HIROSHIMA MATH. J.
38 (2008), 31–35

Maximally differential graded ideals in zero characteristic

Atul GAUR and Alok KUMAR MALOO

(Received July 19, 2006)
(Revised March 23, 2007)

ABSTRACT. A new proof, which is much simpler and which works in more generality, of a structure theorem on maximally differential graded ideals in a Noetherian graded ring containing a field of characteristic zero is given.

1. Introduction

Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded ring and let I be an ideal of R. It was shown in [1] that if $R = R_0[R_1]$ and R_0 is a field of characteristic zero and if I is the maximally D-differential graded ideal of R for some set D of R_0-derivations of R, then there exist a Noetherian graded subring $A = \bigoplus_{n=0}^{\infty} A_n$ of R, elements $x_1, \dots, x_r \in R_1$ such that $x_1, \dots, x_r$ are algebraically independent over A, $R = A[x_1, \dots, x_r]$ and $I = \mathfrak{n}R$, where $\mathfrak{n} = \bigoplus_{n=1}^{\infty} A_n$, the irrelevant maximal ideal of A.

One wonders whether some of the conditions in the hypothesis of above result are indeed required. In other words is the result true without the following assumptions?

(1) R is generated by R_1 as an R_0-algebra.

(2) D is a set of R_0-derivations.

If one analyses the proofs in [1], one realizes that the proof of the above result depends essentially on Lemma 3 and Lemma 5 of the article. Proof of Lemma 3 can be modified to remove both the conditions. However, it is not clear if the same can be done to the proof of Lemma 5.

In this article we give a proof of the above result without the assumptions of (1) and (2). We use some of the ideas of [1], which are modified to suit the situation and we also use some ideas from [4]. The proofs in this article are more direct, much simpler and slicker.

2. Results

By a ring we mean a commutative ring with unity.

The first author is supported by SRF of CSIR, India.
2000 *Mathematics Subject Classification*. Primary 13A02, 13N15.
Key words and phrases. Graded rings, Maximally differential ideals.

We recall the following definition:

Definition 1. Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a graded ring and let D be a set of derivations of R. Let F be the set of all proper graded ideals I which are D-differential, that is, $I \neq R$ and $d(I) \subseteq I$ for all $d \in D$. By a maximally D-differential graded ideal of R, we mean a maximal element of F. If R_0 is a field then it is immediate that R has a unique maximally D-differential graded ideal.

An ideal I is said to be a maximally differential graded ideal if it is the maximally D-differential graded ideal for some set D of derivations of R.

We now prove a few lemmas. The construction in the lemma below is from [2].

Lemma 1. *Let A be a ring and let $B = A[x]$, where x is an indeterminate. Let d be a derivation of B. For $a \in A$ and $i \geq 0$, let $d_i(a)$ denote the coefficient of x^i in the expression of $d(a)$. Then for all $i \geq 0$, d_i is a derivation of A.*

Proof. For all $a \in A$, $d(a) = \sum_{i=0}^{\infty} d_i(a)x^i$. Therefore for all $a, b \in A$, we have

$$d(a+b) = d(a) + d(b) = \sum_{i=0}^{\infty} (d_i(a) + d_i(b))x^i$$

and

$$d(ab) = bd(a) + ad(b) = \sum_{i=0}^{\infty} (bd_i(a) + ad_i(b))x^i.$$

On comparing the coefficient of x^i, we get $d_i(a+b) = d_i(a) + d_i(b)$ and $d_i(ab) = bd_i(a) + ad_i(b)$ for all $i \geq 0$. $\square$

Lemma 2. *Let A be a ring containing a field of characteristic zero and let x be an indeterminate. Let $B = A[x]$ and let I be an ideal of B. Assume that I is d-differential, where $d = d/dx$. Then we have:*

(a) *Let $f = \sum_{i=0}^{m} a_i x^i \in I$ with $a_i \in A$. Then $a_i \in I$ for all $0 \leq i \leq m$.*

(b) *$I = JB$, where $J = I \cap A$.*

Proof. (a) As $d^m(f) \in I$ we have $m!a_m \in I$, that is, $a_m \in I$. Hence $f_1 = \sum_{i=0}^{m-1} a_i x^i \in I$. Now, by induction, it follows that $a_i \in I$ for all $i \geq 0$.

(b) By (a), $I \subseteq JB$ and hence $I = JB$. $\square$

Lemma 3. *Let B be a graded ring containing a field of characteristic zero and let A be a graded subring of B. Let $x \in B$ be a homogeneous element of B such that x is algebraically independent over A and $B = A[x]$. Let D be a set of* *derivations of B and let I be the maximally D-differential graded ideal of B. Let $J = I \cap A$. If, in addition, I is d/dx-differential then J is a maximally differential graded ideal of A and $JB = I$.*

Proof. Since I is d/dx-differential, by Lemma 2, $JB = I$.

For $d \in D$, define d_is as in Lemma 1 and let

$$\tilde{D} = \{d_i \mid i \geq 0, d \in D\}.$$

We show that J is the maximally $\tilde{D}$-differential graded ideal of A. Let $a \in J$, $d \in D$. Then $a \in I$. Hence $d(a) = \sum_{i=0}^{\infty} d_i(a)x^i \in I$. Since I is d/dx-differential, we get $d_i(a) \in I$ for all i. Hence $d_i(a) \in J$ for all $d \in D$ and for all $i \geq 0$. Therefore J is $\tilde{D}$-differential.

To prove that J is a maximal element in the set of proper graded $\tilde{D}$-differential ideals of A, let $\tilde{J}$ be a $\tilde{D}$-differential graded ideal of A such that J is a proper subset of $\tilde{J}$.

Let $a \in \tilde{J}$ and $d \in D$. Then $d(a) = \sum_{i \geq 0} d_i(a)x^i \in \tilde{J}B$. Therefore $\tilde{J}B$ is D-differential. Since $I = JB$ and JB is a proper subset of $\tilde{J}B$, we must have $\tilde{J}B = B$. Therefore $\tilde{J} = A$. $\square$

We now recall the following definition from [1]:

Definition 2. Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring and let r be an integer. A derivation d of R is said to be of weight r if $d(R_i) \subseteq R_{i+r}$ for all $i \geq 0$ (by convention $R_i = 0$ for $i < 0$).

The next lemma is immediate from the definition:

Lemma 4. *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring and let d be a derivation of R of weight r. Then $\ker(d)$ is a graded subring of R.*

Lemma 5 [5, Proposition 2.4]. *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring, where R_0 is a field of characteristic 0. Let $r \geq 1$ be an integer and let δ be a derivation of R of weight $-r$. If x is a homogeneous element of degree r such that $\delta(x) = 1$ then x is algebraically independent over $A = \ker(\delta)$ and $R = A[x]$.*

Proof. The lemma is the same as [5, Proposition 2.4] for $r = 1$. The proof for $r \geq 2$ is also similar. $\square$

Lemma 6. *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a Noetherian graded ring such that R_0 is a field of characteristic 0. Let I be the maximally D-differential graded ideal of R for some set D of derivations of R. Then I is prime.*

Proof. Let P be a minimal prime ideal of I. Then, by [3, 1.3], P is D-differential. As I is graded, so is P. Hence $I = P$, that is, I is a prime ideal of R. $\square$

We now prove the main result.

Theorem [1, Theorem]. *Let* $R = \bigoplus_{i=0}^{\infty} R_i$ *be a Noetherian graded ring such that* R_0 *is a field of characteristic* 0. *Let* I *be a graded ideal of* R *and let* $n = \dim(R/I)$. *Then the following are equivalent:*

(a) *I is a maximally differential graded ideal.*

(b) *There exist a Noetherian graded subring* A *of* R *and homogeneous elements* $x_1, x_2, \dots, x_n$ *of* R *such that* $x_1, x_2, \dots, x_n$ *are algebraically independent over* A, $R = A[x_1, x_2, \dots, x_n]$ *and* $I = \mathfrak{n}R$, *where* $\mathfrak{n}$ *is the irrelevant maximal ideal of* A.

Proof. (b) $\Rightarrow$ (a). Clearly I is the maximally $\{\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n\}$-differential graded ideal.

(a) $\Rightarrow$ (b). Let I be the maximally D-differential graded ideal of R. Then, by Lemma 6, I is prime.

Let $\mathfrak{m}$ denote the irrelevant maximal ideal of R.

We prove the result by induction on $n = \dim(R/I)$.

If $n = 0$ then $I = \mathfrak{m}$ and so we have nothing to prove.

Suppose now that $n \geq 1$. Then $I \neq \mathfrak{m}$. Therefore $\mathfrak{m}$ cannot be D-differential. Hence there exist a homogeneous element $x \in \mathfrak{m}$, say of degree $r \geq 1$, and a derivation $d \in D$ such that $d(x) \notin \mathfrak{m}$.

Define $\delta : R \to R$ as follows: For $y \in R$, let y_i denote the ith homogeneous component of y. Put $\delta(y) = \sum_{i=0}^{\infty} (d(y_i))_{i-r}$. Then δ is a derivation of weight $-r$.

Note that $\delta(x)$ is a unit of R_0. By replacing δ by $\delta(x)^{-1}\delta$ we may assume that $\delta(x) = 1$.

Since I is d-differential and graded, it is also δ-differential.

Let $A = \ker(\delta)$ and let $J = I \cap A$. Now we have:

(1) By Lemma 4, A is a graded subring of R.

(2) By Lemma 5, x is algebraically independent over A and $R = A[x]$.

(3) Construct $\tilde{D}$ as in the proof of Lemma 3. As $\delta = d/dx$, by Lemma 3, J is the maximally $\tilde{D}$-differential graded ideal of A and $I = JR$.

Therefore

$$\frac{R}{I} = \frac{A[x]}{JA[x]} \cong \frac{A}{J}[t],$$

where t is an indeterminate. This implies that $\dim(A/J) = n - 1$. Now, by induction, the result follows. $\square$

The following corollary is an easy consequence of the above theorem:

Corollary. *Let* $R = \bigoplus_{i=0}^{\infty} R_i$ *be a Noetherian graded ring such that* R_0 *is a field of characteristic* 0. *Let* D *be a set of derivations of* R *such that the ideal* (0) *is the only proper* D*-differential graded ideal* R. *Then* R *is isomorphic to a polynomial ring over* R_0.

References

[1] Y. Ishibashi, Maximally differential graded prime ideals. Bull. Fac. Sch. Educ. Hiroshima Univ. Part II, **7** (1984), 39–41.

[2] T. Kimura and H. Niitsuma, On Kunz's Conjecture, J. Math. Soc. Japan, **34** (1982), 371–378.

[3] Y. Lequain, Differential simplicity and Complete Integral Closure, Pacific Journal of Math., **36(3)** (1971), 741–751.

[4] A. K. Maloo, Maximally Differential Ideals, Journal of Algebra, **176** (1995), 806–823.

[5] J. M. Wahl, A cohomological characterisation of $\mathbf{P}^n$, Invent. Math., **72(2)** (1983), 315–322.

Atul Gaur
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur
India
E-mail: atulgaur@iitk.ac.in

Alok Kumar Maloo
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur
India
E-mail: akmaloo@iitk.ac.in