THE CHROMATIC POLYNOMIAL OF $\mathbf{P_2 \times P_n}$ AND $\mathbf{C_3 \times P_n}$

Thomas J. Pfaff and Jasmine Walker

Abstract. We provide formulas for the chromatic polynomials of the Cartesian product of a path on two vertices with a path on n vertices, and a cycle on three vertices with a path on n vertices. The result for the Cartesian product of paths involves the trinomial coefficients. We define a new sequence of numbers, $CP(n,i)$, which are part of the result for the chromatic polynomial of the Cartesian product of the three cycle and the path. In both cases the chromatic polynomial is factored in the form $\lambda(\lambda - 1)$.

1. Introduction. In this paper we provide an instance of the trinomial coefficients, which appear as the coefficients of the chromatic polynomial of $P_2 \times P_n$ in a particular form. We also give a new sequence of numbers that appear as coefficients of the chromatic polynomial of $C_3 \times P_n$.

Let P_n be the path on n vertices and let C_3 be the cycle with 3 vertices. The graph $P_2 \times P_n$, the Cartesian product of the two graphs, is defined by having vertex set $V(P_2) \times V(P_n)$ and there is an edge between (u,v) and (u',v') if either $u = u'$ and $vv' \in E(P_2)$ or $v = v'$ and $uu' \in E(P_n)$. In this case, $P_2 \times P_n$ is known as a ladder. The graph of $C_3 \times P_n$ is defined similarly.

The chromatic polynomial of graph G, $P_G(\lambda)$ gives the number of ways the graph G can be colored with λ colors such that no two adjacent vertices are colored the same. Brualdi [1] provides a number of interesting basic facts about the chromatic polynomial. For instance, the signs of the coefficients of the chromatic polynomial alternate, with the leading coefficient having a positive sign. Connected graphs have nonzero λ terms, but graphs that are not connected have a zero λ term. The constant term is always zero. The absolute value of the λ^{n-1} term is the number of edges of the graph.

One particular chromatic polynomial we will use is that of a tree. Starting with any vertex say v, we can color v with any of the λ colors. Any vertex that is adjacent to v can be colored with any of $\lambda - 1$ colors, leaving out the color that was used to color v. We can continue in this fashion to get the chromatic polynomial of a tree with n vertices, $P_{T_n}(\lambda) = \lambda(\lambda - 1)^{n-1}$.

Here we will focus our attention on $P_{P_2 \times P_n}(\lambda)$ and $P_{C_3 \times P_n}(\lambda)$. As noted by Chia [2], little in general is known about the chromatic polynomial of the Cartesian product of two graphs. Although in our case, we are presenting a new form of $P_{P_2 \times P_n}(\lambda)$ and $P_{C_3 \times P_n}(\lambda)$ as formulas since these chromatic polynomials do exist. It is worth noting that the chromatic polynomial is a specific case of the Potts model partition function in physics which is discussed in [3], while a specific calculation of the ladder can be found in [4]. We will use the deletion contraction algorithm to calculate our chromatic polynomials.

To see how this algorithm works, let G be a graph with edge α connecting vertices a and b. If we delete the edge from G, we call this graph $G_{\ominus\alpha}$. If we contract the edge, in other words we identify the vertices a and b and remove the loop α creates and any multiple edges, we denote the graph by $G_{\otimes\alpha}$; see Figure 1. The point here is that the graphs $G_{\ominus\alpha}$ and $G_{\otimes\alpha}$ are simpler than the original graph G, having either fewer vertices or fewer edges. Now when coloring $G_{\ominus\alpha}$, since a and b are no longer adjacent, they can be either the same color or not. Notice that $P_G(\lambda)$ counts the number of ways that $G_{\ominus\alpha}$ can be colored with λ colors when a and b are distinct colors. On the other hand, $P_{G_{\otimes\alpha}}(\lambda)$ counts the number of ways that $G_{\ominus\alpha}$ can be colored with λ colors when a and b are colored with the same color. Hence, $P_{G_{\ominus\alpha}}(\lambda) = P_G(\lambda) + P_{G_{\otimes\alpha}}(\lambda)$, or, in the form that we use, $P_G(\lambda) = P_{G_{\ominus\alpha}}(\lambda) - P_{G_{\otimes\alpha}}(\lambda)$. Note particularly the negative sign that arises in this formula.

This algorithm is typically used repeatedly to reduce the original graph down to graphs whose chromatic polynomial can be calculated. For instance, *Mathematica* will compute chromatic polynomials but seems to always reduce the graph to a set of null graphs.

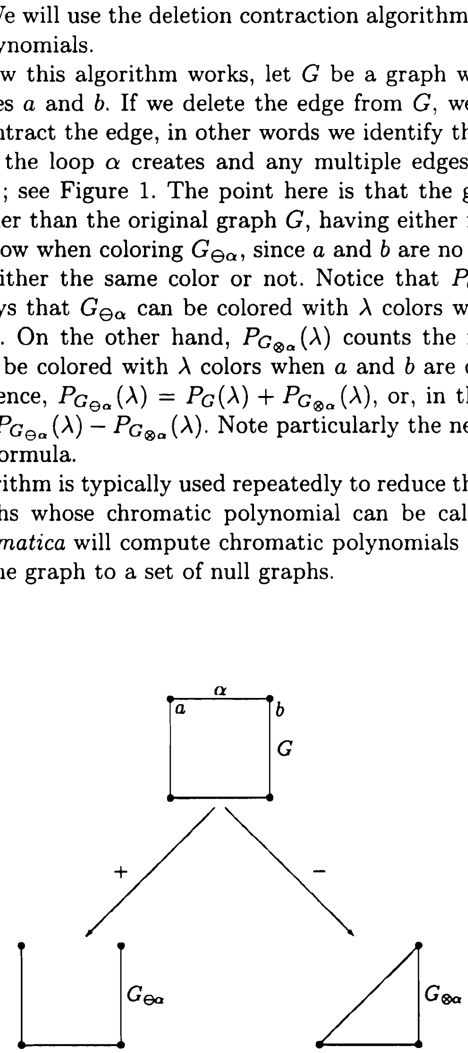

FIGURE 1. The result of deleting and contracting the edge α from G. The + and − signs on the arrows represent the fact that $P_G(\lambda) = P_{G_{\otimes\alpha}}(\lambda) - P_{G_{\ominus\alpha}}(\lambda)$.

As mentioned we will relate our chromatic polynomials to two sequences of numbers. The first are the trinomial coefficients (A027907 in [5]). They can be defined as $T(n, i)$ for $n \geq 0$ and $0 \leq i \leq 2n$, with $T(0, 0) = 1$

and $T(n,i) = T(n-1,i)+T(n-1,i-1)+T(n-1,i-2)$. Weisstein [6] shows that the trinomial coefficients can be found by expanding $(1+x+x^2)^n$ and can also be found by

$$T(n,i) = \sum_{j=0}^{n} \frac{n!}{j!(j+i-n)!(2n-2j-i)!}.$$

We define the second sequence in a similar manner as the trinomial coefficients. Let $CP(n,i)$ be defined for $n \geq 0$ and $0 \leq i \leq 3n+1$, with $CP(0,0) = CP(0,1) = 1$ and $CP(n,i) = CP(n-1,i) + 3CP(n-1,i-1) + 5CP(n-1,i-2) + 4CP(n-1,i-3)$ (the CP stands for cycle path since they arise in the chromatic polynomial of $C_3 \times P_n$). When $n < 0$ we let $CP(n,i) = 0$. Here is a small table of the numbers $CP(n,i)$ (A123531 in [5]):

$n\backslash i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1												
1	1	4	8	9	4									
2	1	7	25	57	87	89	56	16						
3	1	10	51	171	411	735	986	977	684	304	64			
4	1	13	86	378	1219	3027	5930	9254	11485	11185	8304	4448	1536	256

2. Chromatic Polynomial of $\mathbf{P_2 \times P_n}$ with $\mathbf{T(n,i)}$. We can now state and prove the result involving the chromatic polynomial of $P_2 \times P_n$ and the trinomial coefficients $T(n,i)$. The proof is an induction proof using the deletion contraction algorithm. In this case, two deletions and contractions reduces $P_2 \times P_n$ to $P_2 \times P_{n-1}$ with a "tail," which we can deal with and is what provides the recursion that involves the trinomial coefficients.

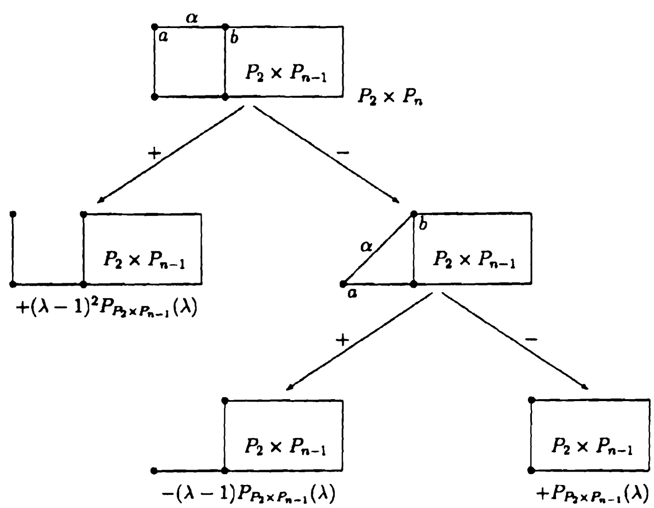

FIGURE 2. The two deletion and contraction steps needed to reduce $P_2 \times P_n$ to $P_2 \times P_{n-1}$ with a "tail". Beneath the final graphs is the chromatic polynomial for that graph.

Theorem 1. For $n \geq 1$ we have

$$P_{P_2\times P_n}(\lambda) = \sum_{i=1}^{2n-1} (-1)^{i+1} T(n-1, i-1)\lambda(\lambda-1)^i. \tag{1}$$

Proof. We first calculate $P_{P_2\times P_1}(\lambda)$. Since $P_2 \times P_1$ is just a tree with two vertices we have $P_{P_2\times P_1}(\lambda) = T(0,0)\lambda(\lambda-1)$, since $T(0,0) = 1$, which is (1) with $n = 1$. Assume (1) holds with $n-1$. Using the deletion contraction algorithm as in Figure 2, we obtain

$$P_{P_2\times P_n}(\lambda) = (\lambda-1)^2 P_{P_2\times P_{n-1}}(\lambda) - (\lambda-1)P_{P_2\times P_{n-1}}(\lambda) + P_{P_2\times P_{n-1}}(\lambda). \tag{2}$$

Recall that $T(n,i) = 0$ whenever $i < 0$ or $i > 2n$ so that

$$\begin{aligned}(\lambda-1)^2 P_{P_2\times P_{n-1}}(\lambda) &= \sum_{i=1}^{2n-3}(-1)^{i+1}T(n-2,i-1)\lambda(\lambda-1)^{i+2}\\ &= \sum_{i=3}^{2n-1}(-1)^{i+1}T(n-2,i-3)\lambda(\lambda-1)^{i}\\ &= \sum_{i=1}^{2n-1}(-1)^{i+1}T(n-2,i-3)\lambda(\lambda-1)^{i},\end{aligned}$$

$$\begin{aligned}(\lambda-1)^1 P_{P_2\times P_{n-1}}(\lambda) &= \sum_{i=1}^{2n-3}(-1)^{i+1}T(n-2,i-1)\lambda(\lambda-1)^{i+1}\\ &= -\sum_{i=2}^{2n-2}(-1)^{i+1}T(n-2,i-2)\lambda(\lambda-1)^{i}\\ &= -\sum_{i=1}^{2n-1}(-1)^{i+1}T(n-2,i-2)\lambda(\lambda-1)^{i},\end{aligned}$$

and

$$\begin{aligned}P_{P_2\times P_{n-1}}(\lambda) &= \sum_{i=1}^{2n-3}(-1)^{i+1}T(n-2,i-1)\lambda(\lambda-1)^{i}\\ &= \sum_{i=1}^{2n-1}(-1)^{i+1}T(n-2,i-1)\lambda(\lambda-1)^{i}.\end{aligned}$$

Putting this all together using $T(n-1,i-1) = T(n-2,i-3) + T(n-$

$2, i-2) + T(n-2, i-1)$ we see that

$$P_{P_2 \times P_n}(\lambda)$$

$$= \sum_{i=1}^{2n-1} (-1)^{i+1}[T(n-2, i-3) + T(n-2, i-2) + T(n-2, i-1)]\lambda(\lambda-1)^i$$

$$= \sum_{i=1}^{2n-1} (-1)^{i+1} T(n-1, i-1)\lambda(\lambda-1)^i.$$

There are other ways to calculate $P_{P_2 \times P_n}(\lambda)$. For instance, Theorem 1.3.2 in [3] states that given two graphs G_1 and G_2 and $G \in \mathcal{G}[G_1 \cup_r G_2]$ then

$$P_G(\lambda) = \frac{P_{G_1}(\lambda) P_{G_2}(\lambda)}{P_{K_r}(\lambda)}.$$

Here, we have K_r representing the complete graph on r vertices, and $\mathcal{G}[G_1 \cup_r G_2]$ representing the set of graphs that are obtained by identifying a K_r subgraph in both G_1 and G_2, or in other words, gluing the two K_r subgraphs together. We can use this theorem repeatedly in the case when $r = 2$, recognizing that $P_2 \times P_n$ can be built by gluing C_4 graphs together at the K_2 rungs of the ladder to show that

$$P_{P_2 \times P_n}(\lambda) = \frac{((\lambda-1)^4 + (\lambda-1))^{n-1}}{(\lambda(\lambda-1))^{n-2}}, \tag{3}$$

but then we lose the clear relationship to the trinomial coefficients.

3. Chromatic Polynomial of $\mathbf{C_3 \times P_n}$ with $\mathbf{CP(n, i)}$. The proof here follows in the same manner as the previous proof. We do not provide a figure here because there are 12 deletions and contractions, and hence, the coefficients in the recursion of $CP(n, i)$ add to 13, to reduce $C_3 \times P_n$ to $C_3 \times P_{n-1}$ with a "tail." In general, if we view $C_3 \times P_n$ as a double ladder with an extra loop at each step, then we start the deletions and contractions by deleting and contracting the top extra loop and then work to separate the top step from the rest of the ladder until we are left with our $C_3 \times P_{n-1}$ with a "tail."

Theorem 2. For $n \geq 2$ we have

$$P_{C_3 \times P_n}(\lambda) = \sum_{i=1}^{3n-1} (-1)^{i+1} CP(n-1, i-1)\lambda(\lambda-1)^{3n-i}. \tag{4}$$

Proof. First, $C_3 \times P_1$ is simply a triangle. One deletion and contraction shows that $P_{C_3 \times P_1}(\lambda) = \lambda(\lambda-1)^2 - \lambda(\lambda-1)$. In the same manner as Figure 2 and equation (2) we obtain

$$\begin{aligned} P_{C_3 \times P_n}(\lambda) &= (\lambda-1)^3 P_{C_3 \times P_{n-1}}(\lambda) - 3(\lambda-1)^2 P_{C_3 \times P_{n-1}}(\lambda) \\ &\quad + 5(\lambda-1) P_{C_3 \times P_{n-1}}(\lambda) - 4 P_{C_3 \times P_{n-1}}(\lambda). \end{aligned} \tag{5}$$

Recall that $CP(n-2, i) = 0$ whenever $i < 0$ or $i > 3n-5$. Following the same type of calculation as in the proof of Theorem 1 we have

$$\begin{aligned} (\lambda-1)^3 P_{C_3 \times P_{n-1}}(\lambda) &= \sum_{i=1}^{3n-4} (-1)^{i+1} CP(n-2, i-1)\lambda(\lambda-1)^{3n-i} \\ &= \sum_{i=1}^{3n-1} (-1)^{i+1} CP(n-2, i-1)\lambda(\lambda-1)^{3n-i}, \end{aligned}$$

$$\begin{aligned} -3(\lambda-1)^2 P_{C_3 \times P_{n-1}}(\lambda) &= -\sum_{i=1}^{3n-4} (-1)^{i+1} 3CP(n-2, i-1)\lambda(\lambda-1)^{3n-i-1} \\ &= -\sum_{i=2}^{3n-3} (-1)^{i} 3CP(n-2, i-2)\lambda(\lambda-1)^{3n-i} \\ &= \sum_{i=1}^{3n-1} (-1)^{i+1} 3CP(n-2, i-2)\lambda(\lambda-1)^{3n-i}, \end{aligned}$$

$$\begin{aligned} 5(\lambda-1) P_{C_3 \times P_{n-1}}(\lambda) &= \sum_{i=1}^{3n-4} (-1)^{i+1} 5CP(n-2, i-1)\lambda(\lambda-1)^{3n-i-2} \\ &= \sum_{i=3}^{3n-2} (-1)^{i+1} 5CP(n-2, i-3)\lambda(\lambda-1)^{3n-i} \\ &= \sum_{i=1}^{3n-1} (-1)^{i+1} 5CP(n-2, i-3)\lambda(\lambda-1)^{3n-i}, \end{aligned}$$

and

$$
\begin{aligned}
-4P_{C_3\times P_{n-1}}(\lambda) &= -\sum_{i=1}^{3n-4}(-1)^{i+1}4CP(n-2,i-1)\lambda(\lambda-1)^{3n-i-3} \\
&= -\sum_{i=4}^{3n-1}(-1)^{i}4CP(n-2,i-4)\lambda(\lambda-1)^{3n-i} \\
&= \sum_{i=1}^{3n-1}(-1)^{i+1}4CP(n-2,i-2)\lambda(\lambda-1)^{3n-i}.
\end{aligned}
$$

Hence, putting this all together with (5), we get

$$
\begin{aligned}
P_{C_3\times P_n}(\lambda) = \sum_{i=1}^{3n-1}(-1)^{i+1}[&CP(n-2,i-1)+3CP(n-2,i-2) \\
&+5CP(n-2,i-3)+4CP(n-2,i-4)]\lambda(\lambda-1)^{3n-i} \\
= \sum_{i=1}^{3n-1}&(-1)^{i+1}CP(n-1,i-1)\lambda(\lambda-1)^{3n-i}.
\end{aligned}
$$

We end by noting that a formula for $P_{C_3\times P_n}(\lambda)$ can be derived in a similar manner to the one for $P_{P_2\times P_n}(\lambda)$ given in (3).

References

1. R. A. Brualdi, *Introductory Combinatorics*, 3rd ed., Prentice-Hall, 1999.
2. G. L. Chia, "Some Problems on Chromatic Polynomials," *Discrete Mathematics*, 172 (1997), 39–44.
3. F. M. Dong, K. M. Koh, K. L. Teo, *Chromatic Polynomials and Chromaticity of Graphs*, World Scientific, 2005.
4. R. Shrock, "Exact Potts Model Partition Functions on Ladder Graphs," *Physica A*, 283 (2000), 388–446.
5. N. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at <http://www.research.att.com/njas/sequences>

6. E. W. Weisstein, "Trinomial Coefficients," from *MathWorld*–A Wolfram Web Resource, <http://mathworld.wolfram.com/TrinomialCoefficient.html>

Mathematics Subject Classification (2000): 05C15, 05C38

Thomas J. Pfaff
Department of Mathematics
Ithaca College
Ithaca, NY 14850
email: tpfaff@ithaca.edu

Jasmine Walker
132 Hayward St.
Burlington, VT 05401
email: jazyrose@gmail.com