^{\infty} \left(\int_t^{\infty} |a(s)|^2 \, ds \right)^{1/2} dt. \end{aligned}$$

It follows from the latter inequality and assumption of the theorem that

$$\lim_{t \to +\infty} x(t) = 0$$

and, correspondingly,

$$x(t) = - \int_t^{\infty} \left(\int_{\tau}^{\infty} a(s) f(x(s)) ds \right) d\tau, \qquad t \geq T. \tag{14}$$

Finally, integrating Eq. (14) and applying again the Cauchy-Schwarz inequality, we conclude that for $t \geq T$

$$\begin{aligned} \int_t^{\infty} |x(s)|^2 \, ds \leq M^2 N^{2(p-1)} & \int_t^{\infty} \left(\int_{\tau}^{\infty} \left(\int_u^{\infty} |a(s)|^2 \, ds \right)^{1/2} \right. \\ & \times \left. \left(\int_u^{\infty} |x(s)|^2 \, ds \right)^{1/2} du \right)^2 d\tau \\ \leq M^2 N^{2(p-1)} & \left(\int_t^{\infty} \varphi^2(s) ds \right) \left(\int_t^{\infty} |x(s)|^2 \, ds \right). \end{aligned}$$

It follows from the latter inequality and assumption on the function $\varphi(t)$ that, for t large enough, one has

$$\int_t^{\infty} |x(s)|^2 \, ds = 0.$$

This means that any bounded L^2-solution of Eq. (5) vanishes eventually. Hence, Eq. (5) cannot have nontrivial bounded L^2-solutions, and thus is in the limit point case. ∎

Corollary 9. *Suppose that all solutions of Eq. (5) are bounded. Assume further that the function $\varphi(t)$ is of class $L^2((T, +\infty), \mathbb{R})$. Then Eq. (5) is in the limit point case.*

We note that assumption of boundedness of solutions in the statement of Theorem 8 can be omitted provided the hypothesis on $a(t)$ is suitably modified.

Theorem 10. *Let $1 \leq p < 2$, and assume that condition*

$$\int_{\tau}^{\infty} \left(\int_t^{\infty} \left(\int_s^{\infty} |a(\tau)|^q \, d\tau \right)^{1/q} ds \right)^2 dt < +\infty$$

is satisfied, where $q > 0$ and $p/2 + 1/q = 1$.

Then Eq. (5) cannot have nontrivial L^2-solutions.

Proof. The proof follows the same lines as that of Theorem 8, but the Hölder inequality is applied instead of the Cauchy-Schwarz inequality. Assume that $x(t)$ is an L^2-solution of Eq. (5). Then, obviously, $|x(t)|^p$ belongs to the class $L^{2/p}((T,+\infty),\mathbb{R})$. Integrating Eq. (5) from t_1 to t_2, where $t_2 \geq t_1 \geq T$, and applying the Hölder inequality, we obtain

$$\begin{aligned}|x'(t_2)-x'(t_1)| &\leq M\int_{t_1}^{t_2}|a(t)|\,|x(t)|^p\,dt\\ &\leq M\left(\int_{t_1}^{t_2}|a(t)|^q\,dt\right)^{1/q}\left(\int_{t_1}^{\infty}|x(t)|^2\,dt\right)^{p/2}\\ &\leq M\left(\|x\|_{L^2}\right)^{p/2}\left(\int_{t_1}^{\infty}|a(t)|^q\,dt\right)^{1/q},\end{aligned}$$

which implies that $\lim\limits_{t\to+\infty}x'(t)=0$. As in Theorem 8, we arrive at (14). Integrating Eq. (14) and applying the Hölder inequality once again, we conclude that for $t\geq T$

$$\begin{aligned}\int_t^\infty |x(s)|^2\,ds \leq \int_t^\infty &\left(\int_\tau^\infty\left(\int_u^\infty |a(s)|^q\,ds\right)^{1/q}du\right)^2 d\tau\\ &\times\left(\int_t^\infty |x(s)|^2\,ds\right)^p.\end{aligned}$$

Since $p\geq 1$, we can consider $t\geq T$ sufficiently large to guarantee that

$$\left(\int_t^\infty |x(s)|^2\,ds\right)^p \leq \int_t^\infty |x(s)|^2\,ds.$$

The rest of the proof resembles that of Theorem 8 and is omitted. ∎

Corollary 11. *Under assumptions of Theorem 10, Eq. (5) is in the limit point case.*

Remark 12. *We conclude by noting that it is not difficult to obtain extensions of the results reported in this paper to nonlinear differential equations (6) and*

$$x''+f(t,x,x')=0$$

where the nonlinearity satisfies, correspondingly,

$$|f(t,x)|\leq\alpha(t)\,|x|^p \qquad \text{and} \qquad |f(t,x,x')|\leq\beta(t)\,|x|^p\,.$$

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