HIROSHIMA MATH. J.
4 (1974), 629–678

On the Stable Homotopy Ring of Moore Spaces

Shichirô OKA

(Received May 17, 1974)

Introduction

Let p be a prime integer $\geqq 5$, $q=2(p-1)$, and $M_t=S^1\cup_t e^2$ be a Moore space of type $(Z_t, 1)$. Denote by $\mathscr{A}_k(M_t)$ the stable track group $\{S^kM_t, M_t\}=\mathrm{Dir}\lim\{[S^{n+k}M_t, S^nM_t], S\}$, S being the suspension functor. Then the direct sum $\mathscr{A}_*(M_{p^r})=\Sigma_k\mathscr{A}_k(M_{p^r})$ is an algebra over Z_{p^r} with the multiplication induced by the composition of maps. The structure of the ring $\mathscr{A}_*(M_{p^r})$ is studied by several authors [4] [6] [13] [14].

N. Yamamoto [14] has calculated the ring structure of $\mathscr{A}_*(M_p)$ for degree $<p^2q-4$, $q=2(p-1)$, from the results [8] on the stable homotopy ring $G_*=\Sigma_kG_k$, $G_k=\mathrm{Dir}\lim\pi_{n+k}(S^n)$, of spheres. P. Hoffman [4] has introduced a differential in $\mathscr{A}_*(M_t)$ and studied the commutativity of the ring $\mathscr{A}_*(M_t)$ using this differential. H. Toda [13] has generalized Hoffman's results and obtained several useful relations involving the elements $\beta_{(t)}\in\mathscr{A}_{(tp+t-1)q-1}(M_p)$.

The purpose of this paper is to determine the ring structure of $\mathscr{A}_*(M_{p^r})$ for any $r\geqq 1$, within the limits of degree less than $(p^2+3p+1)q-6$.

Let $i\ (=i_r)$: $S^1\to M_{p^r}$ and $\pi\ (=\pi_r)$: $M_{p^r}\to S^2$ denote the natural maps and set $\delta\ (=\delta_r)=i\pi\in\mathscr{A}_{-1}(M_{p^r})$. We have in Proposition 2.3 a direct sum decomposition for odd t:

$$\mathscr{A}_k(M_t)\approx G_{k+1}\otimes Z_t+G_k\otimes Z_t+G_k*Z_t+G_{k-1}*Z_t.$$

Let $H\approx Z_{p^s}$ be a summand of G_k generated by an element γ. Then H gives summands Z_{p^m}, $Z_{p^m}+Z_{p^m}$ and Z_{p^m}, $m=\min\{r, s\}$, of $\mathscr{A}_{k+1}(M_{p^r})$, $\mathscr{A}_k(M_{p^r})$ and $\mathscr{A}_{k-1}(M_{p^r})$, via the above decomposition. In § 3, we construct elements $[\gamma]$ $(=[\gamma]_r)\in\mathscr{A}_{k+1}(M_{p^r})$ and $\langle\gamma\rangle$ $(=\langle\gamma\rangle_r)\in\mathscr{A}_k(M_{p^r})$ for γ above, and we see in Lemma 3.3 that we can take the elements $[\gamma]$, $[\gamma]\delta$, $\langle\gamma\rangle$ and $\langle\gamma\rangle\delta$ for the generators of four cyclic summands of $\mathscr{A}_*(M_{p^r})$ given by H. Thus the additive structure of $\mathscr{A}_*(M_{p^r})$ is described by using such elements (Theorem 3.5).

In Propositions 3.8–3.9, we discuss the relations of the products $\langle\alpha\rangle[\beta]$ and $[\alpha][\beta]$ in $\mathscr{A}_*(M_{p^r})$ with the composition $\alpha\beta$ and the Toda bracket $\langle\alpha, p^s, \beta\rangle$ in G_*. By these results and by employing the differential D (see (1.6) for the definition) in $\mathscr{A}_*(M_{p^r})$, we can calculate the ring structure of $\mathscr{A}_*(M_{p^r})$ from the results [5] [6] on G_*.

We define some elements of $\mathscr{A}_*(M_p)$ as follows:

$$\alpha = [\alpha_1]_1 \in \mathscr{A}_q(M_p),$$

$$\beta_{(s)} = [\beta_s]_1 \in \mathscr{A}_{(sp+s-1)q-1}(M_p) \qquad (1 \leqq s \leqq p+1, \quad s \neq p),$$

$$\bar{\varepsilon} = [\varepsilon']_1 \in \mathscr{A}_{(p^2+1)q-2}(M_p),$$

$$\varepsilon = [\varepsilon_1]_1 \in \mathscr{A}_{(p^2+1)q-1}(M_p),$$

$$\bar{\varphi} = \langle \varphi \rangle_1 \in \mathscr{A}_{(p^2+p)q-3}(M_p),$$

where $q=2(p-1)$, and α_1, β_s, ε', ε_1 and φ are the generators of the p-primary part of G_*[6]. The elements α and $\beta_{(s)}$ are the same ones studied in [13] and [14]. For new indecomposable elements, the following Toda bracket formulae are satisfied (Propositions 5.2 and 6.3):

$$\bar{\varepsilon} \in \langle (\beta_{(1)}\delta)^{p-1}\beta_{(1)}, \alpha\delta-\delta\alpha, \alpha\delta-\delta\alpha \rangle,$$

$$\varepsilon \in \langle \alpha, \beta_{(1)}, \delta\alpha(\delta\beta_{(1)})^{p-1} \rangle,$$

$$\bar{\varphi} \in \langle \varepsilon\alpha^{p-3}\delta+\delta\varepsilon\alpha^{p-3}, \alpha\delta-\delta\alpha, \alpha\delta-\delta\alpha \rangle,$$

and the elements $\bar{\varepsilon}$, ε and $\bar{\varphi}$ are uniquely determined by these formulae and the relations $D(\bar{\varepsilon})=D(\varepsilon)=D(\bar{\varphi})=0$ and $\varepsilon\alpha^{p-1}=0$. Then, our results on $\mathscr{A}_*(M_p)$ are summarized as follows:

THEOREM 0.1. *Let p be a prime integer $\geqq 5$. The ring $\mathscr{A}_*(M_p)$ is multiplicatively generated, within the limits of degree less than $(p^2+3p+1)q-6$, $q=2(p-1)$, by the elements*

$$\delta \in \mathscr{A}_{-1}, \quad \alpha \in \mathscr{A}_q, \quad \beta_{(s)} \in \mathscr{A}_{(sp+s-1)q-1} \ (1 \leqq s \leqq p+1, \ s \neq p),$$

$$\bar{\varepsilon} \in \mathscr{A}_{(p^2+1)q-2}, \quad \varepsilon \in \mathscr{A}_{(p^2+1)q-1} \quad \text{and} \quad \bar{\varphi} \in \mathscr{A}_{(p^2+p)q-3}$$

of order p, with the following relations:

(i) $\delta^2=0, \qquad \delta\alpha^2 = -\alpha^2\delta+2\alpha\delta\alpha;$

(ii) $\beta_{(s)}\delta\alpha = \alpha\delta\beta_{(s)}, \qquad \alpha\beta_{(s)} = \beta_{(s)}\alpha = 0;$

(iii) $\beta_{(s)}\delta\beta_{(t)} = (st/(s+t-1))\beta_{(1)}\delta\beta_{(s+t-1)}$ *for* $s+t \neq p, p+1,$

$\beta_{(s)}\delta\beta_{(p-s)} = s^2\beta_{(1)}\delta\beta_{(p-1)}+s(s-1)/2(\beta_{(1)}\beta_{(p-1)}\delta+\delta\beta_{(1)}\beta_{(p-1)}),$

$\beta_{(s)}\delta\beta_{(p+1-s)} = s(s-1)\beta_{(2)}\delta\beta_{(p-1)};$

(iv) $\beta_{(s)}\beta_{(t)}=0$ *for* $s+t \neq p, \quad \beta_{(s)}\beta_{(p-s)} = s\beta_{(1)}\beta_{(p-1)};$

(v) $\alpha(\delta\beta_{(1)})^p = 0, \qquad (\beta_{(1)}\delta)^p\beta_{(2)} = 0;$

(vi) $\alpha\delta\varepsilon = \varepsilon\delta\alpha + \delta\varepsilon\alpha - \varepsilon\alpha\delta, \qquad \alpha\varepsilon = \varepsilon\alpha, \qquad \varepsilon\alpha^{p-1} = 0;$

(vii) $\bar{\varepsilon}\delta\alpha = \bar{\varepsilon}\alpha\delta, \qquad \alpha\delta\bar{\varepsilon} = \delta\bar{\varepsilon}\alpha, \qquad \bar{\varepsilon}\alpha = \alpha\bar{\varepsilon} = \varepsilon\alpha\delta - \varepsilon\delta\alpha;$

(viii) $\beta_{(1)}\beta_{(p-1)} = 2\varepsilon\alpha^{p-4}\delta\alpha - 3\varepsilon\alpha^{p-3}\delta - \delta\varepsilon\alpha^{p-3}$ *up to non zero coefficient*;

(ix) $(s+1)\varepsilon\delta\beta_{(s)} = 2\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(s+1)}\delta + (s-1)\delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(s+1)},$

$(s+1)\beta_{(s)}\delta\varepsilon = (1-s)\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(s+1)}\delta - 2\delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(s+1)},$

$\beta_{(s)}\varepsilon = -\varepsilon\beta_{(s)} = \alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(s+1)} \qquad \textit{for} \quad s = 1, 2;$

(x) $\bar{\varepsilon}\delta\beta_{(1)} = \beta_{(1)}\delta\bar{\varepsilon} - \delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(2)}\delta,$

$$\bar{\varepsilon}\delta\beta_{(2)} = -\beta_{(2)}\delta\bar{\varepsilon} = -\frac{1}{3}\,\delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(3)}\delta,$$

$$\beta_{(s)}\bar{\varepsilon} = \bar{\varepsilon}\beta_{(s)} = -\frac{1}{s+1}\,(\alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(s+1)}\delta - \delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(s+1)}) \qquad \textit{for } s = 1, 2;$$

(xi) $\bar{\varphi}\delta = -\delta\bar{\varphi}, \quad \bar{\varphi}\alpha = \alpha\bar{\varphi} = 0, \quad \beta_{(1)}\bar{\varphi} = -\bar{\varphi}\beta_{(1)},$

$\beta_{(1)}\bar{\varphi} = \alpha\delta\beta_{(2)}\delta\beta_{(p-1)}$ *up to non zero coefficient.*

An additive basis (over Z_p*) for* $\mathscr{A}_k(M_p)$, $k<(p^2+3p+1)q-6$, *is given by the following elements*:

$\delta, \quad 1, \quad \alpha^s\delta^a, \quad \alpha^{s-1}\delta\alpha\delta^a \quad (1 \leqq s \leqq p^2+3p),$

$\delta^a(\beta_{(1)}\delta)^{r-1}\beta_{(1)}\delta^b \quad (1 \leqq r \leqq p+3), \qquad \delta^a(\beta_{(1)}\delta)^r\beta_{(s)}\delta^b \quad ((r,s)\in I),$

$\delta^a(\beta_{(1)}\delta)^r\beta_{(2)}\delta\beta_{(p-1)}\delta^b \quad (r = 0, 1), \qquad \delta^a\alpha(\delta\beta_{(1)})^r\delta^b \quad (1 \leqq r < p),$

$\delta^a\alpha(\delta\beta_{(1)})^r\delta\beta_{(s)}\delta^b \quad ((r,s)\in J), \qquad \delta^a\alpha(\delta\beta_{(1)})^r\delta\beta_{(2)}\delta\beta_{(p-1)}\delta^b \quad (r = 0, 1),$

$\delta^a(\beta_{(1)}\delta)^r\bar{\varepsilon}\delta^b \quad (0 \leqq r \leqq 3), \qquad \delta^a\varepsilon\alpha^i\delta^b \quad (0 \leqq i \leqq p-2),$

$\delta^a\varepsilon\alpha^{i-1}\delta\alpha\delta^b \quad (1 \leqq i \leqq p-3), \qquad \varepsilon\alpha^{p-2}\delta\alpha\delta^a, \qquad \bar{\varphi}\delta^a,$

where a, $b=0$ *or* 1, *and the index sets* I *and* J *are given by*

$$I = \{(r,s) | 0 \leqq r < p, \quad 2 \leqq s \leqq p+1, \quad s \neq p, \quad r+s \leqq p+2\},$$

$$J = I - \{(1, p+1)\}.$$

Now let $\lambda\colon M_p \to M_{p^2}$ and $\rho\colon M_{p^2} \to M_p$ denote the natural maps. Then we define the following elements of $\mathscr{A}_*=\mathscr{A}_*(M_{p^2})$:

$$\delta_2 = i_2\pi_2 \in \mathscr{A}_{-1}, \qquad \xi_s = \lambda\alpha^s\rho \in \mathscr{A}_{sq} \quad (1 \leqq s < p),$$

$$\beta'_{(s)} = \lambda\beta_{(s)}\rho \in \mathscr{A}_{(sp+s-1)p-1} \quad (1 \leqq s \leqq p+1,\ s \neq p),$$

$$\bar{\varepsilon}' = \lambda\bar{\varepsilon}\rho \in \mathscr{A}_{(p^2+1)q-2}, \quad \varepsilon'_{(i)} = \lambda\varepsilon\alpha^{i-1}\rho \in \mathscr{A}_{(p^2+i)q-1} \quad (1 \leqq i \leqq p-1),$$

$$\alpha' = [\alpha'_p]_2 \in \mathscr{A}_{pq}, \qquad \varphi' = [\varphi]_2 \in \mathscr{A}_{(p^2+p)q-2},$$

where α'_p and φ are the generators of the p-parts of G_{pq-1} and $G_{(p^2+p)q-3}$, which are isomorphic to Z_{p^2}.

THEOREM 0.2. *Let p be a prime $\geqq 5$. The ring $\mathscr{A}_*(M_{p^2})$ is an algebra over Z_{p^2} and it is multiplicatively generated, for degree $<(p^2+3p+1)q-6$, by the elements δ_2, α', φ' of order p^2 and the elements ξ_s $(1 \leqq s < p)$, $\beta'_{(s)}$ $(1 \leqq s \leqq p+1,\ s \neq p)$, $\bar{\varepsilon}$ and $\varepsilon'_{(i)}$ $(1 \leqq i \leqq p-1)$ of order p, with the following relations*:

$$(\delta_2)^2 = 0, \qquad \delta_2\alpha'^2 = -\alpha'^2\delta_2 + 2\alpha'\delta_2\alpha', \qquad \alpha'\xi_s = \xi_s\alpha',$$

$$\eta\zeta = 0 \qquad \textit{for} \quad \eta, \zeta \in \{\xi_s, \beta'_{(s)}, \bar{\varepsilon}', \varepsilon'_{(i)}\},$$

$$\alpha'\eta = \eta\alpha' = 0 \qquad \textit{for} \quad \eta \in \{\beta'_{(s)}, \bar{\varepsilon}', \varepsilon'_{(i)}, \varphi'\},$$

$$\varphi'\eta = \eta\varphi' = 0 \qquad \textit{for} \quad \eta \in \{\xi_s, \beta'_{(s)}\},$$

$$\begin{aligned}\eta\delta_2\zeta = \zeta\delta_2\eta = 0 \qquad \textit{for} \quad & (\eta, \zeta) = (\xi_s, \xi_t) \ (s+t \neq p), \quad (\xi_s, \beta'_{(t)}) \ (s \geqq 2), \\ & (\alpha', \beta'_{(s)}), \quad (\xi_s, \bar{\varepsilon}'), \quad (\xi_s, \varphi'), \quad (\alpha', \bar{\varepsilon}'), \\ & (\alpha', \varepsilon'_{(i)}), \quad (\alpha', \varphi'), \quad (\beta'_{(2)}, \bar{\varepsilon}'), \quad (\beta'_{(s)}, \varepsilon'_{(i)}),\end{aligned}$$

$$\xi_s\delta_2\xi_{p-s} = s(\alpha'\delta_2 - \delta_2\alpha'), \quad \alpha'\delta_2\xi_s = \alpha'\xi_s\delta_2, \quad \xi_s\delta_2\alpha' = \delta_2\alpha'\xi_s,$$

$$\beta'_{(s)}\delta_2\xi_1 = \xi_1\delta_2\beta'_{(s)}, \quad \bar{\varepsilon}'\delta_2\beta'_{(1)} = \beta'_{(1)}\delta_2\bar{\varepsilon}',$$

$$\xi_s\delta_2\varepsilon'_{(i)} = \varepsilon'_{(i)}\delta_2\xi_s = \begin{cases} s\varepsilon'_{(i+s)}\delta_2\xi_1 & \textit{for} \quad i+s \leqq p-3, \\ sp\varphi' & \textit{for} \quad i+s = p-1, \\ 0 & \textit{for} \quad i+s = p-2 \ \textit{and for} \ i+s \geqq p, \end{cases}$$

$$\beta'_{(s)}\delta_2\beta'_{(t)} = \begin{cases} (st/(s+t-1))\beta'_{(1)}\delta_2\beta'_{(s+t-1)} & \textit{for} \quad s+t \neq p+1, \\ s(s-1)\beta'_{(2)}\delta_2\beta'_{(p-1)} & \textit{for} \quad s+t = p+1, \end{cases}$$

$$\beta'_{(1)}\delta_2\varphi' = \varphi'\delta_2\beta'_{(1)} = \xi_1\delta_2\beta'_{(2)}\delta_2\beta'_{(p-1)} \quad \textit{up to non zero coefficient},$$

$$\xi_1(\delta_2\beta'_{(1)})^p = 0, \quad (\beta'_{(1)}\delta_2)^p\beta'_{(2)} = 0.$$

Also the group $\mathscr{A}_k(M_{p^2})$, $k<(p^2+3p+1)q-6$, *is the direct sum of cyclic groups generated by the following elements* $(a, b=0$ *or* $1)$:

(i) δ_2, 1, $\alpha'^s\delta_2^a$, $\alpha'^{s-1}\delta_2\alpha'\delta_2^a$ $(1 \leqq s \leqq p+3)$, $\delta_2^a\varphi'\delta_2^b$;

(ii) $\delta_2^a\alpha'^s\xi_t\delta_2^b$ $(0 \leqq s \leqq p+2,\ 1 \leqq t < p)$,

$\delta_2^a(\beta'_{(1)}\delta_2)^{r-1}\beta'_{(1)}\delta_2^b$ $(1 \leqq r \leqq p+3)$, $\quad \delta_2^a(\beta'_{(1)}\delta_2)^r\beta'_{(s)}\delta_2^b$ $((r, s) \in I)$,

$\delta_2^a(\beta'_{(1)}\delta_2)^r\beta'_{(2)}\delta_2\beta'_{(p-1)}\delta_2^b$ $(r = 0, 1)$, $\quad \delta_2^a\xi_1(\delta_2\beta'_{(1)})^r\delta_2^b$ $(1 \leqq r < p)$,

$\delta_2^a\xi_1(\delta_2\beta'_{(1)})^r\delta_2\beta'_{(s)}\delta_2^b$ $((r, s) \in J)$,

$\delta_2^a\xi_1(\delta_2\beta'_{(1)})^r\delta_2\beta'_{(2)}\delta_2\beta'_{(p-1)}\delta_2^b$ $(r = 0, 1)$,

$\delta_2^a(\beta'_{(1)}\delta_2)^r\bar{\varepsilon}'\delta_2^b$ $(0 \leqq r \leqq 3)$, $\quad \delta_2^a\varepsilon'_{(i)}\delta_2^b$ $(1 \leqq i \leqq p-1)$,

$\delta_2^a\varepsilon'_{(i)}\delta_2\xi_1\delta_2^b$ $(1 \leqq i \leqq p-3)$;

where the elements in (i) *and* (ii) *are of order* p^2 *and* p *respectively, and* I *and* J *in* (ii) *are given in Theorem* 0.1.

There exists an element α''_{p^2} of G_{p^2q-1} of order p^3 [5] and it is the only element of order $\geqq p^3$ in G_k for $k<(p^2+3p+1)q-5$ [6]. So we only introduce an element

$$\alpha'' = [\alpha''_{p^2}]_3 \in \mathscr{A}_{p^2q}(M_{p^3}) \qquad \textit{of order } p^3$$

to describe the structure of $\mathscr{A}_*(M_{p^r})$ for $r \geqq 3$. Let B_1 be the set of the elements α'^s $(1 \leqq s \leqq p+3,\ s \neq p)$ and φ' of (i) in Theorem 0.2, and B_2 be the set of the elements of (ii) for $a=b=0$ in Theorem 0.2. Then the group $\mathscr{A}_k(M_{p^r})$, $k<(p^2+3p+1)q-6$, $r \geqq 3$, is the direct sum of cyclic groups generated by the following elements:

$\delta_r = i_r\pi_r \in \mathscr{A}_{-1}(M_{p^r})$,	$1 \in \mathscr{A}_0(M_{p^r})$	*of order*	p^r;
$\delta_r^a\lambda^{r-3}\alpha''\rho^{r-3}\delta_r^b \in \mathscr{A}_{p^2q-a-b}(M_{p^r})$		*of order*	p^3;
$\delta_r^a\lambda^{r-2}\eta\rho^{r-2}\delta_r^b$	*for* $\eta \in B_1$	*of order*	p^2;
$\delta_r^a\lambda^{r-2}\eta\rho^{r-2}\delta_r^b$	*for* $\eta \in B_2$	*of order*	p;

where a, $b=0$ or 1, and λ^t: $M_{p^s} \to M_{p^{s+t}}$ and ρ^t: $M_{p^{s+t}} \to M_{p^s}$ denote the natural maps $(\lambda^0=\rho^0=1)$. Also the multiplicative structure of $\mathscr{A}_*(M_{p^r})$, $r \geqq 3$, can be determined similarly as Theorem 0.2 using Theorem 4.4, and the detailed results are stated in Theorem 7.5.

This paper is organized as follows: In §1, we introduce a differential D

in the ring $\mathscr{A}_*(M_t)$ due to P. Hoffman [4] and H. Toda [13]. In § 2, we discuss the relations among the differentials in $\mathscr{A}_*(M_t)$ and $\mathscr{A}_*(M_{t'})$ for $t' \equiv 0 \mod t$ (Proposition 2.2), and the direct sum decomposition for the group $\mathscr{A}_k(M_t)$ stated above is proved (Proposition 2.3). In § 3, we construct and study the above elements $[\gamma]_r$ and $\langle\gamma\rangle_r$ of $\mathscr{A}_*(M_{p^r})$. The results in § 3 are useful to determine the ring $\mathscr{A}_*(M_{p^r})$. In § 4, we consider the subring of $\mathscr{A}_*(M_{p^r})$, $1 \leqq r \leqq 3$, related with the family $\{\alpha_r\}$ of G_* due to J. F. Adams [1] and H. Toda [9]. For $r=1$, this is the subring generated by two elements δ and α, and its structure was determined by N. Yamamoto [14]. Our results for $r=2$, 3 (Theorems 4.3–4.4) are more complicated than the case $r=1$. In § 5, we introduce the known relations among the elements $\beta_{(s)}$ from [13], and give the elements $\bar{\varepsilon}$ and ε of $\mathscr{A}_*(M_p)$. In § 6, the ring structure of $\mathscr{A}_*(M_p)$ is calculated and Theorem 0.1 is proved. In § 7, we treat the ring $\mathscr{A}_*(M_{p^r})$, $r \geqq 2$. In the first half of § 7 Theorem 0.2 is proved, and in the second half the results for $\mathscr{A}_*(M_{p^r})$, $r \geqq 3$, are stated and proved. In the final section, § 8, several relations on the stable Toda brackets in G_* are proved. For example, we obtain in Proposition 8.1 the following formulae from Theorem 0.1:

$$\langle(\beta_1)^p, \alpha_r, \alpha_s\rangle = \pm rs\varepsilon_{r+s-2}\alpha_1 \qquad \textit{for} \quad r \geqq 1,\ s \geqq 2,\ r+s \leqq p+1,$$

$$\langle(\beta_1)^p, \alpha_1, \alpha'_p\rangle = \pm\varepsilon_{p-1}\alpha_1 .$$

§ 1. A differential in the ring $\mathscr{A}_*(M_t)$.

For any based finite CW-complexes X and Y, the smash product of X and Y is denoted by $X \wedge Y$, and the n-fold suspension S^nX of X is defined by the smash product $S^n \wedge X$ of the n-sphere S^n and X. For X and Y, one can form the stable track group

$$\{X, Y\}_k = \operatorname{Dir\,lim}\{[S^{n+k}X, S^nY], S\},$$

where $[X, Y]$ denotes the set of based homotopy classes of maps of X to Y. For a map $f\colon S^{n+k}X \to S^nY$, we denote usually by the same letter f its homotopy class in $[S^{n+k}X, S^nY]$ and its stable class in $\{X, Y\}_k$. Especially, $1=1_X\colon S^nX \to S^nX$ denotes the identity map of S^nX and its classes in $[S^nX, S^nX]$ and in $\{X, X\}_0$. For $\alpha \in \{X, Y\}_k$ and $\beta \in \{W, X\}_l$, we denote by $\alpha\beta \in \{W, Y\}_{k+l}$ the element represented by a composition fg, where $f \in [S^{n+k}X, S^nY]$ and $g \in [S^{n+k+l}W, S^{n+k}X]$ represent α and β respectively. $\alpha_*\colon \{W, X\}_l \to \{W, Y\}_{k+l}$ and $\beta^*\colon \{X, Y\}_k \to \{W, Y\}_{k+l}$ denote the homomorphisms defined by $\alpha_*(\beta)=\alpha\beta$ and $\beta^*(\alpha)=\alpha\beta$.

We also denote by

$$\mathscr{A}_k(X) = \{X, X\}_k .$$

Then the direct sum $\mathscr{A}_*(X)=\Sigma_k\mathscr{A}_k(X)$ forms a graded ring with the multiplication as above; $1_X\in\mathscr{A}_0(X)$ being the unit. When $X=S^0$,

$$G_k = \mathscr{A}_k(S^0) \quad (\text{resp. } G_* = \mathscr{A}_*(S^0))$$

is the stable homotopy group (resp. ring) of spheres.

In the following of this section and the next section, t denotes a fixed odd integer. Let

$$M_t(=M) = S^1 \cup_t e^2$$

be a Moore space of type $(Z_t, 1)$. Then, there is a cofibering

$$S^1 \xrightarrow{i} M_t \xrightarrow{\pi} S^2, \tag{1.1}$$

and we have the following short exact sequences for finite CW-complexes X and Y (cf. [2; (1.7), (1.7)′]):

$$\begin{aligned}(1.2)\quad & 0\longrightarrow\{X, Y\}_{k-1}\otimes Z_t\xrightarrow{(i\wedge 1_Y)_*}\{X, M_t\wedge Y\}_k\xrightarrow{(\pi\wedge 1_Y)_*}\{X, Y\}_{k-2}*Z_t\longrightarrow 0,\\ & 0\longrightarrow\{X, Y\}_{k+2}\otimes Z_t\xrightarrow{(\pi\wedge 1_X)^*}\{M_t\wedge X, Y\}_k\xrightarrow{(i\wedge 1_X)^*}(X, Y\}_{k+1}*Z_t\longrightarrow 0.\end{aligned}$$

Since $t\not\equiv 2 \bmod 4$, $1_M\in\mathscr{A}_0(M_t)$ is of order t (cf. e.g. [2; Th. 1.1]), and so $\{X, M_t\wedge Y\}_k$ and $\{M_t\wedge X, Y\}_k$ are modules over Z_t for any finite CW-complexes X and Y (cf. e.g. [2; (1.8)]), and in particular $\mathscr{A}_*(M_t)$ is an algebra over Z_t. Equivalently the smash product $M_t\wedge M_t$ is stably homotopy equivalent to the wedge $SM_t\vee S^2M_t$ (cf. e.g. [2; (4.5)–(4.6)]), and hence there are splittings

$$\mu\in\{M_t\wedge M_t, M_t\}_1 \text{ and } \phi\in\{M_t, M_t\wedge M_t\}_{-2}$$

such that

$$\begin{aligned}(1.3)\quad & \mu(i\wedge 1_M)=1_M, \qquad (\pi\wedge 1_M)\phi=1_M, \qquad \mu\phi=0,\\ & (i\wedge 1_M)\mu+\phi(\pi\wedge 1_M)=1_{M\wedge M} \qquad (M=M_t),\end{aligned}$$

(cf. [3; (7.6)–(7.8)]). Since t is odd, $\mathscr{A}_1(M_t)=0$ and so μ and ϕ are unique. Also μ and ϕ are commutative by [3; Th. 7.10]:

$$\mu T=-\mu \quad \textit{and} \quad T\phi=\phi, \tag{1.4}$$

where $T\in\mathscr{A}_0(M_t\wedge M_t)$ denotes the element represented by a map switching factors. Referring to [12; Th. 6] (cf. [13; Prop. 2.1]), if $t\not\equiv 3 \bmod 9$, μ and ϕ satisfy a sort of associativity:

(1.5) *If* $t\not\equiv 3 \bmod 9$, *then*

$$\mu(1_M \wedge \mu)+\mu(\mu \wedge 1_M) = 0 \qquad in \quad \{M_t \wedge M_t \wedge M_t, M_t\}_2,$$

$$(1_M \wedge \phi)\phi-(\phi \wedge 1_M)\phi = 0 \qquad in \quad \{M_t, M_t \wedge M_t \wedge M_t\}_{-4}.$$

We define a linear map

$$(1.6) \qquad D\colon \mathscr{A}_k(M_t) \longrightarrow \mathscr{A}_{k+1}(M_t) \qquad by \quad D(\xi) = \mu(\xi \wedge 1_M)\phi.$$

Then, $-D\sigma=\sigma D$ coincides with D of P. Hoffman [4] for the map σ defined by $\sigma(\xi)=(-1)^{\deg \xi}\xi$. Also our D coincides with $\lambda_M=-\theta$ of H. Toda [13] if t is a prime integer. According to [4; Th. A] (cf. [13; Th. 2.2]), D is a derivation and the associativity (1.5) implies that D is a differential:

$$(1.7) \qquad D(\xi\eta) = D(\xi)\eta+(-1)^{\deg \xi}\xi D(\eta),$$

$$(1.8) \qquad If\ t \not\equiv 3 \bmod 9,\ D^2(\xi) = 0.$$

For i and π of (1.1), we put

$$\delta = i\pi \in \mathscr{A}_{-1}(M_t).$$

Then, δ generates $\mathscr{A}_{-1}(M_t) \approx Z_t$ and we have immediately

$$(1.9) \qquad \delta^2 = 0, \quad D(\delta) = 1_M \quad and \quad D(1_M) = 0.$$

The following formula is Proposition 2.1 (a) of [4] (cf. [13; Th. 2.4 (iii)]).

$$(1.10) \qquad D(\xi)\eta+(-1)^{kl}D(\eta)D(\delta\xi) = (-1)^{(k+1)l}\eta D(\xi)+D(\xi\delta)D(\eta)$$

for $\xi \in \mathscr{A}_k(M_t)$ *and* $\eta \in \mathscr{A}_l(M_t)$.

This formula is useful in connection with the commutativity of the ring $\mathscr{A}_*(M_t)$, that is, we have the following two corollaries of (1.10).

(1.11) ([4; Th. A (b)]) *The subring* Ker D *of* $\mathscr{A}_*(M_t)$ *is commutative, i.e.,*

$$\xi\eta = (-1)^{kl}\eta\xi$$

for $\xi \in \mathscr{A}_k(M_t)$ *and* $\eta \in \mathscr{A}_l(M_t)$ *with* $D(\xi)=0$ *and* $D(\eta)=0$.

(1.12) ([4; Prop. 2.1 (d)])

$$(\xi\delta-(-1)^k\delta\xi)\eta = (-1)^{(k-1)l}\eta(\xi\delta-(-1)^k\delta\xi)$$

for any $\xi \in \mathscr{A}_k(M_t) \cap \operatorname{Ker} D$ *and* $\eta \in \mathscr{A}_l(M_t)$.

REMARK TO THE CASE $t\equiv 3 \bmod 9$. Let $\alpha_1 \in G_3$ be an element of order 3, and put $a(t)=i\alpha_1\pi \in \mathscr{A}_2(M_t)$. The elements α_1 and $a(t)$ ($t\equiv 0 \bmod 3$) generate the 3-primary components of G_3 and $\mathscr{A}_2(M_t)$ respectively, and $a(t)=0$

for $t \not\equiv 0 \bmod 3$. By [12; Th. 6] and [13; Prop. 2.1], for the case $t \equiv 3 \bmod 9$, the element $a(t)$ is the obstruction to the associativity of μ and ϕ, that is, the left sides of the equalities (1.5) are equal to $\pm a(t)(\pi \wedge \pi \wedge 1_M)$ and $\pm(i \wedge i \wedge 1_M)a(t)$ respectively. Also, by [13; Th. 6.1 (i)], we have the following formula corresponding to (1.8):

(1.8)′ *If* $t \equiv 3 \bmod 9$, *then* $D^2(\xi) = \pm(a(t)\xi - \xi a(t))$.

§ 2. Relation of $\mathscr{A}_*(M_t)$ and $\mathscr{A}_*(M_{t'})$, $t' \equiv 0 \bmod t$.

In this section, let t and t' be odd integers such that

$$t' \equiv 0 \bmod t,$$

and we denote by

$$i',\ \pi',\ \mu' \quad \text{and} \quad \phi'$$

the elements for $M_{t'}$ in (1.1) and (1.3). Since $t' \equiv 0 \bmod t$, there are elements

$$\lambda \in \{M_t, M_{t'}\}_0 \quad \text{and} \quad \rho \in \{M_{t'}, M_t\}_0 \tag{2.1}$$

such that

$$\begin{aligned} \lambda i &= (t'/t)i', & i &= \rho i', \\ \pi &= \pi'\lambda, & \pi\rho &= (t'/t)\pi', \\ \rho\lambda &= (t'/t)\cdot 1, & \lambda\rho &= (t'/t)\cdot 1', \end{aligned} \tag{2.2}$$

where 1 and 1′ denote the identity maps of M_t and $M_{t'}$.

We notice that λ and ρ generate $\{M_t, M_{t'}\}_0 \approx Z_t$ and $\{M_{t'}, M_t\}_0 \approx Z_t$ respectively and so these are unique.

LEMMA 2.1. *The following equalities hold.*

(i) $(i' \wedge 1)\mu(\rho \wedge 1) + (\lambda \wedge 1)\phi(\pi' \wedge 1) = 1' \wedge 1$.

(ii) $\mu'(1' \wedge \lambda) = \lambda\mu(\rho \wedge 1)$, $(1' \wedge \rho)\phi' = (\lambda \wedge 1)\phi\rho$.

(iii) $\mu'(\lambda \wedge 1') = \lambda\mu(1 \wedge \rho)$, $(\rho \wedge 1')\phi' = (1 \wedge \lambda)\phi\rho$.

(iv) $\mu(\rho \wedge \rho) = \rho\mu'$, $\mu'(\lambda \wedge \lambda) = (t'/t)\lambda\mu$,

$(\lambda \wedge \lambda)\phi = \phi'\lambda$, $(\rho \wedge \rho)\phi' = (t'/t)\phi\rho$.

Here 1 *and* 1′ *denote the identity maps of* M_t *and* $M_{t'}$.

PROOF. Put $\mu''=\mu(\rho\wedge 1)$ and $\phi''=(\lambda\wedge 1)\phi$. Then $\mu''(i'\wedge 1)=\mu(\rho i'\wedge 1)=\mu(i\wedge 1)=1$, $(\pi'\wedge 1)\phi''=(\pi'\lambda\wedge 1)\phi=(\pi\wedge 1)\phi=1$ and $\mu''\phi''=\mu(\rho\lambda\wedge 1)\phi=(t'/t)\mu\phi=0$ by (2.2) and (1.3). Hence for the element $\xi=(i'\wedge 1)\mu''+\phi''(\pi'\wedge 1)-1'\wedge 1$, we have $(\pi'\wedge 1)_*\xi=0$ and $(i'\wedge 1)^*\xi=0$. Since $\mathscr{A}_0(M_t)=Z_t$, generated by 1, and $\mathscr{A}_1(M_t)=0$, it follows from (1.2) for $M_{t'}$ that $\xi=0$. This proves (i).

Since $\{M_t, M_{t'}\}_1=0$ and $\{M_{t'}, M_t\}_1=0$, we have $\mu'(\lambda\wedge\lambda)\phi=\mu'(1'\wedge\lambda)\phi''=0$ and $\mu(\rho\wedge\rho)\phi'=\mu''(1'\wedge\rho)\phi'=0$. Then, $\mu'(1'\wedge\lambda)=\mu'(1'\wedge\lambda)((i'\wedge 1)\mu''+\phi''(\pi'\wedge 1))=\mu'(i'\wedge 1)\lambda\mu''=\lambda\mu''$ by (1.3) for $M_{t'}$. Similarly $(1'\wedge\rho)\phi'=\phi''\rho$, and (ii) is proved.

Let $T\in\mathscr{A}_0(M_t\wedge M_t)$, $T'\in\mathscr{A}_0(M_{t'}\wedge M_{t'})$ and $T''\in\{M_t\wedge M_{t'}, M_{t'}\wedge M_t\}_0$ be the elements represented by switching maps. Then, by (1.4) for $M_{t'}$, (ii) and (1.4) for M_t, we have $\mu'(\lambda\wedge 1')=-\mu'T'(\lambda\wedge 1')=-\mu'(1'\wedge\lambda)T''=-\lambda\mu(\rho\wedge 1)T''=-\lambda\mu T(1\wedge\rho)=\lambda\mu(1\wedge\rho)$ and similarly $(\rho\wedge 1')\phi'=(\rho\wedge 1')T'\phi'=T''(1'\wedge\rho)\phi'=T''(\lambda\wedge 1)\phi\rho=(1\wedge\lambda)T\phi\rho=(1\wedge\lambda)\phi\rho$. Thus, (iii) is obtained.

For (iv), we have

$$\begin{aligned}
\mu(\rho\wedge\rho) &= \mu(\rho\wedge\rho)((i'\wedge 1')\mu'+\phi'(\pi'\wedge 1'))\\
&= \mu(\rho\wedge\rho)(i'\wedge 1')\mu' = \mu(i\wedge\rho)\mu' = \rho\mu',\\
\mu'(\lambda\wedge\lambda) &= \mu'(\lambda\wedge\lambda)((i\wedge 1)\mu+\phi(\pi\wedge 1))\\
&= (t'/t)\mu'(i'\wedge\lambda)\mu = (t'/t)\lambda\mu,\\
(\lambda\wedge\lambda)\phi &= ((i'\wedge 1')\mu'+\phi'(\pi'\wedge 1'))(\lambda\wedge\lambda)\phi = \phi'(\pi'\lambda\wedge\lambda)\phi = \phi'\lambda,
\end{aligned}$$

and $$(\rho\wedge\rho)\phi' = ((i\wedge 1)\mu+\phi(\pi\wedge 1))(\rho\wedge\rho)\phi' = \phi(\pi\rho\wedge\rho)\phi' = (t'/t)\phi\rho.$$

q.e.d.

PROPOSITION 2.2. *The following equalities hold.*

(i) $D(\xi')\lambda = \lambda D(\rho\xi'\lambda)$, $\rho D(\xi') = D(\rho\xi'\lambda)\rho$ *for* $\xi'\in\mathscr{A}_*(M_{t'})$.

(ii) $D(\xi''\rho) = \lambda D(\rho\xi'')\rho$ *for* $\xi''\in\{M_t, M_{t'}\}_*$,

$D(\lambda\xi'') = \lambda D(\xi''\lambda)\rho$ *for* $\xi''\in\{M_{t'}, M_t\}_*$.

(iii) $D(\lambda\xi\rho) = (t'/t)\lambda D(\xi)\rho$ *for* $\xi\in\mathscr{A}_*(M_t)$.

(iv) $\rho D(\xi')\lambda = (t'/t)D(\rho\xi'\lambda)$ *for* $\xi'\in\mathscr{A}_*(M_{t'})$.

PROOF. (i) $$\begin{aligned}
D(\xi')\lambda &= \mu'(\xi'\wedge 1')\phi'\lambda\\
&= \mu'(\xi'\wedge 1')(\lambda\wedge\lambda)\phi \qquad \text{by Lemma 2.1 (iv)}\\
&= \mu'(1'\wedge\lambda)(\xi'\lambda\wedge 1)\phi
\end{aligned}$$

$$= \lambda\mu(\rho \wedge 1)(\xi'\lambda \wedge 1)\phi \qquad \text{by Lemma 2.1 (ii)}$$

$$= \lambda\mu(\rho\xi'\lambda \wedge 1)\phi = \lambda D(\rho\xi'\lambda),$$

and similarly

$$\rho D(\xi') = \mu(\rho \wedge \rho)(\xi' \wedge 1')\phi' = \mu(\rho\xi' \wedge 1)(\lambda \wedge 1)\phi\lambda = D(\rho\xi'\lambda)\rho.$$

(ii) $D(\xi''\rho) = \mu'(\xi''\rho \wedge 1')\phi' = \mu'(\xi'' \wedge 1')(\rho \wedge 1')\phi'$

$$= \mu'(\xi'' \wedge 1')(1 \wedge \lambda)\phi\rho \qquad \text{by Lemma 2.1 (iii)}$$

$$= \mu'(1' \wedge \lambda)(\xi'' \wedge 1)\phi\rho$$

$$= \lambda\mu(\rho \wedge 1)(\xi'' \wedge 1)\phi\rho \qquad \text{by Lemma 2.1 (ii)}$$

$$= \lambda\mu(\rho\xi'' \wedge 1)\phi\rho = \lambda D(\rho\xi'')\rho,$$

and similarly

$$D(\lambda\xi'') = \lambda\mu(\xi'' \wedge 1)(1' \wedge \rho)\phi' = \lambda\mu(\xi'' \wedge 1)(\lambda \wedge 1)\phi\rho = \lambda D(\xi''\lambda)\rho.$$

(iii) and (iv) follows immediately from (ii) and (i) by using $(t'/t)1' = \lambda\rho$ and $(t'/t)1 = \rho\lambda$, respectively. q.e.d.

By (1.2), we have the following short exact sequences:

(2.3) $$0 \longrightarrow G_{k+2} \otimes Z_t \xrightarrow{\pi^*} \{M_t, S^0\}_k \xrightarrow{i^*} G_{k+1} * Z_t \longrightarrow 0,$$

(2.3)* $$0 \longrightarrow G_{k-1} \otimes Z_t \xrightarrow{i_*} \{S^0, M_t\}_k \xrightarrow{\pi_*} G_{k-2} * Z_t \longrightarrow 0,$$

(2.4) $$0 \longrightarrow \{S^0, M_t\}_{k+2} \xrightarrow{\pi^*} \mathscr{A}_k(M_t) \xrightarrow{i^*} \{S^0, M_t\}_{k+1} \longrightarrow 0,$$

(2.4)* $$0 \longrightarrow \{M_t, S^0\}_{k-1} \xrightarrow{i_*} \mathscr{A}_k(M_t) \xrightarrow{\pi_*} \{M_t, S^0\}_{k-2} \longrightarrow 0.$$

PROPOSITION 2.3. *The above sequences are split, and hence $\mathscr{A}_k(M_t)$ is additively isomorphic to the direct sum*

$$G_{k+1} \otimes Z_t + G_k \otimes Z_t + G_k * Z_t + G_{k-1} * Z_t.$$

PROOF. Let γ be any element of $G_{k+1} * Z_t \subset G_{k+1}$ (resp. $\{S^0, M_t\}_{k+1}$). The order s of γ is a divisor of t. There is an element $\bar{\gamma} \in \{M_s, S^0\}_k$ (resp. $\{M_s, M_t\}_k$) such that $\bar{\gamma} i_s = \gamma$ for the inclusion $i_s(=i)\colon S^1 \to M_s$. For $\rho\colon M_t \to M_s$ of (1.13), the element $\bar{\gamma}\rho \in \{M_t, S^0\}_k$ (resp. $\mathscr{A}_k(M_t)$) satisfies $s\bar{\gamma}\rho = 0$ and $i^*(\bar{\gamma}\rho) = \gamma$ by (2.2). This means that (2.3) (resp. (2.4)) is split.

Next let γ be any element of $G_{k-2} * Z_t$ (resp. $\{M_t, S^0\}_{k-2}$) of order s dividing t. There is an element $\tilde{\gamma} \in \{S^0, M_s\}_k$ (resp. $\{M_t, M_s\}_k$) such that $\pi_s\tilde{\gamma} = \gamma$ for $\pi_s(=\pi)\colon M_s \to S^2$. Then, for $\lambda\colon M_s \to M_t$, the element $\lambda\tilde{\gamma} \in \{S^0, M_t\}_k$ (resp. $\mathscr{A}_k(M_t)$)

satisfies $s\lambda\tilde{\gamma}=0$ and $\pi_*(\lambda\tilde{\gamma})=\gamma$. Hence (2.3)* (resp. (2.4)*) is split. q.e.d.

PROPOSITION 2.4. *Let r and s be relatively prime odd integers. Then $\mathscr{A}_*(M_{rs})$ is isomorphic, as a ring, to the direct sum $\mathscr{A}_*(M_r)+\mathscr{A}_*(M_s)$. If $\xi+\eta\in\mathscr{A}_k(M_r)+\mathscr{A}_k(M_s)$ corresponds to $\zeta\in\mathscr{A}_k(M_{rs})$ via this isomorphism, then $sD(\xi)+rD(\eta)$ corresponds to $D(\zeta)$.*

PROOF. Let a and b be integers such that $as+br=1$. Let $\lambda_1\in\{M_r, M_{rs}\}_0$, $\lambda_2\in\{M_s, M_{rs}\}_0$, $\rho_1\in\{M_{rs}, M_r\}_0$ and $\rho_2\in\{M_{rs}, M_s\}_0$ be the elements of (2.1). Set $\lambda'_1=a\lambda_1$ and $\lambda'_2=b\lambda_2$. Since $\{M_r, M_s\}_*=\{M_s, M_r\}_*=0$ by $(r, s)=1$, we have $\rho_2\lambda'_1=0$ and $\rho_1\lambda'_2=0$. Also $\rho_1\lambda'_1=1$, $\rho_2\lambda'_2=1$ and $\lambda'_1\rho_1+\lambda'_2\rho_2=1$ by (2.2). Define $f\colon\mathscr{A}_k(M_{rs})\to\mathscr{A}_k(M_r)+\mathscr{A}_k(M_s)$ and $g\colon\mathscr{A}_k(M_r)+\mathscr{A}_k(M_s)\to\mathscr{A}_k(M_{rs})$ by $f(\zeta)=\rho_1\zeta\lambda'_1+\rho_2\zeta\lambda'_2$ and $g(\xi+\eta)=\lambda'_1\xi\rho_1+\lambda'_2\eta\rho_2$. Then, we see easily that f is a desired ring isomorphism and g is its inverse. q.e.d.

§3. Some elements defined from G_*.

In this section, we treat the case that t and t' in the previous sections are powers of a fixed odd prime p. Henceforward we set λ and ρ the generators of $\{M_{p^r}, M_{p^{r+1}}\}_0$ and $\{M_{p^{r+1}}, M_{p^r}\}_0$ in (2.1). So the s-fold iterations $\lambda^s=\lambda\ldots\lambda$ and $\rho^s=\rho\ldots\rho$ are the elements of (2.1) for $t=p^r$ and $t'=p^{r+s}$. The elements i, π and δ of (1.1) and (1.9) for $t=p^r$ are sometimes denoted by i_r, π_r and δ_r if it is necessary, and so, for $t=p^r$ and $t'=p^{r+s}$ (2.2) is paraphrased as

$$(3.1)\qquad \begin{aligned} &\lambda^s i_r = p^s i_{r+s}, \quad i_r = \rho^s i_{r+s},\\ &\pi_r = \pi_{r+s}\lambda^s, \quad \pi_r\rho^s = p^s\pi_{r+s},\\ &\rho^s\lambda^s = p^s\cdot 1_M\ (M=M_{p^r}), \quad \lambda^s\rho^s = p^s\cdot 1_{M'}\ (M'=M_{p^{r+s}}). \end{aligned}$$

We put

$$\delta_{r,s} = i_r\pi_s\in\{M_{p^s}, M_{p^r}\}_{-1}.$$

It is well known (cf. [2; p. 80]) that there exists a sequence of cofiberings:

$$M_{p^r}\xrightarrow{\lambda^s}M_{p^{r+s}}\xrightarrow{\rho^r}M_{p^s}\xrightarrow{\delta_{r,s}}SM_{p^r},$$

and so we have, for any finite CW-complex X, the following exact sequences:

$$(3.2)\quad \cdots\longrightarrow\{X, M_{p^r}\}_k\xrightarrow{\lambda^s_*}\{X, M_{p^{r+s}}\}_k\xrightarrow{\rho^r_*}\{X, M_{p^s}\}_k\xrightarrow{\delta_{r,s*}}\{X, M_{p^r}\}_{k-1}\longrightarrow\cdots;$$

$$(3.2)^*\ \cdots\longrightarrow\{M_{p^s}, X\}_k\xrightarrow{\rho^{r*}}\{M_{p^{r+s}}, X\}_k\xrightarrow{\lambda^{s*}}\{M_{p^r}, X\}_k\xrightarrow{\delta^*_{r,s}}\{M_{p^s}, X\}_{k-1}\longrightarrow\cdots.$$

Now let γ be any element of G_k. Then we define

$$\text{(3.3)} \qquad \langle\gamma\rangle_r \ (=\langle\gamma\rangle) = \gamma\wedge 1_M \in \mathscr{A}_k(M_{p^r}),\ M = M_{p^r}.$$

The following lemma is easily proved from definitions.

LEMMA 3.1. *The following equalities hold.*

(i) $D\langle\gamma\rangle = 0$.

(ii) $\langle\gamma\rangle i = (-1)^{\deg\gamma} i\gamma, \quad \pi\langle\gamma\rangle = \gamma\pi$.

(iii) $\langle\gamma\rangle\xi = (-1)^{\deg\gamma\deg\xi}\xi\langle\gamma\rangle$ *for any* $\xi \in \mathscr{A}_*(M_{p^r})$.

(iv) $\lambda^s\langle\gamma\rangle_r = \langle\gamma\rangle_{r+s}\lambda^s, \quad \langle\gamma\rangle_r\rho^s = \rho^s\langle\gamma\rangle_{r+s}$.

Next let γ be an element of G_k of order p^s and suppose that γ generates a direct summand of G_k. Then there is an extension $\bar\gamma \in \{M_{p^s}, S^0\}_{k-1}$ of γ such that $\bar\gamma i_s = \gamma$. This element $\bar\gamma$ is determined modulo the subgroup $G_{k+1}\pi_s$. We define an element $[\gamma]_r\ (=[\gamma])$ of $\mathscr{A}_{k+1}(M_{p^r})$ by

$$\text{(3.4)} \qquad [\gamma]_r = \begin{cases} D(i_r\bar\gamma\lambda^{s-r}) & \text{for} \quad r \leqq s, \\ D(i_r\bar\gamma\rho^{r-s}) & \text{for} \quad r > s. \end{cases}$$

For any $\gamma' \in G_{k+1}$, we have $D(i_r\gamma'\pi_s\lambda^{s-r}) = D(i_r\gamma'\pi_r) = D(\delta_r\langle\gamma'\rangle_r) = \langle\gamma'\rangle_r$ for $r \leqq s$ and $D(i_r\gamma'\pi_s\rho^{r-s}) = p^{r-s}D(i_r\gamma'\pi_r) = p^{r-s}\langle\gamma'\rangle_r$ for $r > s$, by (3.1), Lemma 3.1 (ii), (i), (1.7) and (1.9). So, the element $[\gamma]_r$ of (3.4) is determined modulo the subgroup $p^{r-\min\{r,s\}}G_{k+1}\wedge 1_M$.

LEMMA 3.2. *The element* $[\gamma]_r$ *above is of order* p^m, $m=\min\{r,s\}$, *and the following equalities hold.*

(i) $D[\gamma] = 0$, *if* $(p,r) \neq (3,1)$.

(ii) $\pi[\gamma]i = \begin{cases} p^{s-r}\gamma & \text{for} \quad r < s, \\ \gamma & \text{for} \quad r \geqq s. \end{cases}$

(iii) $[\gamma]\xi = (-1)^{\deg[\gamma]\deg\xi}\xi[\gamma]$ *for any* $\xi \in \mathscr{A}_*(M_{p^r}) \cap \operatorname{Ker} D$, *if* $(p,r) \neq (3,1)$.

(iv) $\lambda[\gamma]_r = [\gamma]_{r+1}\lambda, \quad [\gamma]_r\rho = \rho[\gamma]_{r+1}$ *for* $r < s$,

$[\gamma]_{r+1} = \lambda[\gamma]_r\rho$ *for* $r \geqq s$.

PROOF. (i) follows from (1.8), and (ii) follows from (1.3), (1.6), Lemma

2.1 and (3.1). By (ii), $p^{m-1}\pi[\gamma]i=p^{s-1}\gamma\neq 0$, and hence $[\gamma]$ is of order p^m. (iii) follows from (i) and (1.11), and (iv) follows from Proposition 2.2. q.e.d.

LEMMA 3.3. *Assume that G_k has a direct summand Z_{p^s} generated by an element γ and that $r\geqq 2$ if $p=3$, and let $m=\min\{r, s\}$. Then, $\mathscr{A}_{k+1}(M_{p^r})$ has a direct summand Z_{p^m} generated by the element $[\gamma]$ of (3.4), $\mathscr{A}_k(M_{p^r})$ has a summand $Z_{p^m}+Z_{p^m}$ generated by $[\gamma]\delta$ and $\langle\gamma\rangle$ of (3.3), and $\mathscr{A}_{k-1}(M_{p^r})$ has a summand Z_{p^m} generated by $\langle\gamma\rangle\delta$. These summands are the ones obtained from the summand Z_{p^s}, generated by γ, via the direct sum decomposition in Proposition 2.3.*

The following relations hold in $\mathscr{A}_k(M_{p^r})$ and $\mathscr{A}_{k-1}(M_{p^r})$:

$$(3.5)\qquad [\gamma]\delta+(-1)^k\delta[\gamma]=\begin{cases}p^{s-r}\langle\gamma\rangle & \text{for } r\leqq s,\\ \langle\gamma\rangle & \text{for } r>s,\end{cases}$$

$$(3.6)\qquad \delta[\gamma]\delta=\begin{cases}p^{s-r}\delta\langle\gamma\rangle=(-1)^k p^{s-r}\langle\gamma\rangle\delta & \text{for } r\leqq s,\\ \delta\langle\gamma\rangle=(-1)^k\langle\gamma\rangle\delta & \text{for } r>s,\end{cases}$$

and in particular

$$(3.6)'\qquad \delta[\gamma]\delta=0 \quad \text{if } 2r\leqq s.$$

To prove the lemma, we prepare the following elementary lemma.

LEMMA 3.4. *Let G be a finitely generated abelian group and x be an element of order p^r, $r\geqq 1$, where p is a prime. Then, x generates a direct summand of G if and only if $p^{r-1}x$ is not divisible by p^r.*

PROOF OF LEMMA 3.3. By lemma 3.2, $[\gamma]$ generates a cyclic subgroup of order p^m. For $s\leqq r$, $p^{m-1}\pi[\gamma]i=p^{s-1}\gamma$ is not divisible by $p^s=p^m$, and so $[\gamma]$ generates a direct summand, by Lemma 3.4. For $s>r$, $\mathscr{A}_*(M_{p^r})$ is a Z_{p^r}-module and $[\gamma]$ has the highest order. Hence $[\gamma]$ generates a summand. We have $D([\gamma]\delta)=(-1)^{k+1}[\gamma]$ by (1.7), (1.9) and Lemma 3.2 (i), and hence we see that $[\gamma]\delta$ is of order p^m and generates a summand Z_{p^m}. By definition, $p^m\langle\gamma\rangle=0$. By Lemma 3.1 (ii), $p^{m-1}\pi\langle\gamma\rangle=p^{m-1}\gamma\pi\neq 0$ since $p^{m-1}\gamma$ is not p^r-divisible. Hence $\langle\gamma\rangle$ is of order p^m. For $s>r$, $\langle\gamma\rangle$ has the highest order and generates a summand Z_{p^m}. For $s\leqq r$, we have $p^{s-1}\pi_r\langle\gamma\rangle_r\lambda^{r-s}=p^{s-1}\gamma\pi_r\lambda^{r-s}=p^{s-1}\gamma\pi_s\neq 0$ by Lemma 3.1 (ii) and (3.1). So, $p^{s-1}\langle\gamma\rangle$ is not p^s-divisible by $p^s\lambda^{r-s}=0$. Hence $\langle\gamma\rangle$ generates a summand Z_{p^m}, by Lemma 3.4. Since $D(\langle\gamma\rangle\delta)=(-1)^k\langle\gamma\rangle$ by (1.7), (1.9) and Lemma 3.1 (i), $\langle\gamma\rangle\delta$ generates a summand Z_{p^m} of $\mathscr{A}_{k-1}(M_{p^r})$. Let $x\langle\gamma\rangle+y[\gamma]\delta=0$ in $\mathscr{A}_k(M_{p^r})$. Then, $x\langle\gamma\rangle\delta=0$ and $x\equiv 0 \bmod p^m$, so $y[\gamma]\delta=0$ and $y\equiv 0 \bmod p^m$. This means that $\langle\gamma\rangle$ and $[\gamma]\delta$ generates a summand

$Z_{p^m}+Z_{p^m}$.

Using Lemma 3.1 (ii) and Lemma 3.2 (ii), we see immediately by the split exact sequences (2.3)–(2.4)* that these four summands Z_{p^m} of $\mathscr{A}_*(M_{p^r})$ are regarded as the ones obtained from the summand Z_{p^s} generated by γ via the direct sum decomposition of Proposition 2.3.

We have $\delta[\gamma]\delta=i\pi[\gamma]i\pi=p^{s-m}i\gamma\pi=p^{s-m}\delta\langle\gamma\rangle$ by Lemma 3.1 (ii) and Lemma 3.2 (ii), and (3.6) is proved by Lemma 3.1 (iii). Applying D to (3.6), we obtain (3.5). q.e.d.

Combining Proposition 2.3 with Lemma 3.3, we obtain the following theorem, which determine the additive structure of $\mathscr{A}_*(M_{p^r})$.

THEOREM 3.5. *Let the p-primary parts of G_{k-1}, G_k and G_{k+1} be isomorphic to direct sums of cyclic subgroups generated by $\alpha_1,\dots,\alpha_l\in G_{k-1}$, $\beta_1,\dots,\beta_m\in G_k$ and $\gamma_1,\dots,\gamma_n\in G_{k+1}$; the orders of the elements α_i, β_i and γ_i being p^{a_i}, p^{b_i} and p^{c_i}, respectively. Then if $p^r\neq 3$, we have the direct sum decomposition*

$$\mathscr{A}_k(M_{p^r}) = \textstyle\sum_{i=1}^{l}H_i+\sum_{i=1}^{m}K_i+\sum_{i=1}^{m}K'_i+\sum_{i=1}^{n}L_i\,,$$

where H_i, K_i, K'_i and L_i are the cyclic subgroups generated by the elements $[\alpha_i]_r$, $[\beta_i]_r\delta$, $\langle\beta_i\rangle_r$ and $\langle\gamma_i\rangle_r\delta$ of order $p^{\min\{a_i,r\}}$, $p^{\min\{b_i,r\}}$, $p^{\min\{b_i,r\}}$ and $p^{\min\{c_i,r\}}$, respectively. Further if $b_i\leqq r$ (resp. $c_i\leqq r$), we can replace the element $\langle\beta_i\rangle_r$ (resp. $\langle\gamma_i\rangle_r\delta$) by $\delta[\beta_i]_r$ (resp. $\delta[\gamma_i]_r\delta$).

COROLLARY 3.6. *Let $\xi\in\mathscr{A}_k(M_{p^r})$, $p^r\neq 3$. Then, there exists an element $\gamma\in G_k$ such that $\xi=\langle\gamma\rangle_r$ if and only if ξ satisfies $D(\xi)=0$ and $\pi\xi i=0$.*

PROOF. The *only if* part is obvious by Lemma 3.1. Put $\xi=\Sigma a_i[\alpha_i]+\Sigma b_i[\beta_i]\delta+\Sigma b'_i\langle\beta_i\rangle+\Sigma c_i\langle\gamma_i\rangle\delta$ for the decomposition of Theorem 3.5. Then, $D(\xi)=0$ implies $b_i=c_i=0$ and $\pi\xi i=0$ implies $a_i=0$. Hence, for $\gamma=\Sigma b'_i\beta_i$, we have $\xi=\langle\gamma\rangle$. q.e.d.

COROLLARY 3.7. *The properties* (i) *and* (ii) *of Lemma* 3.2 *characterize the elements $[\gamma]_r$ for $r\leqq s$, that is, if γ is a generator of a summand Z_{p^s} of G_k, $r\leqq s$, and an element $\xi\in\mathscr{A}_{k+1}(M_{p^r})$, $p^r\neq 3$, satisfies $D(\xi)=0$ and $\pi\xi i=p^{s-r}\gamma$, then $\xi=[\gamma]_r$ for a suitable choice of $\bar\gamma$ of* (3.4).

PROOF. Let $\bar\gamma$ be an extension of γ. By Corollary 3.6, we have $\xi-D(i_r\bar\gamma\lambda^{s-r})=\langle\gamma'\rangle$ for some $\gamma'\in G_{k+1}$. Then, the element $\bar{\bar\gamma}=\bar\gamma+\gamma'\pi_s$ is also an extension of γ and we have $\xi=D(i_r\bar{\bar\gamma}\lambda^{s-r})=[\gamma]_r$ for this extension $\bar{\bar\gamma}$. q.e.d.

REMARK FOR $\mathscr{A}_*(M_3)$. For the case $(p,r)=(3,1)$, by virtue of (1.8)′ the formula (i) in Lemma 3.2 is replaces by $D[\gamma]=\pm 3^{s-1}i\gamma\alpha_1\pi$. By using this corrected formula, we can see that the above results (Lemma 3.3, Theorem 3.5 and Corollaries 3.6–3.7) hold without the assumption $p^r \neq 3$. Also the last formula in (3.8) and Proposition 3.9 below hold for the case $p^r = 3$ by adding a minor suitable assumption on the elements α and β.

Next we consider the products $\langle\alpha\rangle\langle\beta\rangle$, $\langle\alpha\rangle[\beta]$ and $[\alpha][\beta]$. From (3.3), it follows immediately that

$$(3.7) \qquad \langle\alpha\rangle\langle\beta\rangle = \langle\alpha\beta\rangle .$$

By Lemma 3.1 (iii), and Lemma 3.2 (iii),

$$(3.8) \quad \langle\beta\rangle\langle\alpha\rangle = (-1)^{kl}\langle\alpha\rangle\langle\beta\rangle, \quad [\beta]\langle\alpha\rangle = (-1)^{k(l+1)}\langle\alpha\rangle[\beta]. \quad \textit{If } p^r \neq 3, \quad [\beta][\alpha] = (-1)^{(k+1)(l+1)}[\alpha][\beta] \qquad (k = \deg\alpha,\ l = \deg\beta).$$

PROPOSITION 3.8. *Let α and β be elements of G_k and G_l such that β generates a summand Z_{p^s}, $s \geqq 1$. Assume that the product $\alpha\beta \in G_{k+l}$ is of order p^s and there is an element $\gamma \in G_{k+l}$ satisfying $\alpha\beta = p^u\gamma$, $u \geqq 0$, and generating a summand $Z_{p^{s+u}}$ of G_{k+l}. Then, for suitable choices of γ, $\bar\beta$ and $\bar\gamma$ defining $[\beta]_r$ and $[\gamma]_r$, we have*

$$\langle\alpha\rangle_r[\beta]_r = \begin{cases} [\gamma]_r & \textit{for} \quad r \leqq s, \\ p^{r-s}[\gamma]_r & \textit{for} \quad s < r \leqq s+u, \\ p^u[\gamma]_r & \textit{for} \quad r > s+u . \end{cases}$$

In particular, $\langle\alpha\rangle_r[\beta]_r = [\alpha\beta]_r$ if $\alpha\beta$ generates a summand.

PROOF. Let $\bar\beta \in \{M_{p^s}, S^0\}_{l-1}$ be an extension with $\bar\beta i_s = \beta$. Since $\delta^*_{s,u}(\alpha\bar\beta) = \alpha\bar\beta i_s\pi_u = \alpha\beta\pi_u = \gamma(p^u\pi_u) = 0$, it follows from (3.2)* for $X = S^0$ that $\alpha\bar\beta = \lambda^{u*}\xi$ for some $\xi \in \{M_{p^{s+u}}, S^0\}_{k+l-1}$. We have $p^u i^*_{s+u}\xi = i^*_s\lambda^{u*}\xi = \alpha\beta$, and so we can take $i^*_{s+u}\xi = \gamma$ replacing γ modulo the subgroup $G_{k+l}{}^*Z_{p^u}$ of G_{k+l}. For this γ, $[\gamma]_r = D(i_r\xi\lambda^{s+u-r})$ for $r \leqq s+u$, $= D(i_r\xi\rho^{r-s-u})$ for $r > s+u$, by (3.4). We have, for $r \leqq s$,

$$\begin{aligned} \langle\alpha\rangle_r[\beta]_r &= \langle\alpha\rangle_r D(i_r\bar\beta\lambda^{s-r}) = (-1)^k D(\langle\alpha\rangle_r i_r\bar\beta\lambda^{s-r}) \\ &\qquad\qquad \text{by (1.7) and Lemma 3.1 (i)} \\ &= D(i_r\alpha\bar\beta\lambda^{s-r}) \qquad \text{by Lemma 3.1 (ii)} \\ &= D(i_r\xi\lambda^{s+u-r}) = [\gamma]_r , \end{aligned}$$

and for $r > s$,

$$\langle\alpha\rangle_r[\beta]_r = \langle\alpha\rangle_r D(i_r\bar\beta\rho^{r-s}) = D(i_r\alpha\bar\beta\rho^{r-s}) \quad \text{by (1.7) and Lemma 3.1 (i)–(ii)}$$

$$= D(i_r\xi\lambda^u\rho^{r-s})$$

$$=\begin{cases} p^{r-s}D(i_r\xi\lambda^{u+s-r}) = p^{r-s}[\gamma]_r & \text{for } r-s \leqq u, \\ p^u D(i_r\xi\rho^{r-s-u}) = p^u[\gamma]_r & \text{for } r-s > u. \end{cases}$$

q.e.d.

Now before considering the product $[\alpha][\beta]$ we are concerned with the stable Toda bracket (secondary composition). Let W, X, Y and Z be either M_t or S^0, and let $\alpha \in \{X, W\}_k$, $\beta \in \{Y, X\}_l$ and $\gamma \in \{Z, Y\}_m$ be elements such that $\alpha\beta = 0$ in $\{Y, W\}_{k+l}$ and $\beta\gamma = 0$ in $\{Z, X\}_{l+m}$. Take representatives $f \in [S^{n+k}X, S^n W]$, $g \in [S^{n+k+l}Y, S^{n+k}X]$ and $h \in [S^{n+k+l+m}Z, S^{n+k+l}Y]$ of α, β and γ such that $fg = 0$ and $gh = 0$. Then, the usual Toda bracket $\{f, g, h\}(\subset [S^{n+k+l+m+1}Z, S^n W])$ is defined as a coset of $f_*[S^{n+k+l+m+1}Z, S^{n+k}X] + (Sh)^*[S^{n+k+l+1}Y, S^n W]$ as in [10; pp. 9–10]. Put $\varepsilon(W) = 0$ if $W = S^0$, $=1$ if $W = M_t$. Then, the stable Toda bracket

$$\langle\alpha, \beta, \gamma\rangle \qquad (\subset \{Z, W\}_{k+l+m+1})$$

is defined to be the limit of $(-1)^{n-1+\varepsilon(W)}\{f, g, h\}$ (cf. [10; p. 32] for the case $W = X = Y = Z = S^0$ and [14; p. 52] for the case $W = X = Y = Z = M_p$). It is a coset of the subgroup $\alpha_*\{Z, X\}_{l+m+1} + \gamma^*\{Y, W\}_{k+l+1}$. If this subgroup is zero, the bracket consists of a single element, say σ, and we denote simply by $\sigma = \langle\alpha, \beta, \gamma\rangle$. In [10; p. 33] the case $W = X = Y = Z = S^0$ is treated and several properties of the bracket are proved. The linearity [10; (3.8)] and the formula [10; (3.5), 0)–iii)] are also valid for our bracket. The formulae (3.5) iv) and (3.6) of [10] are translated to the following

$$\alpha\langle\beta, \gamma, \delta\rangle \subset (-1)^{k+\varepsilon}\langle\alpha\beta, \gamma, \delta\rangle,$$

$$\alpha\langle\beta, \gamma, \delta\rangle = (-1)^{k+1+\varepsilon}\langle\alpha, \beta, \gamma\rangle\delta,$$

where $\alpha \in \{X, W\}_k$ and $\varepsilon = \varepsilon(W) + \varepsilon(X)$. Also, Propositions 1.7–1.9 of [10] can be translated to our situation.

PROPOSITION 3.9. *Let α and β be generators of direct summands of G_k and G_l of order p^s and p^t, $s \geqq t \geqq 1$, respectively. Suppose that the following are satisfied*:

(i) *the subgroup $I = \alpha G_{l+1} + \beta G_{k+1}$ of G_{k+l+1} is trivial*;

(ii) *the element $\gamma = \langle\alpha, p^s, \beta\rangle$[1] is of order p^t*;

1) We identify G_0 with Z and the element of G_0 represented by a map of degree n is denoted by the integer n.

(iii) *there is an element* $\varepsilon \in G_{k+l+1}$ *such that* $p^u \varepsilon = \gamma$ *for some* $u \geqq 0$ *and that* ε *generates a summand.*

Then, for suitable choices of ε, $\bar{\alpha}$, $\bar{\beta}$ *and* $\bar{\varepsilon}$, *the product* $[\alpha]_r[\beta]_r \in \mathscr{A}_{k+l+2}(M_{p^r})$, $p^r \neq 3$, *is equal to*

$$\begin{array}{ll} p^{r-s+u}[\varepsilon]_r & \textit{if} \quad r > \max\{s, t+u\}, \\ p^u[\varepsilon]_r & \textit{if} \quad s > t+u \quad \textit{and} \quad t+u < r \leqq s, \\ p^{2r-s-t}[\varepsilon]_r & \textit{if} \quad s \leqq t+u \quad \textit{and} \quad s < r \leqq t+u, \\ p^{r-t}[\varepsilon]_r & \textit{if} \quad t < r \leqq \min\{s, t+u\}, \\ [\varepsilon]_r & \textit{if} \quad r \leqq t. \end{array}$$

In particular, if $\gamma = \langle \alpha, p^s, \beta \rangle$ *generates a summand, then*

$$[\alpha]_r[\beta]_r = p^{r-\min\{r,s\}}[\gamma]_r.$$

Proof. Consider the element

$$\xi = \pi_s[\alpha]_s \lambda^{s-t}[\beta]_t \in \{M_t, S^0\}_{k+l}.$$

Then $\xi i_t = \pi_s[\alpha]_s \lambda^{s-t}[\beta]_t \rho^{s-t} i_s = (\pi_s[\alpha]_s)([\beta]_s i_s)$ by (3.1) and Lemma 3.2 (iv). Since $(\pi_s[\alpha]_s) i_s = \alpha$ and $\pi_s([\beta]_s i_s) = \beta$ by Lemma 3.2 (ii), it follows from definition that ξi_t belongs to $\langle \alpha, p^s, \beta \rangle$. Thus $\xi i_t = \gamma$ by (i). Since $\delta^*_{t,u}(\xi) = \gamma \pi_u = p^u \varepsilon \pi_u = 0$ by (iii), there is an element $\eta \in \{M_{p^{t+u}}, S^0\}_{k+l}$ satisfying $\lambda^{u*}(\eta) = \xi$ by (3.2)*. Replacing ε modulo the subgroup $G_{k+l+1} * Z_{p^u}$, we have $\eta i_{t+u} = \varepsilon$. For $r \leqq t$ we have

$$\begin{aligned} [\alpha]_r[\beta]_r &= D(i_r \pi_r [\alpha]_r [\beta]_r) && \text{by (1.7), (1.9) and Lemma 3.2 (i)} \\ &= D(i_r \pi_s [\alpha]_s \lambda^{s-t} [\beta]_t \lambda^{t-r}) && \text{by (3.1) and Lemma 3.2 (iv)} \\ &= D(i_r \xi \lambda^{t-r}) \\ &= D(i_r \eta \lambda^{t+u-r}) = [\varepsilon]_r. \end{aligned}$$

Next consider the case $r > t$. We put

$$\zeta = \lambda^{r-t}[\alpha]_t[\beta]_t \rho^{r-t} \in \mathscr{A}_{k+l+2}(M_{p^r}).$$

By Proposition 2.2 and Lemma 3.2 (i), $D(\zeta) = 0$. So we have

$$\begin{aligned} \zeta &= D(i_r \pi_r \zeta) = D(i_r \pi_t [\alpha]_t [\beta]_t \rho^{r-t}) \\ &= D(i_r \xi \rho^{r-t}) = D(i_r \eta \lambda^u \rho^{r-t}) \end{aligned}$$

$$= \begin{cases} p^{r-t}[\varepsilon]_r & \text{for} \quad t < r \leqq t+u\,, \\ p^u[\varepsilon]_r & \text{for} \quad r \geqq t+u\,. \end{cases}$$

On the other hand,

$$\lambda^{r-t}[\alpha]_t = \begin{cases} [\alpha]_r\lambda^{r-t} & \text{for} \quad r \leqq s\,, \\ \lambda^{r-s}[\alpha]_s\lambda^{s-t} & \text{for} \quad r > s\,, \end{cases}$$

and hence

$$[\alpha]_r[\beta]_r = \lambda^{r-s}[\alpha]_s\rho^{r-s}\lambda^{r-s}\lambda^{s-t}[\beta]_t\rho^{r-t} = p^{r-s}\zeta \qquad \text{for} \quad r \leqq s\,,$$

$$[\alpha]_r[\beta]_r = [\alpha]_r\lambda^{r-t}[\beta]_t\rho^{r-t} = \zeta \qquad \text{for} \quad r > s\,.$$

Thus we have the proposition. q.e.d.

§ 4. Some subrings parallel to a subring due to Yamamoto.

Let p denote a fixed odd prime integer, and set $q=2(p-1)$.

According to J. F. Adams [1] and H. Toda [8][9], there exists a family $\{\alpha_k \in G_{kq-1}\,;\ k=1, 2,\dots\}$, called *the α-series,* of elements of the p-primary component of G_*, for any p, satisfying the following

(4.1) *α_k is of order p, not divisible by p if $k \not\equiv 0 \bmod p$, and defined inductively by $\alpha_k \in \langle \alpha_{k-1}, p, \alpha_1 \rangle$.*

Following N. Yamamoto [14], we set

$$\alpha = [\alpha_1]_1 \in \mathscr{A}_q(M_p)\,.$$

This is uniquely determined, since G_q has no p-torsion. Let $A(\alpha, \delta)$ be the subring of $\mathscr{A}_*(M_p)$ generated by α and $\delta=\delta_1$. Then, N. Yamamoto has shown [14; Th. II, Cor. 5.1–5.2] (cf. [13; Prop. 2.3]) that the following results.

THEOREM 4.1. *The subring $A(\alpha, \delta)$, $\delta=\delta_1$, has the following fundamental relations*

(4.2) $$\delta^2 = 0$$

(4.3) $$\delta\alpha^2 = -\alpha^2\delta + 2\alpha\delta\alpha$$

and is freely generated (over Z_p) by the elements

$$\delta,\ 1,\ \alpha^{k-1}\delta\alpha\delta,\ \alpha^{k-1}\delta\alpha,\ \alpha^k\delta,\ \alpha^k, \qquad k = 1, 2,\dots.$$

Also, (4.2) *and* (4.3) *imply the following relations:*

(4.4) (i) $\alpha^k\delta\alpha^l = l\alpha^{k+l-1}\delta\alpha+(1-l)\alpha^{k+l}\delta$;
(ii) $\alpha^k\delta\alpha^l\delta\alpha^m = l\alpha^{k+l+m-1}\delta\alpha\delta$;
(iii) *any monomial involving three or more δ's is zero.*

PROOF. Since $\mathscr{A}_{q+1}(M_p)=0$, $D(\alpha)=0$. So, by (1.11) and (1.12), we have

(4.5) $\alpha\xi = \xi\alpha$ *for any* $\xi \in \mathscr{A}_*(M_p)$ *with* $D(\xi)=0$,

$(\alpha\delta-\delta\alpha)\xi = (-1)^{\deg\xi}\xi(\alpha\delta-\delta\alpha)$ *for any* $\xi\in\mathscr{A}_*(M_p)$.

In particular, (4.3) follows. (4.4) (i) follows from (4.2) and (4.3) by the induction on l, and (4.4) (ii)–(iii) follows from (4.4) (i).

The elements α_k of (4.1) can be taken such that

$$\alpha_k = \pi\alpha^k i$$

[1; Prop. 12.7], and so

$$[\alpha_k]_1 = \alpha^k \qquad \text{if} \quad k \not\equiv 0 \bmod p. \tag{4.6}$$

We have also

$$\langle\alpha_k\rangle = \alpha^k\delta-\delta\alpha^k = k(\alpha^k\delta-\alpha^{k-1}\delta\alpha). \tag{4.7}$$

For (4.7), $\delta\langle\alpha_k\rangle_1 = i\alpha_k\pi = \delta\alpha^k\delta$ and $\langle\alpha_k\rangle_1 = D(\delta\langle\alpha_k\rangle) = D(\delta\alpha^k\delta) = \alpha^k\delta-\delta\alpha^k$.

Thus, by Theorem 3.5, when $k\not\equiv 0$ mid p, the four summands of $\mathscr{A}_*(M_p)$ obtained from the element α_k are spanned by the elements α^k, $\alpha^k\delta$, $\alpha^{k-1}\delta\alpha$ ($\equiv -(1/k)\langle\alpha_k\rangle \bmod \alpha^k\delta$) and $\alpha^{k-1}\delta\alpha\delta$ ($=-(1/k)\langle\alpha_k\rangle\delta$).

When $k\equiv 0 \bmod p$, the elements α^k, $\alpha^k\delta$, $\alpha^{k-1}\delta\alpha$ and $\alpha^{k-1}\delta\alpha\delta$ are also linearly independent since so are the elements α^{k+1}, $\alpha^{k+1}\delta$, $\alpha^k\delta\alpha$ and $\alpha^k\delta\alpha\delta$.

q.e.d.

Now, by [8] the element α_p is divisible by p and not by p^2, and the element α'_p satisfying $p\alpha'_p=\alpha_p$ generates the p-component of G_{pq-1}. We define

$$\alpha' = [\alpha'_p]_2 \in \mathscr{A}_{pq}(M_{p^2}).$$

This is uniquely determined for a fixed α'_p. We have

$$\alpha^p = [\alpha'_p]_1$$

by Corollary 3.7, and so

$$\lambda\alpha^{kp} = \alpha'^k\lambda \quad \text{and} \quad \alpha^{kp}\rho = \rho\alpha'^k \tag{4.8}$$

by Lemma 3.2 (iv). We put

$$\alpha'_{kp} = \pi_2\alpha'^k i_2 \in G_{kpq-1}.$$

Then, $p\alpha'_{kp}=\alpha_{kp}$ by (4.8) and $\alpha'_{kp}\in\langle\alpha'_{(k-1)p}, p^2, \alpha'_p\rangle$ by definition. J. F. Adams [1] defined an invariant e: $G_*\to Q/Z$ such that $e(\alpha_k)=\pm 1/p$. The values of e on G_{kp^jq-1}, $k\not\equiv 0 \bmod p$, are integer multiples of a rational $1/p^{j+1}b$, $b\not\equiv 0 \bmod p$. Thus, α_{kp} is not divisible by p^2 and α'_{kp} generates a summand, if $k\not\equiv 0 \bmod p$. We see therefore

$$[\alpha'_{kp}]_1 = \alpha^{kp} \quad\text{and}\quad [\alpha'_{kp}]_2 = \alpha'^k \qquad \text{if} \quad k \not\equiv 0 \bmod p,$$

$$\langle\alpha'_{kp}\rangle_2 = \alpha'^k\delta_2-\delta_2\alpha'^k = k(\alpha'^k\delta_2-\alpha'^{k-1}\delta_2\alpha'). \tag{4.7$'$}$$

For the last equality, we use the relation

$$\delta_2\alpha'^2 = -\alpha'^2\delta_2+2\alpha'\delta_2\alpha', \tag{4.3$'$}$$

which is obtained in the same way as (4.3). Furthermore, the following is proved in the same way as Theorem 4.1.

PROPOSITION 4.2. *The subring $A(\alpha', \delta_2)$, generated by α' and δ_2, of $\mathscr{A}_*(M_{p^2})$ has the fundamental relations $(\delta_2)^2=0$ and* (4.3)$'$, *and is additively generated by the following elements of order p^2:*

$$\delta_2,\ 1,\ \alpha'^{k-1}\delta_2\alpha'\delta_2,\ \alpha'^{k-1}\delta_2\alpha',\ \alpha'^k\delta_2,\quad \alpha'^k, \qquad k = 1, 2,\dots.$$

Also the relations (4.4) *with the replacement of α and δ by α' and δ_2 hold.*

According to H. Toda [8; Th. 4.14], the element α'_p belongs to $-\langle\alpha_{p-1}, \alpha_1, p\rangle$, so we have $\pi_1\langle\alpha'_p\rangle_1=\alpha'_p\pi_1=-\langle\alpha_{p-1}, \alpha_1, p\rangle\pi_1=-\alpha_{p-1}\pi_1\alpha=-\pi_1\langle\alpha_{p-1}\rangle_1\alpha$ $=\pi_1(\alpha^p\delta_1-\alpha^{p-1}\delta_1\alpha)$ by Lemma 3.1 and (4.7), and $\langle\alpha'_p\rangle_1=D(\delta_1\langle\alpha'_p\rangle_1)=\alpha^p\delta_1-\alpha^{p-1}\delta_1\alpha$. Hence, by Lemma 3.1 and (4.7)$'$, $\lambda\langle\alpha'_{kp}\rangle_1=k\lambda(\alpha^{kp}\delta_1-\alpha^{kp-1}\delta_1\alpha)$. Since $\operatorname{Ker} D\cap\operatorname{Ker}\lambda^*=0$, we obtain

$$\langle\alpha'_{kp}\rangle_1 = k(\alpha^{kp}\delta_1-\alpha^{kp-1}\delta_1\alpha).$$

We have immediately from Lemma 3.2 and (3.5) that

$$[\alpha_k]_2 = \lambda\alpha^k \quad\text{and}\quad \langle\alpha_k\rangle_2 = \lambda\alpha^k\rho\delta_2-\delta_2\lambda\alpha^k\rho \qquad \text{if} \quad k\not\equiv 0 \bmod p.$$

From the above discussions, considering the submodule (dierct summand) of $\mathscr{A}_*(M_{p^2})$ obtained from the elements α_k and α'_{kp}, we get the following result.

THEOREM 4.3. *Let $A(\alpha, \alpha', \delta_2)$ be the subring of $\mathscr{A}_*(M_{p^2})$ generated by the submodule $\lambda_*\rho^*A(\alpha, \delta_1)$ and the subring $A(\alpha', \delta_2)$, and set $\xi_k=\lambda\alpha^k\rho\in A(\alpha, \alpha', \delta_2)$. Then, $A(\alpha, \alpha', \delta_2)$ is the direct sum of cyclic groups generated by the elements*

$$\delta_2,\ 1,\ \alpha'^k,\ \alpha'^k\delta_2,\ \alpha'^{k-1}\delta_2\alpha',\ \alpha'^{k-1}\delta_2\alpha'\delta_2, \qquad k = 1, 2,\dots.$$

of order p^2 and the elements

$$\alpha'^k\xi_l,\ \delta_2\alpha'^k\xi_l,\ \alpha'^k\xi_l\delta_2,\ \delta_2\alpha'^k\xi_l\delta_2, \qquad k=1,2,\dots,\ 1\leqq l<p,$$

of order p. The elements $\delta_2, \alpha', \xi_1,\dots,\xi_{p-1}$ generate the ring $A(\alpha,\alpha',\delta_2)$, and the relations among these elements are given by

$$(\delta_2)^2=0,\quad \delta_2\alpha'^2=-\alpha'^2\delta_2+2\alpha'\delta_2\alpha',\quad \alpha'\xi_k=\xi_k\alpha',\quad \xi_k\xi_l=0,$$

$$\xi_k\delta_2\xi_l=0 \quad \text{for} \quad k+l\neq p,\quad \xi_k\delta_2\xi_{p-k}=kp(\alpha'\delta_2-\delta_2\alpha'),$$

$$\alpha'\delta_2\xi_k=\delta_2\alpha'\xi_k,\quad \xi_k\delta_2\alpha'=\alpha'\xi_k\delta_2.$$

For the homomorphisms $\lambda_*\rho^*\colon \mathscr{A}_*(M_p)\to\mathscr{A}_*(M_{p^2})$ and $\rho_*\lambda^*\colon \mathscr{A}_*(M_{p^2})\to \mathscr{A}_*(M_p)$ the following equalities hold:

$$\lambda_*\rho^*(\alpha^{kp})=p\alpha'^k,\quad \lambda_*\rho^*(\alpha^{kp+l})=\alpha'^k\xi_l,\quad \lambda_*\rho^*(\alpha^k\delta_1)=0,$$

$$\lambda_*\rho^*(\alpha^{k-1}\delta_1\alpha)=0 \quad \text{for} \quad k\not\equiv 0 \bmod p,\quad \lambda_*\rho^*(\alpha^{kp-1}\delta_1\alpha)$$

$$=p(\alpha'^{k-1}\delta_2\alpha'-\alpha'^k\delta_2),\quad \lambda_*\rho^*(\alpha^{k-1}\delta_1\alpha\delta_1)=0;$$

$$\rho_*\lambda^*(\alpha'^k)=0,\quad \rho_*\lambda^*(\alpha'^k\delta_2)=\rho_*\lambda^*(\alpha'^{k-1}\delta_2\alpha')=\alpha^{kp}\delta_1,$$

$$\rho_*\lambda^*(\alpha'^{k-1}\delta_2\alpha'\delta_2)=\alpha^{kp-1}\delta_1\alpha\delta_1,\quad \rho_*\lambda^*(\alpha'^k\xi_l)=0,\quad \rho_*\lambda^*(\delta_2\alpha'^k\xi_l)$$

$$=\rho_*\lambda^*(\alpha'^k\xi_l\delta_2)=0,\quad \rho_*\lambda^*(\delta_2\alpha'^k\xi_l\delta_2)=l\alpha^{kp+l-1}\delta_1\alpha\delta_1.$$

PROOF. It suffices to show the all equalities except the first two.
By (4.8), $\alpha'\xi_k=\lambda\alpha^{p+k}=\xi_k\alpha'$. By (3.1)–(3.2), we have

$$\text{(A)} \qquad \rho\lambda=0,\quad \lambda\rho=p,\quad \rho\delta_2\lambda=\delta_1,\quad \delta_1\rho=0 \quad \text{and} \quad \lambda\delta_1=0.$$

By (4.4) (i), $(k+l)\alpha^k\delta_1\alpha^l=k\alpha^{k+l}\delta_1+l\delta_1\alpha^{k+l}$ and so, $\xi_k\delta_2\xi_l=0$ for $k+l\neq p$, by (A). By (4.4) (ii), $\delta_1\alpha^k\delta_1=k\alpha^{k-1}\delta_1\alpha\delta_1$, and $\rho_*\lambda^*(\delta_2\alpha'^k\xi_l\delta_2)=\rho\delta_2\alpha'^k\lambda\alpha^l\rho\delta_2\lambda=\delta_1\alpha^{kp+l}\delta_1$ $=l\alpha^{kp+l-1}\delta_1\alpha\delta_1$ by (4.8) and (A). We have

$$\begin{aligned}
\text{(B)}\quad (\alpha^{kp-1}\delta_1\alpha)\rho &= -\alpha^{(k-1)p}(\alpha^p\delta_1-\alpha^{p-1}\delta_1\alpha)\rho && \text{by (A)}\\
&= -\alpha^{(k-1)p}\langle\alpha'_p\rangle_1\rho=-\alpha^{(k-1)p}\rho\langle\alpha'_p\rangle_2 && \text{by Lemma 3.1 (iv)}\\
&= -\rho\alpha'^{k-1}(\alpha'\delta_2-\delta_2\alpha') && \text{by (4.7)}' \text{ and (4.8)}\\
&= \rho(\alpha'^{k-1}\delta_2\alpha'-\alpha'^k\delta_2)
\end{aligned}$$

and similarly

$$\text{(C)} \qquad \lambda(\alpha^{kp-1}\delta_1\alpha)=(\alpha'^{k-1}\delta_2\alpha'-\alpha'^k\delta_2)\lambda.$$

Hence, $\alpha'^{l}\xi_k\delta_2\alpha'^{m}\xi_{p-k}=\lambda\alpha^{lp+k}\delta_1\alpha^{(m+1)p-k}\rho=-k\lambda\alpha^{(l+m+1)p-1}\delta_1\alpha\rho=-kp(\alpha'^{l+m}\delta_2\alpha'-\alpha'^{l+m+1}\delta_2)$ by (A), (4.8), (4.4) (i) and (B), and this determines $\xi_k\delta_2\xi_{p-k}$ and $\lambda_*\rho^*(\alpha^{kp-1}\delta_1\alpha)$.

The other equalities follow from (4.4), (A)–(C) and (4.8). q.e.d.

By [5], the p-primary part of G_{p^2q-1} is Z_{p^3}, so the element α'_{p^2} is divisible by p, not by p^2. Let α''_{p^2} be an element such that $p\alpha''_{p^2}=\alpha'_{p^2}$, and define

$$\alpha''=[\alpha''_{p^2}]_3\in\mathscr{A}_{p^2q}(M_{p^3}),$$

$$\alpha''_{kp^2}=\pi_3\alpha''^{k}i_3\in G_{kp^2q-1}.$$

Then, $p\alpha''_{kp^2}=\alpha'_{kp^2}$ and α''_{kp^2} generates a summand Z_{p^3} if $k\not\equiv 0 \bmod p$. The following relations are proved similarly as the previous discussions ($k\not\equiv 0 \bmod p$, l: arbitrary):

$$[\alpha''_{kp^2}]_1=\alpha^{kp^2},\quad [\alpha''_{kp^2}]_2=\alpha'^{kp},\quad [\alpha''_{kp^2}]_3=\alpha''^{k},$$

$$[\alpha'_{kp}]_3=\lambda\alpha'^{k}\rho,\quad [\alpha_k]_3=\lambda^2\alpha^k\rho^2;$$

$$\langle\alpha''_{lp^2}\rangle_3=l(\alpha''^{l}\delta_3-\alpha''^{l-1}\delta_3\alpha''),\quad \langle\alpha''_{lp^2}\rangle_2=l(\alpha'^{lp}\delta_2-\alpha'^{lp-1}\delta_2\alpha'),$$

$$\langle\alpha''_{lp^2}\rangle_1=l(\alpha^{lp^2}\delta_1-\alpha^{lp^2-1}\delta_1\alpha),\quad \langle\alpha'_{kp}\rangle_3=\lambda\alpha'^{k}\rho\delta_3-\delta_3\lambda\alpha'^{k}\rho,$$

$$\langle\alpha_k\rangle_3=\lambda^2\alpha^k\rho^2\delta_3-\delta_3\lambda^2\alpha^k\rho^2.$$

Here, to describe the elements $\langle\alpha''_{kp^2}\rangle_1$ and $\langle\alpha''_{kp^2}\rangle_2$ above, we use the formula $\alpha''_{p^2}\in-\langle\alpha_{p^2-1},\alpha_1,p\rangle$ for a suitable choice of α''_{p^2}, which is proved by a similar way as [8; Th. 4.14].

Let $A(\alpha'',\delta_3)$ be the subring of $\mathscr{A}_*(M_{p^3})$ generated by α'' and δ_3, and let $A(\alpha,\alpha',\alpha'',\delta_3)$ by the subring generated by thd submodule $\lambda^*\rho^*A(\alpha,\alpha',\delta_2)$ and the subring $A(\alpha'',\delta_3)$. Then, from the above discussions we obtain the following result corresponding to the previous theorems.

THEOREM 4.4. *$A(\alpha,\alpha',\alpha'',\delta_3)$ is the direct sum of cyclic groups generated by the elements*

$$\delta_3,\ 1,\ \alpha''^{k},\ \alpha''^{k}\delta_3,\ \alpha''^{k-1}\delta_3\alpha'',\ \alpha''^{k-1}\delta_3\alpha''\delta_3\quad (k\geqq 1)\quad \textit{of order } p^3;$$

$$\alpha''^{k}\xi_{lp},\quad \alpha''^{k}\xi_{lp}\delta_3,\quad \delta_3\alpha''^{k}\xi_{lp},\quad \delta_3\alpha''^{k}\xi_{lp}\delta_3$$
$$(k\geqq 0, 1\leqq l<p)\quad \textit{of order } p^2;$$

$$\alpha''^{k}\xi_{lp+m},\ \alpha''^{k}\xi_{lp+m}\delta_3,\ \delta_3\alpha''^{k}\xi_{lp+m},\ \delta_3\alpha''^{k}\xi_{lp+m}\delta_3$$
$$(k\geqq 0, 0\leqq l<p, 1\leqq m<p)\quad \textit{of order } p;$$

where $\xi_{lp}=\lambda\alpha'^l\rho$ *and* $\xi_{lp+m}=\lambda\alpha'^l\lambda\alpha^m\rho^2=\lambda^2\alpha^{lp+m}\rho^2$ $(1\leqq m<p)$. *The ring* $A(\alpha, \alpha', \alpha'', \delta_3)$ *is generated multiplicatively by* δ_3, α'' *and* ξ_s $(1\leqq s<p^2)$ *with the following relations* $(l, l', m, m'<p, 1\leqq s, t<p^2)$:

$$(\delta_3)^2=0,\quad \delta_3\alpha''^2=-\alpha''^2\delta_3+2\alpha''\delta_3\alpha'',\quad \alpha''\xi_s=\xi_s\alpha'',$$

$$\xi_s\xi_t=0 \qquad \textit{for}\quad s\not\equiv 0 \bmod p \quad \textit{or}\quad t\not\equiv 0 \bmod p,$$

$$\begin{aligned}\xi_{lp}\xi_{l'p}&=p\xi_{(l+l')p} && \textit{for}\quad l+l'<p,\\ &=p^2\alpha'' && \textit{for}\quad l+l'=p,\\ &=p\alpha''\xi_{(l+l'-p)p} && \textit{for}\quad l+l'>p,\end{aligned}$$

$$\begin{aligned}\xi_{lp+m}\delta_3\xi_{l'p+m'}&=mp^2(\alpha''\delta_3-\delta_3\alpha'') && \textit{for}\quad l+l'+1=m+m'=p,\\ &=0 && \textit{otherwise},\end{aligned}$$

$$\xi_{lp+m}\delta_3\xi_{l'p}=\xi_{l'p}\delta_3\xi_{lp+m}=0,$$

$$\begin{aligned}\xi_{lp}\delta_3\xi_{l'p}&=(p/(l+l'))(l\xi_{(l+l')p}\delta_3+l'\delta_3\xi_{(l+l')p}) && \textit{for}\quad l+l'<p,\\ &=lp\alpha''\delta_3+l'p\delta_3\alpha'' && \textit{for}\quad l+l'=p,\\ &=(p/(l+l'))(l\alpha''\xi_{(l+l'-p)p}\delta_3+l'\delta_3\alpha''\xi_{(l+l'-p)p}) && \textit{for}\quad l+l'>p,\end{aligned}$$

$$\begin{aligned}\alpha''\delta_3\xi_t&=\delta_3\alpha''\xi_t && \textit{for}\quad t\not\equiv 0 \bmod p,\\ &=(1/(l+p))(p\alpha''\xi_{lp}\delta_3+l\delta_3\alpha''\xi_{lp}) && \textit{for}\quad t=lp,\end{aligned}$$

$$\begin{aligned}\xi_t\delta_3\alpha''&=\alpha''\xi_t\delta_3 && \textit{for}\quad t\not\equiv 0 \bmod p,\\ &=(1/(l+p))(l\alpha''\xi_{lp}\delta_3+p\delta_3\alpha''\xi_{lp}) && \textit{for}\quad t=lp.\end{aligned}$$

§ 5. Some elements of $\mathscr{A}_*(M_p)$.

In the rest of this paper, we shall calculate the ring $\mathscr{A}_*(M_{p^r})$ up to some range of degree from the results on the p-primary component of G_* in [5] and [6], where p will be always greater than 3.

L. Smith [7] has discovered an another family $\{\beta_k;\ k=1, 2,\ldots\}$, called *the β-series*, of the p-component of G_*, the first $p-1$ elements of which had been obtained by H. Toda [8], and H. Toda [13] has studied multiplicative properties of this family and a related family $\{\beta_{(k)};\ k=1, 2,\ldots\}$ of $\mathscr{A}_*(M_p)$.

Let f: $S^{n+q}M_p\to S^nM_p$ be a map representing the element $\alpha\in\mathscr{A}_q(M_p)$, where $q=2(p-1)$. Let C_f be the mapping cone of f, and i_1: $S^nM_p\to C_f$ and π_1: $C_f\to S^{n+q+1}M_p$ be the natural maps. For the suitable generator β of $\mathscr{A}_{(p+1)q}(C_f)=$

Z_p, the elements

(5.1) $$\beta_{(k)} \in \mathscr{A}_{(kp+k-1)q-1}(M_p), \qquad k = 1, 2, \dots,$$

are defined by $\beta_{(k)} = \pi_1 \beta^k i_1$. Then, it follows immediately from definition that

(5.2) ([13; Th. 5.1 (i) and (vi)], cf. [14; Prop. 5.3])

(i) $\alpha\beta_{(k)} = \beta_{(k)}\alpha = 0$,

(ii) $\beta_{(k)} \in \langle \beta_{(k-1)}, \alpha, \beta_{(1)} \rangle$.

H. Toda has also obtained the following relations [13; (3.7)′, Th. 5.1 and (5.4)] (cf. [14; Prop. 6.1 and 7.4]);

(5.3) $$D(\beta_{(k)}) = 0;$$

(5.4) $$\beta_{(k)}\delta\alpha = \alpha\delta\beta_{(k)};$$

(5.5) $$\beta_{(k)}\beta_{(l)} = k\beta_{(1)}\beta_{(k+l-1)}, \text{ which is zero if } k+l \not\equiv 0 \bmod p;$$

(5.6) $$\beta_{(k)}\delta\beta_{(l)} = -k(k-2)\beta_{(1)}\delta\beta_{(k+l-1)} + \binom{k}{2}\beta_{(2)}\delta\beta_{(k+l-2)},$$

which is equal to

$$(kl/(k+l-1))\beta_{(1)}\delta\beta_{(k+l-1)} \qquad \text{if} \quad k+l \not\equiv 0, 1 \bmod p,$$

$$(kl/(k+l-2))\beta_{(2)}\delta\beta_{(k+l-2)} \qquad \text{if} \quad k+l \not\equiv 0, 2 \bmod p.$$

By (5.3) and (1.11)–(1.12), we have

(5.7) (i) $\beta_{(k)}\xi = (-1)^{\deg \xi}\xi\beta_{(k)}$ *for any* $\xi \in \mathscr{A}_*(M_p) \cap \operatorname{Ker} D$.

(ii) $(\beta_{(k)}\delta + \delta\beta_{(k)})\xi = \xi(\beta_{(k)}\delta + \delta\beta_{(k)})$ *for any* $\xi \in \mathscr{A}_*(M_p)$.

Repeating (4.4), (5.2) (i) and (5.4)–(5.7), we obtain the following

(5.8) *Every monomial on δ, α and $\beta_{(k)}$'s involving two or more α's and one or more $\beta_{(k)}$'s is zero. Furthermore every monomial involving at least one $\beta_{(k)}$ is equal to some multiple of one of the following monomials*:

$$\delta^a\beta_{(k_1)}\delta\dots\delta\beta_{(k_r)}\delta^b, \quad \delta^a\beta_{(lp)}\delta^b,$$

$$\delta^a\alpha\delta\beta_{(k_1)}\delta\dots\delta\beta_{(k_r)}\delta^b, \quad \delta^a\alpha\delta\beta_{(lp)}\delta^b,$$

$$\delta^a(\beta_{(1)}\delta)^{r-1}\beta_{(1)}\beta_{(lp-1)}\delta^b,$$

where $a, b \in \{0, 1\}$, $r \geqq 1$, $k_j \geqq 1$, $k_j \not\equiv 0 \bmod p$ and $l \geqq 1$, and in particular we have a relation

$$\beta_{(k_1)}\delta\ldots\delta\beta_{(k_r)} = \frac{k_1\ldots k_r}{k-r+1}(\beta_{(1)}\delta)^{r-1}\beta_{(k-r+1)}$$

$$\text{if} \quad k \leqq r+p-3,\ k = k_1+\cdots+k_r.$$

Now the elements

$$\beta_k \in G_{(kp+k-1)q-2}, \qquad k = 1, 2, \ldots,$$

are defined by

$$\beta_k = \pi\beta_{(k)}i,$$

and L. Smith [7] has proved $\beta_k \neq 0$ (hence $\beta_{(k)} \neq 0$) for arbitrary k.

The following equalities follow immediately from Lemma 3.1, Corollary 3.7 and (5.3).

(5.9) $\quad [\beta_k]_1 = \beta_{(k)}$ *if* β_k *generates a summand.*

(5.10) $\quad \langle\beta_k\rangle_1 = \beta_{(k)}\delta + \delta\beta_{(k)}$.

Applying Proposition 3.8 and using the relations (5.2) (i), (5.4)–(5.5) and (5.8), we have also

(5.11)
$$\begin{aligned}
&[(\beta_1)^k\beta_l]_1 = (\beta_{(1)}\delta)^k\beta_{(l)}, \quad l \not\equiv -1 \bmod p,\\
&[(\beta_1)^k\beta_2\beta_{p-1}]_1 = (\beta_{(1)}\delta)^k\beta_{(2)}\delta\beta_{(p-1)},\\
&[\alpha_1(\beta_1)^k\beta_l]_1 = \alpha\delta(\beta_{(1)}\delta)^k\beta_{(l)},\\
&[\alpha_1(\beta_1)^k\beta_2\beta_{p-1}]_1 = \alpha\delta(\beta_{(1)}\delta)^k\beta_{(2)}\delta\beta_{(p-1)}.
\end{aligned}$$

Here each equality holds when the left side is defined, i.e., the element γ in the left side $[\gamma]_1$ generates a summand.

(5.12)
$$\begin{aligned}
&\langle(\beta_1)^k\beta_l\rangle_1 = (\beta_{(1)}\delta)^k\beta_{(l)}\delta + \delta(\beta_{(1)}\delta)^k\beta_{(l)},\\
&\langle(\beta_1)^k\beta_2\beta_{p-1}\rangle_1 = (\beta_{(1)}\delta)^k\beta_{(2)}\delta\beta_{(p-1)}\delta + \delta(\beta_{(1)}\delta)^k\beta_{(2)}\delta\beta_{(p-1)},\\
&\langle\alpha_1(\beta_1)^k\beta_l\rangle_1 = \alpha\delta(\beta_{(1)}\delta)^k\beta_{(l)}\delta - \delta\alpha(\delta\beta_{(1)})^k\delta\beta_{(l)},\\
&\langle\alpha_1(\beta_1)^k\beta_2\beta_{p-1}\rangle_1 = \alpha\delta(\beta_{(1)}\delta)^k\beta_{(2)}\delta\beta_{(p-1)}\delta - \delta\alpha(\delta\beta_{(1)})^k\delta\beta_{(2)}\delta\beta_{(p-1)}.
\end{aligned}$$

The following relations are Corollary 5.6 of [13].

(5.13) $\quad \alpha(\delta\beta_{(1)})^p = (\beta_{(1)}\delta)^p\alpha = 0.$

(5.14) $\quad (\beta_{(1)}\delta)^p\beta_{(2)} = \beta_{(2)}(\delta\beta_{(1)})^p = 0.$

Now, we recall the results of the p-component ${}_pG_k$ of the group G_k, from

[5](cf. [8]). We denote by

$$Z_{p^r}\{\xi_1,\dots,\xi_s\}$$

the Z_{p^r}-module generated freely by the elements $\xi_1,\dots,\xi_s$.

(5.15) (i) $_pG_k = Z_p\{\alpha_s\}$ *for* $k = sq-1,\ s\not\equiv 0 \bmod p,\ 1\leqq s<p^2,$

$= Z_{p^2}\{\alpha'_{sp}\}$ *for* $k = spq-1,\ 1\leqq s<p-1,$

$= Z_{p^2}\{\alpha'_{(p-1)p}\}+Z_p\{\alpha_1(\beta_1)^{p-1}\}$ *for* $k=(p-1)pq-1,$

$= Z_{p^3}\{\alpha''_{p^2}\}$ *for* $k=p^2q-1,$

$= Z_p\{(\beta_1)^r\beta_s\}$ *for* $k=((r+s)p+s-1)q-2r-2,$

$0\leqq r<p,\ 1\leqq s<p,\ r+s<p$ *and* $(r,s)=(p-1,1),\ (p-2,2),$

$= Z_p\{\alpha_1(\beta_1)^r\beta_s\}$ *for* $k=((r+s)p+s)q-2r-3,$

$0\leqq r<p,\ 1\leqq s<p,\ r+s<p,\ (r,s)\neq(p-2,1),\ (p-1,1),$

$= 0$ *otherwise for* $k<(p^2+1)q-3.$

(ii) $_pG_k = Z_p\{\varepsilon'\}$ *for* $k=(p^2+1)q-3,$

$= Z_p\{\varepsilon_1\}$ *for* $k=(p^2+1)q-2,$

$= Z_p\{\alpha_{p^2+1},\alpha_1(\beta_1)^{p-2}\beta_2\}$ *for* $k=(p^2+1)q-1,$

$= Z_p\{(\beta_1)^{p-3}\beta_3\}$ *for* $k=(p^2+1)q+2,$

$= Z_p\{\varepsilon_1\alpha_1\}$ *for* $k=(p^2+2)q-3,$

$= Z_p\{\varepsilon_2\}$ *for* $k=(p^2+2)q-2,$

$= Z_p\{\alpha_{p^2+2}\}$ *for* $k=(p^2+2)q-1,$

$= 0$ *otherwise for* $(p^2+1)q-3\leqq k\leqq(p^2+2)q.$

Applying Theorem 3.5 to the results (5.15) (i) and using Theorem 4.1 and relations (5.2) (i) and (5.4)–(5.13), we can easily see the ring structure of $\mathscr{A}_*(M_p)$ for degree $<(p^2+1)q-4$.

THEOREM 5.1 (cf. [14; Th. I and II], [13; Th. 5.1 and 5.2]). *Within the limits of degree less than $(p^2+1)q-4$, the ring $\mathscr{A}_*(M_p)$ is generated by the elements $\delta\in\mathscr{A}_{-1}$, $\alpha\in\mathscr{A}_q$ and $\beta_{(s)}\in\mathscr{A}_{(sp+s-1)q-1}$, $1\leqq s<p$, with the relations:*

$$\delta^2=0,\ \delta\alpha^2=-\alpha^2\delta+2\alpha\delta\alpha,\ \beta_{(s)}\delta\alpha=\alpha\delta\beta_{(s)},\ \alpha\beta_{(s)}=\beta_{(s)}\alpha=0,$$

$$\beta_{(s)}\delta\beta_{(t)}=(st/(s+t-1))\beta_{(1)}\delta\beta_{(s+t-1)},\ \beta_{(s)}\beta_{(t)}=0,\ \alpha(\delta\beta_{(1)})^p=0;$$

and it is the direct sum of cyclic groups of order p generated by the following elements:

$$\delta,\ 1,\ \alpha^{s-1}\delta\alpha\delta,\ \alpha^{s-1}\delta\alpha,\ \alpha^s\delta,\ \alpha^s \qquad (1 \leqq s \leqq p^2),$$

$$\delta^a(\beta_{(1)}\delta)^r\beta_{(s)}\delta^b \qquad (a,\ b = 0 \text{ or } 1,\ 0 \leqq r < p,\ 1 \leqq s < p,\ r+s < p \text{ and } (r, s) = (p-1, 1), (p-2, 2)),$$

$$\delta^a\alpha\delta(\beta_{(1)}\delta)^r\beta_{(s)}\delta^b \qquad (a,\ b = 0 \text{ or } 1,\ 0 \leqq r < p,\ 1 \leqq s < p,\ r+s < p).$$

Next we introduce the following new generators $\bar{\varepsilon}$ and ε.

PROPOSITION 5.2. *There exist elements* $\bar{\varepsilon} \in \mathscr{A}_{(p^2+1)q-2}(M_p)$ *and* $\varepsilon \in \mathscr{A}_{(p^2+1)q-1}(M_p)$ *such that*

(5.16) (i) $D(\bar{\varepsilon}) = 0,\ \pi_* i^*(\bar{\varepsilon}) = \varepsilon'$;

(ii) $\bar{\varepsilon} \in \langle(\beta_{(1)}\delta)^{p-1}\beta_{(1)},\ \alpha\delta-\delta\alpha,\ \alpha\delta-\delta\alpha\rangle$;

and that

(5.17) (i) $D(\varepsilon) = 0,\ \pi_* i^*(\varepsilon) = \varepsilon_1$;

(ii) $\varepsilon \in \langle\alpha,\ \beta_{(1)},\ \delta\alpha(\delta\beta_{(1)})^{p-1}\rangle$;

where $\varepsilon' \in {}_pG_{(p^2+1)q-3} = Z_p$ *and* $\varepsilon_1 \in {}_pG_{(p^2+1)q-2} = Z_p$ *are the generators (see* (5.15) (ii)). *The element* $\bar{\varepsilon}$ *is uniquely determined and the element* ε *is determined modulo the subgroup* Z_p, *generated by* $\alpha^{p^2+1}\delta - \alpha^{p^2}\delta\alpha$, *of* $\mathscr{A}_{(p^2+1)q-1}(M_p)$. *For these elements, we have*

$$\mathscr{A}_{(p^2+1)q-4}(M_p) = Z_p\{\delta\bar{\varepsilon}\delta\},$$

$$\mathscr{A}_{(p^2+1)q-3}(M_p) = Z_p\{\bar{\varepsilon}\delta,\ \delta\bar{\varepsilon},\ \delta\varepsilon\delta\},$$

$$\mathscr{A}_{(p^2+1)q-2}(M_p) = Z_p\{\bar{\varepsilon},\ \varepsilon\delta,\ \delta\varepsilon,\ \delta\alpha\delta(\beta_{(1)}\delta)^{p-2}\beta_{(2)}\delta,\ \alpha^{p^2}\delta\alpha\delta\},$$

$$\mathscr{A}_{(p^2+1)q-1}(M_p) = Z_p\{\varepsilon,\ \alpha\delta(\beta_{(1)}\delta)^{p-2}\beta_{(2)}\delta,\ \delta\alpha(\delta\beta_{(1)})^{p-2}\delta\beta_{(2)},\ \alpha^{p^2+1}\delta,\ \alpha^{p^2}\delta\alpha\},$$

$$\mathscr{A}_{(p^2+1)q}(M_p) = Z_p\{\alpha\delta(\beta_{(1)}\delta)^{p-2}\beta_{(2)},\ \alpha^{p^2+1}\}.$$

PROOF. We put $n = (p^2+1)q$ and $\gamma = \alpha\delta(\beta_{(1)}\delta)^{p-2}\beta_{(2)}$. By Corollary 3.7, any elements $\bar{\varepsilon}$ and ε satisfying (5.16) (i) and (5.17) (i) are obtained by setting $\bar{\varepsilon} = [\varepsilon']_1$ and $\varepsilon = [\varepsilon_1]_1$. For these $\bar{\varepsilon}$ and ε, the assertions on the additive structure of $\mathscr{A}_k(M_p)$, $n-4 \leqq k \leqq n$, are proved by applying (5.15) (ii) to Theorem 3.5 and by using (4.6) and (5.11)–(5.12).

By (5.2) (i), (5.4), (5.13) and (4.3), the stable Toda brackets $A=\langle(\beta_{(1)}\delta)^{p-1}\cdot\beta_{(1)}, \alpha\delta-\delta\alpha, \alpha\delta-\delta\alpha\rangle$ and $B=\langle\alpha, \beta_{(1)}, \delta\alpha(\delta\beta_{(1)})^{p-1}\rangle$ are well defined. We see from Theorem 5.1 that A is a coset of the subgroup Q generated by $\delta\gamma\delta$ and $\alpha^{p^2}\delta\alpha\delta$ and that B is a coset of the subgroup R generated by $\gamma\delta$, $\alpha^{p^2+1}\delta$ and $\alpha^{p^2}\delta\alpha$.

Now we recall the definition [6; (12.3)–(12.4)] of the element ε_1 of (5.15) (ii), that is, we defined $\varepsilon_1=\pi_* i^*\langle\alpha, \beta_{(1)}\delta, (\beta_{(1)}\delta)^{p-1}\alpha\rangle$. Hence $\varepsilon_1=\pi_* i^* B$. Let ξ be any element of B. Then $D(\xi)\in\mathscr{A}_n$ and $D(\xi)=x\gamma+y\alpha^{p^2+1}$ for some $x, y\in Z_p$. Then $\xi'=\xi-x\gamma\delta-y\alpha^{p^2+1}\delta\equiv\xi \bmod R$ satisfies $D(\xi')=0$ and $\xi'\in B$. Hence, B contains an element satisfying (5.17) (i), and the element ε is obtained. This is determined modulo $R\cap\operatorname{Ker}\pi_* i^*\cap\operatorname{Ker}D$, which is spanned by $\alpha^{p^2+1}\delta-\alpha^{p^2}\delta\alpha$.

Next, in [5; (6.2)], the element ε' of (5.15) (ii) was defined by $\varepsilon'=\langle(\beta_1)^p, \alpha_1, \alpha_1\rangle$. We have $\pi_* i^* A=-\langle(\beta_1)^p, \alpha_1, \alpha_1\rangle=-\varepsilon'$ by $\pi(\beta_{(1)}\delta)^{p-1}\beta_{(1)}i=(\beta_1)^p$ and $(\alpha\delta-\delta\alpha)i=-i\alpha_1$. Here the sign $-$ occurs from the first formula in p. 645 of the stable bracket. So we change the sign of ε' so that $\pi_* i^* A=\varepsilon'$. Let ξ be any element of A. Put $D(\xi)=x\varepsilon+y\gamma\delta+z\delta\gamma+u\alpha^{p^2+1}\delta+v\alpha^{p^2}\delta\alpha$. Then $0=D^2(\xi)=(z+y)\gamma+(u+v)\alpha^{p^2+1}$ and $D(\xi)=x\varepsilon+yD(\delta\gamma\delta)+uD(\alpha^{p^2}\delta\alpha\delta)\equiv x\varepsilon \bmod D(Q)$, and hence there exists uniquely $\xi'\in A$ such that $\pi^* i^*\xi'=\varepsilon'$ and $D(\xi')=x\varepsilon$, where the uniqueness follows from $Q\cap\operatorname{Ker}\pi_* i^*\cap\operatorname{Ker}D=0$.

To prove $x=0$, we calculate the group $\mathscr{A}_{n+q-1}(M_p)$. First we have from (1.11) that

$$\varepsilon\xi=(-1)^{\deg\xi}\xi\varepsilon \qquad \textit{for any}\ \ \xi\in\mathscr{A}_*(M_p)\cap\operatorname{Ker}D,$$

and in particular

$$\varepsilon\alpha=\alpha\varepsilon. \tag{5.16}$$

The generator ε_2 of ${}_pG_{n+q-2}$ in (5.15) (ii) is defined by $\varepsilon_2=\langle\varepsilon_1, p, \alpha_1\rangle$ [5; (6.3)], where the bracket consists of a single element. By Proposition 3.9, we can take $[\varepsilon_2]_1=\varepsilon\alpha$ and we see that

$$\mathscr{A}_{n+q-1}(M_p)=Z_p\{\varepsilon\alpha, \alpha^{p^2+2}\delta, \alpha^{p^2+1}\delta\alpha\}. \tag{$*$}$$

Since $\langle\alpha, (\beta_{(1)}\delta)^{p-1}\beta_{(1)}, \alpha\rangle$ has full indeterminacy and $\langle\alpha, (\beta_{(1)}\delta)^{p-1}\alpha_{(1)}, \alpha\delta\rangle$ has R as its indeterminacy, it follows that

$$B\subset\langle\alpha, (\beta_{(1)}\delta)^{p-1}\beta_{(1)}, \delta\alpha\rangle=-\langle\alpha, (\beta_{(1)}\delta)^{p-1}\beta_{(1)}, \alpha\delta-\delta\alpha\rangle.$$

Here the last bracket $-B'$ is a coset of the subgroup R' generated by R and $\delta\gamma$.

Now we prove $x=0$ in $D(\xi')=x\varepsilon$. We have

$$\varepsilon(\alpha\delta-\delta\alpha)\in-B'(\alpha\delta-\delta\alpha)=\alpha A\ni\alpha\xi',$$

and hence

$$\varepsilon(\alpha\delta-\delta\alpha) \equiv \alpha\xi' \bmod R'(\alpha\delta-\delta\alpha) = Z_p\{\alpha^{p^2+1}\delta\alpha\delta\}.$$

Put $\varepsilon(\alpha\delta-\delta\alpha)=\alpha\xi'+y\alpha^{p^2+1}\delta\alpha\delta$. Applying D to this, we have $0=x\alpha\varepsilon+y(\alpha^{p^2+2}\delta-\alpha^{p^2+1}\delta\alpha)$, and so $x=y=0$ by (*) and (5.16). q.e.d.

From the above discussion on $x=y=0$, we have a relation

$$\alpha\bar{\varepsilon} = \varepsilon\alpha\delta-\varepsilon\delta\alpha. \tag{5.17}$$

The following are consequences of (1.11)–(1.12):

$$(\varepsilon\delta+\delta\varepsilon)\xi = \xi(\varepsilon\delta+\delta\varepsilon) \qquad \textit{for any} \quad \xi\in\mathscr{A}_*(M_p);$$

$$\bar{\varepsilon}\xi = \xi\bar{\varepsilon} \qquad \textit{for any} \quad \xi\in\mathscr{A}_*(M_p)\cap \operatorname{Ker} D;$$

$$(\bar{\varepsilon}\delta-\delta\bar{\varepsilon})\xi = (-1)^{\deg\xi}\xi(\bar{\varepsilon}\delta-\delta\bar{\varepsilon}) \qquad \textit{for any} \quad \xi\in\mathscr{A}_*(M_p).$$

In particular we have

$$\alpha\delta\varepsilon = \varepsilon\delta\alpha+\delta\varepsilon\alpha-\varepsilon\alpha\delta, \tag{5.18}$$

$$\bar{\varepsilon}\alpha = \alpha\bar{\varepsilon}. \tag{5.19}$$

Also we have

$$\bar{\varepsilon}\delta\alpha = \bar{\varepsilon}\alpha\delta = -\varepsilon\delta\alpha\delta, \quad \alpha\delta\bar{\varepsilon} = \delta\bar{\varepsilon}\alpha = \delta\varepsilon\alpha\delta-\delta\varepsilon\delta\alpha. \tag{5.20}$$

For this, we have $\bar{\varepsilon}(\alpha\delta-\delta\alpha)=0$ since $\bar{\varepsilon}(\alpha\delta-\delta\alpha)\in(\beta_{(1)}\delta)^{p-1}\beta_{(1)}\langle\alpha\delta-\delta\alpha, \alpha\delta-\delta\alpha, \alpha\delta-\delta\alpha\rangle\subset(\beta_{(1)}\delta)^{p-1}\beta_{(1)}\mathscr{A}_{3q-2}(M_p)=Z_p\{(\beta_{(1)}\delta)^{p-1}\beta_{(1)}\alpha^2\delta\alpha\delta\}=0$. So the first follows from (5.17) and (5.19). By (4.5), $(\alpha\delta-\delta\alpha)\bar{\varepsilon}=0$ and the second also follows.

§ 6. The ring structure of $\mathscr{A}_*(M_p)$.

We first introduce the results on ${}_pG_*$ from [5] and [6].

(6.1)

$${}_pG_k = Z_p\{\alpha_s\} \qquad \textit{for } k = sq-1, \quad p^2+3 \leqq s \leqq p^2+3p-1,$$
$$s \neq p^2+p+2, \quad p^2+2p+3,$$

$$Z_{p^2}\{\alpha'_{sp}\} \qquad \textit{for } k=spq-1, \quad p+1 \leqq s \leqq p+3 \textit{ except } (s, p)=(p+3, 5),$$

$$Z_p\{\alpha_{p^2+sp+s+1}, \alpha_1(\beta_1)^{p-2}\beta_{s+2}\} \qquad \textit{for } k = (p^2+sp+s+1)q-1,$$
$$s = 1 \textit{ or } 2,$$

$$Z_{p^2}\{\alpha'_{p^2+3p}\}+Z_p\{(\beta_1)^3\varepsilon'\} \quad \textit{for } k = (p^2+3p)q-1, \quad p = 5,$$

$Z_p\{(\beta_1)^{p-s}\beta_s\}$ *for* $k=(p^2+s-2)q+2s-4,\quad 4\leqq s<p,$

$Z_p\{(\beta_1)^{p+1-s}\beta_s\}$ *for* $k=(p^2+p+s-2)q+2s-6,\quad 1\leqq s\leqq p+1,\ s\neq p,$

$Z_p\{(\beta_1)^{p+2-s}\beta_s\}$ *for* $k=(p^2+2p+s-2)q+2s-8,\quad 1\leqq s\leqq p+1,$

$s\neq 2,\quad p,$

$Z_p\{(\beta_1)^{p+3}\}$ *for* $k=(p^2+3p-1)q-8,$

$Z_p\{\beta_2\beta_{p-1}\}$ *for* $k=(p^2+2p-1)q-4,$

$Z_p\{\beta_1\beta_2\beta_{p-1}\}$ *for* $k=(p^2+3p-1)q-6,$

$Z_p\{\alpha_1(\beta_1)^{p-s}\beta_s\}$ *for* $k=(p^2+s-1)q+2s-5,\quad 3\leqq s<p,$

$Z_p\{\alpha_1(\beta_1)^{p+1-s}\beta_s\}$ *for* $k=(p^2+p+s-1)q+2s-7,$

$2\leqq s\leqq p+1,\quad s\neq 3,\quad p,$

$Z_p\{\alpha_1(\beta_1)^{p+2-s}\beta_s\}$ *for* $k=(p^2+2p+s-1)q+2s-9,$

$3\leqq s\leqq p-1,\quad s\neq 4,$

$Z_p\{\alpha_1\beta_2\beta_{p-1}\}$ *for* $k=(p^2+2p)q-5,$

$Z_p\{\alpha_1\beta_1\beta_2\beta_{p-1}\}$ *for* $k=(p^2+3p)q-7,$

$Z_p\{\varepsilon_i\}$ *for* $k=(p^2+i)q-2,\quad 3\leqq i<p,$

$Z_p\{\varepsilon_i\alpha_1\}$ *for* $k=(p^2+i+1)q-3,\quad 2\leqq i\leqq p-2,$

$Z_{p^2}\{\varphi\}$ *for* $k=(p^2+p)q-3,$

$Z_p\{(\beta_1)^r\varepsilon'\}$ *for* $k=(p^2+rp+1)q-2r-3,\quad 1\leqq r\leqq 3$

except $(r,p)=(3,5),$

0 *otherwise for* $(p^2+2)q<k<(p^2+3p+1)q-5.$

Here we denote by

$$Z_{p^r}\{\xi\}$$

the cyclic group of order p^r generated by the element ξ, and by

$$Z_{p^r}\{\xi_1,\dots,\xi_s\}$$

the direct sum $Z_{p^r}\{\xi_1\}+\cdots+Z_{p^r}\{\xi_s\}$.

The elements ε_i, $2\leqq i<p$, in (5.15) (ii) and (6.1) are defined inductively by

$$\varepsilon_i = \langle \varepsilon_{i-1}, p, \alpha_1 \rangle,$$

where the bracket consists of a single element [5; (6.3), (7.3)]. Hence by Proposition 3.9 and (5.17), we can take

(6.2) $$[\varepsilon_i]_1 = \varepsilon\alpha^{i-1} = \alpha^{i-1}\varepsilon \quad \textit{for} \quad 1 \leqq i \leqq p-1.$$

Also by (3.5),

(6.2)′ $$\langle \varepsilon_i \rangle_1 = \varepsilon\alpha^{i-1}\delta + \delta\varepsilon\alpha^{i-1}, \qquad 1 \leqq i \leqq p-1,$$

and by Proposition 3.8, we have

(6.3) $$[\varepsilon_i\alpha_1] = \langle \varepsilon_i \rangle_1 \alpha \equiv \varepsilon\alpha^{i-1}\delta\alpha \bmod \delta[\varepsilon_{i+1}]_1, \qquad 1 \leqq i \leqq p-3.$$

The following relation will be proved in § 8.

LEMMA 6.1. *The bracket* $\langle \varepsilon_{p-1}, p, \alpha_1 \rangle$ *consists of a single element* 0.
The following proposition determines uniquely the element ε.

PROPOSITION 6.2. *We can take the element* ε *of* (5.17) *such that*

(6.4) $$\varepsilon\alpha^{p-1} = \alpha^{p-1}\varepsilon = 0,$$

and ε *is uniquely determined by this and* (5.17).

PROOF. The element $\varepsilon\alpha^{p-1}$ belongs to $\mathscr{A}_{(p^2+p)q-1}(M_p)$, and we have $\mathscr{A}_{(p^2+p)q-1}(M_p) = Z_p\{(\beta_{(1)}\delta)^{p-1}\beta_{(2)}, \alpha^{p^2+p}\delta, \alpha^{p^2+p-1}\delta\alpha\}$ by Theorem 3.5, (6.1), Theorem 4.1 and (5.11). Since $D(\varepsilon\alpha^{p-1})=0$, $D((\beta_{(1)}\delta)^{p-1}\beta_{(2)})=0$ and $D(\alpha^{p^2+p}\delta) = D(\alpha^{p^2+p-1}\delta\alpha) = \alpha^{p^2+p}$, we can put $\varepsilon\alpha^{p-1} = x(\beta_{(1)}\delta)^{p-1}\beta_{(2)} + y(\alpha^{p^2+p}\delta - \alpha^{p^2+p-1}\delta\alpha)$. Since $\pi_* i^* \varepsilon\alpha^{p-1} \in \langle \varepsilon_{p-1}, p, \alpha_1 \rangle = 0$ by Lemma 6.1, and since $\pi_* i^*(\beta_{(1)}\delta)^{p-1}\beta_{(2)} = (\beta_1)^{p-1}\beta_2 \neq 0$ and $\pi_* i^*(\alpha^{p^2+p}\delta - \alpha^{p^2+p-1}\delta\alpha) = 0$, it follows that $x=0$ and $\varepsilon\alpha^{p-1} = y(\alpha^{p^2+1}\delta - \alpha^{p^2}\delta\alpha)\alpha^{p-1}$ by (4.4). Replacing ε with $\varepsilon + y(\alpha^{p^2+1}\delta - \alpha^{p^2}\delta\alpha)$ and using (5.17), we have the proposition. q.e.d.

Next we consider the element $\varphi \in {}_pG_{(p^2+p)q-3} = Z_{p^2}$ of (6.1). This is the only element of order $\geqq p^2$ in ${}_pG_k$ for $0<k<(p^2+3p+1)q-5$, except the elements α'_{sp} and α''_{p^2}. By [5; Th. 7.9], φ is defined by

(6.5) (i) $$\varphi \in \langle \varepsilon_{p-2}, \alpha_1, \alpha_1 \rangle$$

and there is a relation

(6.5) (ii) $$p\varphi = \varepsilon_{p-1}\alpha_1.$$

By Proposition 3.8, (6.5) (ii), (6.2)′ and (6.4),

$$(6.6) \qquad [\varphi]_1 = (\varepsilon\alpha^{p-2}\delta+\delta\varepsilon\alpha^{p-2})\alpha = \varepsilon\alpha^{p-2}\delta\alpha$$

and by (3.5) $\delta[\varphi]_1=[\varphi]_1\delta$, i.e.,

$$(6.7) \qquad \delta\varepsilon\alpha^{p-2}\delta\alpha = \varepsilon\alpha^{p-2}\delta\alpha\delta.$$

We define

$$(6.8)\ \text{(i)} \qquad \bar{\varphi} = \langle\varphi\rangle_1 \in \mathscr{A}_{(p^2+p)q-3}(M_p)\,.$$

This satisfies

$$(6.8)\ \text{(ii)} \qquad D\bar{\varphi} = 0,\ i^*\bar{\varphi} = -i_*\varphi,\quad \pi_*\varphi = \pi^*\varphi;$$

$$\text{(iii)} \qquad \bar{\varphi}\xi = (-1)^{\deg\xi}\xi\bar{\varphi} \qquad \textit{for any } \xi\in\mathscr{A}_*(M_p);$$

by Lemma 3.1. In particular,

$$(6.9) \qquad \bar{\varphi}\delta = -\delta\bar{\varphi}.$$

PROPOSITION 6.3. *The element $\bar{\varphi}$ belongs to the bracket*

$$\langle\varepsilon\alpha^{p-3}\delta+\delta\varepsilon\alpha^{p-3},\alpha\delta-\delta\alpha,\alpha\delta-\delta\alpha\rangle\,,$$

and conversely this and $D\bar{\varphi}=0$ determine uniquely the element $\bar{\varphi}$.

PROOF. Since $\langle\varepsilon_{p-2}\rangle=\varepsilon\alpha^{p-3}\delta+\delta\varepsilon\alpha^{p-3}$ and $\langle\alpha_1\rangle=\alpha\delta-\delta\alpha$, it follows from (6.5) (i) that $\bar{\varphi}\in\langle\langle\varepsilon_{p-2}\rangle,\langle\alpha_1\rangle,\langle\alpha_1\rangle\rangle=\langle\varepsilon\alpha^{p-3}\delta+\delta\varepsilon\alpha^{p-3},\ \alpha\delta-\delta\alpha,\ \alpha\delta-\delta\alpha\rangle$. We have the following results:

$$\mathscr{A}_{2q-1}(M_p) = Z_p\{\alpha^2\delta,\alpha\delta\alpha\}\,,$$

$$\mathscr{A}_{(p^2+p-1)q-2}(M_p) = Z_p\{\varepsilon\alpha^{p-2}\delta,\delta\varepsilon\alpha^{p-2},\alpha^{p^2+p-2}\delta\alpha\delta\}\,,$$

$$\mathscr{A}_{(p^2+p)q-3}(M_p) = Z_p\{\bar{\varphi},\varepsilon\alpha^{p-2}\delta\alpha\delta,\delta(\beta_{(1)}\delta)^{p-1}\beta_{(2)}\delta\}\,,$$

$$\mathscr{A}_{(p^2+p)q-2}(M_p) = Z_p\{\varepsilon\alpha^{p-2}\delta\alpha,(\beta_{(1)}\delta)^{p-1}\beta_{(2)}\delta,(\delta\beta_{(1)})^{p-1}\delta\beta_{(2)},\alpha^{p^2+p-1}\delta\alpha\delta\}\,,$$

from Theorem 5.1, Theorem 3.5, (6.1), Theorem 4.1, (5.11), (6.2), (6.6) and (6.8) (i). Hence by using (4.4), (6.4) and (6.7) the above bracket is a ocset of $Z_p\{\varepsilon\alpha^{p-2}\delta\alpha\delta\}$, whose D-image is $Z_p\{\varepsilon\alpha^{p-2}\delta\alpha\}$. Thus, we see that $\langle\varepsilon\alpha^{p-3}\delta+\delta\varepsilon\alpha^{p-3},\ \alpha\delta-\delta\alpha,\ \alpha\delta-\delta\alpha\rangle\cap\operatorname{Ker} D$ consists of a single element. This shows the uniqueness of $\bar{\varphi}$. q.e.d.

PROPOSITION 6.4. *The assertions on the additive structure of $\mathscr{A}_*(M_p)$ in Theorem 0.1 hold, namely, for $k<(p^2+3p+1)q-6$ a Z_p-basis for $\mathscr{A}_k(M_p)$ is given by the following elements*

$$\delta,\ 1,\ \alpha^s\delta^a,\quad \alpha^{s-1}\delta\alpha\delta^a\ \ (1 \leqq s \leqq p^2+3p),$$

$$\delta^a(\beta_{(1)}\delta)^{r-1}\beta_{(1)}\delta^b\ \ (1 \leqq r \leqq p+3),\qquad \delta^a(\beta_{(1)}\delta)^r\beta_{(s)}\delta^b((r, s) \in I),$$

$$\delta^a(\beta_{(1)}\delta)^r\beta_{(2)}\delta\beta_{(p-1)}\delta^b\ \ (r = 0, 1),\qquad \delta^a\alpha(\delta\beta_{(1)})^r\delta^b\ \ (1 \leqq r < p),$$

$$\delta^a\alpha(\delta\beta_{(1)})^r\delta\beta_{(s)}\delta^b\ \ ((r, s) \in J),\qquad \delta^a\alpha(\delta\beta_{(1)})^r\delta\beta_{(2)}\delta\beta_{(p-1)}\delta^b\ \ (r = 0, 1),$$

$$\delta^a(\beta_{(1)}\delta)^r\bar{\varepsilon}\delta^b\ \ (0 \leqq r \leqq 3),\qquad \delta^a\varepsilon\alpha^i\delta^b\ \ (0 \leqq i \leqq p-2),$$

$$\delta^a\varepsilon\alpha^{i-1}\delta\alpha\delta^b\ \ (1 \leqq i \leqq p-3),\qquad \varepsilon\alpha^{p-2}\delta\alpha\delta^a,\qquad \bar{\varphi}\delta^a,$$

where a, $b=0$ or 1, and the index sets I and J are given by

$$I = \{(r, s) | 0 \leqq r < p,\quad 2 \leqq s \leqq p+1,\quad s \neq p,\quad r+s \leqq p+2\},$$

$$J = I - \{(1, p+1)\}.$$

PROOF. For $k \leqq (p^2+1)q$ the proposition is already proved in Theorem 5.1 and Proposition 5.2. So we consider the case $k>(p^2+1)q$. To apply Theorem 3.5 we determine $[\gamma]$ and $\langle\gamma\rangle$ for any generator γ in (5.15) (ii) and (6.1). For $\gamma=\alpha_s$ and α'_{sp}, this is done in §4, and the subring $A(\alpha, \delta)$ of Theorem 4.1 is obtained. For $\gamma=\varepsilon_i$ and φ this is done in (6.2), (6.2)′, (6.6) and (6.8) (i). For other γ, we have $\langle\gamma\rangle=[\gamma]\delta+(-1)^k\delta[\gamma]$ by (3.5) because γ is of order p. Hence we only determine $[\gamma]$. For $\gamma=(\beta_1)^r\beta_s$ $(s \neq p-1)$, $(\beta_1)^r\beta_2\beta_{p-1}$, $\alpha_1(\beta_1)^r\beta_s$ and $\alpha_1(\beta_1)^r\beta_2\beta_{p-1}$, this is done in (5.11) and (6.3).

Using Proposition 3.8, we have the following values of $[\gamma]$ for other γ:

$$[\varepsilon_i\alpha_1] \equiv \varepsilon\alpha^{i-1}\delta\alpha \bmod \delta[\varepsilon_{i+1}]\quad \text{by (6.3)},$$

$$\begin{aligned}[(\beta_1)^r\beta_{p-1}] &= ((\beta_{(1)}\delta)^r+(\delta\beta_{(1)})^r)\beta_{(p-1)}\\ &\equiv (\beta_{(1)}\delta)^r\beta_{(p-1)} \bmod (\delta\beta_{(1)})^r\beta_{(p-1)},\quad r \geqq 1,\\ [(\beta_1)^r\varepsilon'] &= ((\beta_{(1)}\delta)^r+(\delta\beta_{(1)})^r)\bar{\varepsilon}\\ &\equiv (\beta_{(1)}\delta)^r\bar{\varepsilon} \bmod (\delta\beta_{(1)})^r\bar{\varepsilon},\quad r \geqq 1.\end{aligned}$$

Thus, the proposition with the replacement of $(\beta_{(1)}\delta)^r\beta_{(p-1)}$ and $(\beta_{(1)}\delta)^r\bar{\varepsilon}$ by $((\beta_{(1)}\delta)^r+(\delta\beta_{(1)})^r)\beta_{(p-1)}$ and $((\beta_{(1)}\delta)^r+(\delta\beta_{(1)})^r)\bar{\varepsilon}$, $r \geqq 1$, is established by Theorem 3.5. In particular we have

$$\mathscr{A}_{(p^2+p-2)q-2}(M_p) = Z_p\{\varepsilon\alpha^{p-4}\delta\alpha,\ \varepsilon\alpha^{p-3}\delta,\ \delta\varepsilon\alpha^{p-3},\ \alpha^{p^2+p-3}\delta\alpha\delta\},$$

$$\mathscr{A}_{(p^2+p+1)q-3}(M_p) = Z_p\{\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(2)}\delta,\ \delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(2)}\}.$$

We have therefore

$$(*)\qquad \begin{aligned} &\delta\beta_{(1)}\beta_{(p-1)} \in Z_p\{\delta\varepsilon\alpha^{p-4}\delta\alpha,\ \delta\varepsilon\alpha^{p-3}\delta\} = Z_p\{\delta[\varepsilon_{p-3}\alpha_1],\ \delta[\varepsilon_{p-2}]\delta\}\,,\\ &\delta\beta_{(1)}\bar{\varepsilon} \in Z_p\{\delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(2)}\delta\} = Z_p\{\delta[\alpha_1(\beta_1)^{p-1}\beta_2]\delta\}\,,\end{aligned}$$

and hence we can replace $(\beta_{(1)}\delta+\delta\beta_{(1)})\beta_{(p-1)}$ and $(\beta_{(1)}\delta+\delta\beta_{(1)})\bar{\varepsilon}$ by $\beta_{(1)}\delta\beta_{(p-1)}$ and $\beta_{(1)}\delta\bar{\varepsilon}$ respectively. Since $\beta_{(1)}\delta\varepsilon\alpha \in \mathscr{A}_{(p^2+p+1)q-3}(M_p)\alpha=0$ by (5.2) (i), (5.4) and (5.6), we see from (*) $(\delta\beta_{(1)})^r\beta_{(p-1)}=0$ for $r\geqq 2$. Also, $(\delta\beta_{(1)})^r\bar{\varepsilon}=0$ for $r\geqq 2$ by (5.4), (5.13) and (*).

Thus, the proposition is established entirely. q.e.d.

In the rest of this section, we study the multiplicative structure of $\mathscr{A}_*(M_p)$.

Proposition 6.5. *There is a relation*

$$\varepsilon\alpha^{p-3}\delta\alpha = 2\varepsilon\alpha^{p-2}\delta+\delta\varepsilon\alpha^{p-2}\,. \tag{6.10}$$

Proof. First we have from Proposition 6.4

$$\mathscr{A}_{(p^2+p-1)q-2}(M_p) = Z_p\{\varepsilon\alpha^{p-2}\delta,\ \delta\varepsilon\alpha^{p-2},\ \alpha^{p^2+p-2}\delta\alpha\delta\}\,,$$

$$\mathscr{A}_{(p^2+p-1)q-1}(M_p) = Z_p\{\varepsilon\alpha^{p-2},\ \alpha^{p^2+p-1}\delta,\ \alpha^{p^2+p-2}\delta\alpha\}\,.$$

Put $\varepsilon\alpha^{p-3}\delta\alpha = x\varepsilon\alpha^{p-2}\delta+y\delta\varepsilon\alpha^{p-2}+z\alpha^{p^2+p-2}\delta\alpha\delta$. Then, $-\varepsilon\alpha^{p-2}=D(\varepsilon\alpha^{p-3}\delta\alpha)=(-x+y)\varepsilon\alpha^{p-2}+z(\alpha^{p^2+p-1}\delta-\alpha^{p^2+p-2}\delta\alpha)$ and $y=x-1$, $z=0$. Hence $\varepsilon\alpha^{p-3}\delta\alpha=x\varepsilon\alpha^{p-2}\delta+(x-1)\delta\varepsilon\alpha^{p-2}$. By (4.4) and (6.4), $2\varepsilon\alpha^{p-2}\delta\alpha=2\delta\alpha^{p-2}\delta\alpha-\varepsilon\alpha^{p-1}\delta=\varepsilon\alpha^{p-3}\delta\alpha^2=x\varepsilon\alpha^{p-2}\delta\alpha+(x-1)\delta\varepsilon\alpha^{p-1}=x\varepsilon\alpha^{p-2}\delta\alpha$, and so $x=2$. q.e.d.

Corollary 6.6. *The following relations hold.*

(i) $\alpha^i\varepsilon\alpha^j = \varepsilon\alpha^r$ *for* $r\leqq p-2$.

(ii) $\alpha^i\delta\alpha^j\varepsilon\alpha^k = i\varepsilon\alpha^{r-1}\delta\alpha-i\varepsilon\alpha^r\delta+\delta\varepsilon\alpha^r$ *for* $r\leqq p-3$,
$= i\varepsilon\alpha^{p-2}\delta+(i+1)\delta\varepsilon\alpha^{p-2}$ *for* $r\leqq p-2$,
$= i\varepsilon\alpha^{p-2}\delta\alpha$ *for* $r=p-1$.

(iii) $\alpha^i\varepsilon\alpha^j\delta\alpha^k = k\varepsilon\alpha^{r-1}\delta\alpha+(1-k)\varepsilon\alpha^r\delta$ *for* $r\leqq p-3$,
$= (k+1)\varepsilon\alpha^{p-2}\delta+k\delta\varepsilon\alpha^{p-2}$ *for* $r=p-2$,
$= k\varepsilon\alpha^{p-2}\delta\alpha$ *for* $r=p-1$.

(iv) $\alpha^i\bar{\varepsilon}\alpha^j = \varepsilon\alpha^r\delta-\varepsilon\alpha^{r-1}\delta\alpha$ *for* $r\leqq p-3$,
$= -\varepsilon\alpha^{p-2}\delta-\delta\varepsilon\alpha^{p-2}$ *for* $r=p-2$,
$= -\varepsilon\alpha^{p-2}\delta\alpha$ *for* $r=p-1$.

$$
\begin{aligned}
&(\mathrm{v}) && \alpha^i\delta\alpha^j\delta\alpha^k\varepsilon\alpha^l = j\delta\varepsilon\alpha^{r-1}\delta\alpha - j\delta\varepsilon\alpha^r\delta && \textit{for } r \leqq p-3, \\
& && = j\delta\varepsilon\alpha^{p-2}\delta && \textit{for } r = p-2, \\
& && = j\varepsilon\alpha^{p-2}\delta\alpha\delta && \textit{for } r = p-1. \\
&(\mathrm{vi}) && \alpha^i\delta\alpha^j\varepsilon\alpha^k\delta\alpha^l = i\varepsilon\alpha^{r-1}\delta\alpha\delta + l\delta\varepsilon\alpha^{r-1}\delta\alpha + (1-l)\delta\varepsilon\alpha^r\delta && \textit{for } r \leqq p-3, \\
& && = (i+l+1)\delta\varepsilon\alpha^{p-2}\delta && \textit{for } r = p-2, \\
& && = (i+l)\varepsilon\alpha^{p-2}\delta\alpha\delta && \textit{for } r = p-1. \\
&(\mathrm{vii}) && \alpha^i\varepsilon\alpha^j\delta\alpha^k\delta\alpha^l = k\varepsilon\alpha^{r-1}\delta\alpha\delta && \textit{for } r \leqq p-3 \textit{ and for } r = p-1, \\
& && = k\delta\varepsilon\alpha^{p-2}\delta && \textit{for } r = p-2. \\
&(\mathrm{viii}) && \alpha^i\delta\alpha^j\bar{\varepsilon}\alpha^k = \delta\varepsilon\alpha^r\delta - \delta\varepsilon\alpha^{r-1}\delta\alpha && \textit{for } r \leqq p-3, \\
& && = -\delta\varepsilon\alpha^{p-2}\delta && \textit{for } r = p-2, \\
& && = \varepsilon\alpha^{p-2}\delta\alpha\delta && \textit{for } r = p-1. \\
&(\mathrm{ix}) && \alpha^i\bar{\varepsilon}\alpha^j\delta\alpha^k = -\varepsilon\alpha^{r-1}\delta\alpha\delta && \textit{for } r \leqq p-3 \textit{ and for } r = p-1, \\
& && = -\delta\varepsilon\alpha^{p-2}\delta && \textit{for } r = p-2. \\
&(\mathrm{x}) && \alpha^i\delta\alpha^j\delta\alpha^k\varepsilon\alpha^l\delta\alpha^m = j\delta\varepsilon\alpha^{r-1}\delta\alpha\delta && \textit{for } r \leqq p-3. \\
&(\mathrm{xi}) && \alpha^i\delta\alpha^j\varepsilon\alpha^k\delta\alpha^l\delta\alpha^m = l\delta\varepsilon\alpha^{r-1}\delta\alpha\delta && \textit{for } r \leqq p-3. \\
&(\mathrm{xii}) && \alpha^i\delta\alpha^j\bar{\varepsilon}\alpha^k\delta\alpha^l = -\delta\varepsilon\alpha^{r-1}\delta\alpha\delta && \textit{for } r \leqq p-3.
\end{aligned}
$$

(xiii) *Other monomials on δ, α and ε involving just one ε are zero, and other monomials on δ, α and $\bar{\varepsilon}$ involving just one $\bar{\varepsilon}$ are zero.*

Here, in every equality r indicates the sum of exponents of α in the left side.

PROOF. These are easy consequences of (4.4), (5.16)–(5.20), (6.4), (6.7) and (6.10). q.e.d.

By Proposition 6.4, Corollary 6.6 and (4.4), the kernel of $\alpha^*\colon \mathscr{A}_{(p^2+p-2)q-2}(M_p)\to\mathscr{A}_{(p^2+p-1)q-2}(M_p)$ is spanned by the element $2\varepsilon\alpha^{p-4}\delta\alpha-3\varepsilon\alpha^{p-3}\delta-\delta\varepsilon\alpha^{p-3}$, which is equal to $(\bar{\varepsilon}-\varepsilon\delta-\delta\varepsilon)\alpha^{p-3}=\alpha^{p-3}(\bar{\varepsilon}-\varepsilon\delta-\delta\varepsilon)$. The element $\beta_{(1)}\beta_{(p-1)}=-\beta_{(p-1)}\beta_{(1)}$ belongs to this kernel by (5.2) (i), and H. Toda [13; Remark 5.4] showed that $\beta_{(1)}\beta_{(p-1)}\neq 0$. So, we have a relation

$$
\begin{aligned}
(6.11)\qquad \beta_{(1)}\beta_{(p-1)} &= -\beta_{(p-1)}\beta_{(1)} = 2\varepsilon\alpha^{p-4}\delta\alpha-3\delta\alpha^{p-3}\varepsilon-\delta\varepsilon\alpha^{p-3} \\
&= (\bar{\varepsilon}-\varepsilon\delta-\delta\varepsilon)\alpha^{p-3} = \alpha^{p-3}(\bar{\varepsilon}-\varepsilon\delta-\delta\varepsilon),
\end{aligned}
$$

where the second equality holds up to non zero coefficient.

By (5.5) and this relation, we can know the elements $\beta_{(s)}\beta_{(t)}$ in our calculations, and by (5.6) we know the elements $\beta_{(s)}\delta\beta_{(t)}$ for $s+t\neq p$. We have also

$$\beta_{(s)}\delta\beta_{(p-s)} = s^2\beta_{(1)}\delta\beta_{(p-1)}+(s(s-1)/2)(\beta_{(1)}\beta_{(p-1)}\delta+\delta\beta_{(1)}\beta_{(p-1)}). \tag{6.12}$$

For this, put $B_s=\beta_{(s)}\delta\beta_{(p-s)}$. By (5.5), $\beta_{(s)}\beta_{(p-s)}=s\beta_{(1)}\beta_{(p-1)}$. According to [13; (5.4)], $(2/s)B_s=-2(s+2)B_{p-1}+(s+1)B_{p-2}=-2(s-2)B_1+(s-1)B_2$. So, by (5.7), $B_{p-s}=B_s+s(\beta_{(1)}\beta_{(p-1)}\delta+\delta\beta_{(1)}\beta_{(p-1)})$, and (6.12) follows.

LEMMA 6.7. *The bracket $\langle\varphi, \alpha_1, p\rangle$ contains zero.*

The proof of this lemma will be given in § 8.

From Proposition 6.4 we have the following results:

$$\begin{aligned}
(6.13)\qquad & \mathscr{A}_{(p^2+p)q-3}(M_p) = Z_p\{\bar{\varphi}, (\delta\beta_{(1)})^{p-1}\delta\beta_{(2)}\delta\}, \\
& \mathscr{A}_{(p^2+p+1)q-3}(M_p) = Z_p\{\alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(2)}\delta, \delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(2)}\}, \\
& \mathscr{A}_{(p^2+2p)q-4}(M_p) = Z_p\{\alpha\delta\beta_{(2)}\delta\beta_{(p-1)}\}, \\
& \mathscr{A}_{(p^2+2p+2)q-4}(M_p) = Z_p\{\delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(3)}\delta\}, \\
& \mathscr{A}_{(p^2+2p+2)q-3}(M_p) = Z_p\{\alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(3)}\delta, \delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(3)}\}, \\
& \mathscr{A}_{(p^2+2p+2)q-2}(M_p) = Z_p\{\alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(3)}, \alpha^{p^2+2p+1}\delta\alpha\delta\}.
\end{aligned}$$

PROPOSITION 6.8. *The following holds.*

$$\bar{\varphi}\alpha = \alpha\bar{\varphi} = 0. \tag{6.14}$$

PROOF. By (6.8) (iii), $\alpha\bar{\varphi}=\bar{\varphi}\alpha$. By (6.13), we can put $\bar{\varphi}\alpha=x\xi\delta+y\delta\xi$, $\xi=\alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(2)}$. Then, $0=D(\bar{\varphi}\alpha)=(x+y)\xi$ and $\bar{\varphi}\alpha=x(\xi\delta-\delta\xi)$. By (6.8) (ii) and (5.11), $\varphi\pi\alpha=x\alpha_1(\beta_1)^{p-1}\beta_2\pi$. On the other hand, $\varphi\pi\alpha=\langle\varphi, \alpha_1, p\rangle\pi$. Hence $x=0$ by Lemma 6.7. q.e.d.

PROPOSITION 6.9. *The following relations hold.*

$$\beta_{(s)}\varepsilon = -\varepsilon\beta_{(s)} = \alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(s+1)} \qquad \textit{for any} \quad s\geqq 1. \tag{6.15}$$

$$\varepsilon\delta\beta_{(1)} = \alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(2)}\delta, \qquad \beta_{(1)}\delta\varepsilon = -\delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(2)}. \tag{6.16}$$

PROOF. By (5.17) (ii), (5.2) (ii), (5.4) and (5.6), we have

$$\begin{aligned}
\beta_{(s)}\varepsilon &\in \beta_{(s)}\langle\alpha, \beta_{(1)}, \delta\alpha(\delta\beta_{(1)})^{p-1}\rangle \\
&= \langle\beta_{(s)}, \alpha, \beta_{(1)}\rangle\delta\alpha(\delta\beta_{(1)})^{p-1} \\
&\ni \beta_{(s+1)}\delta\alpha(\delta\beta_{(1)})^{p-1} = \alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(s+1)},
\end{aligned}$$

and (6.15) follows from $\beta_{(s)}\mathscr{A}_{(p+1)q-4}(M_p)\delta\alpha(\delta\beta_{(1)})^{p-1}=0$ and (5.7).

Next we have

$$\begin{aligned}\varepsilon\delta\beta_{(1)} &\in \langle \alpha, \beta_{(1)}, \delta\alpha(\delta\beta_{(1)})^{p-1}\rangle\delta\beta_{(1)} \\ &= -\alpha\langle \beta_{(1)}, \delta\alpha(\delta\beta_{(1)})^{p-1}, \delta\beta_{(1)}\rangle \subset \alpha\mathscr{A}_{(p^2+p)q-3}(M_p),\end{aligned}$$

and hence $\varepsilon\delta\beta_{(1)}=x\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(2)}\delta$ by (6.13)–(6.14). Applying D and using (6.15), we obtain $x=1$. By (5.7), the second formula of (6.16) is obtained. q.e.d.

LEMMA 6.10. *Suppose that $s\not\equiv -1 \bmod p$. Then, the bracket*

$$\langle \alpha\delta\beta_{(s)}\delta, \alpha, \beta_{(1)}\rangle$$

contains $(1/(s+1))\delta\alpha\delta\beta_{(s+1)}+(s/(s+1))\alpha\delta\beta_{(s+1)}\delta$ with the indeterminacy $\beta^*_{(1)}\mathscr{A}_{(sp+s+1)q-2}(M_p)$. Further if $s<p-1$, the bracket consists of a single element.

PROOF. To prove the lemma, we introduce some notations and results from H. Toda [13]. Let $V(1)$ be the mapping cone of α, and $i_1\in\{M, V(1)\}_0$ and $\pi_1\in\{V(1), M\}_{-q-1}$ be the natural maps [13; pp. 216–217]. There is an element $\beta\in\mathscr{A}_{(p+1)q}(V(1))$ which defines $\beta_{(s)}$ by $\beta_{(s)}=\pi_1\beta^s i_1$ [13; pp. 217–218]. Also there exists an element $\alpha''\in\mathscr{A}_{q-2}(V(1))$ such that $\alpha''i_1=-i_1\delta\alpha\delta$ and $\pi_1\alpha''=-\delta\alpha\delta\pi_1$ [13; Lemma 3.1 and (5.6)]. These elements satisfy $\beta^r\alpha''\beta^s=s\beta^{r+s-1}\alpha''\beta+(1-s)\beta^{r+s}\alpha''$ [13; Prop. 4.7 (ii)].

Now, this relation implies $(s+1)\beta^s\alpha''\beta=s\beta^{s+1}\alpha''+\alpha''\beta^{s+1}$. Since $(-\pi_1\beta^s\alpha'')i_1=\beta_{(s)}\delta\alpha\delta=\alpha\delta\beta_{(s)}\delta$ and $\pi_1(\beta i_1)=\beta_{(1)}$, the bracket contains an element $(-\pi_1\beta^s\alpha'')\cdot(\beta i_1)$ which is equal to

$$\begin{aligned}&-(s/(s+1))\pi_1\beta^{s+1}\alpha''i_1-(1/(s+1))\pi_1\alpha''\beta^{s+1}i_1 \\ &= (s/(s+1))\beta_{(s+1)}\delta\alpha\delta+(1/(s+1))\delta\alpha\delta\beta_{(s+1)} \\ &= (s/(s+1))\alpha\delta\beta_{(s+1)}\delta+(1/(s+1))\delta\alpha\delta\beta_{(s+1)}.\end{aligned}$$

The rest of the assertions is proved by an easy calculation. q.e.d.

PROPOSITION 6.11. *The following relations hold.*

$$\beta_{(2)}\delta\varepsilon = -\frac{1}{3}\alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(3)}\delta-\frac{2}{3}\delta\alpha(\delta\beta_{(1)})\delta\beta_{(3)}, \tag{6.17}$$

$$\varepsilon\delta\beta_{(2)} = \frac{2}{3}\alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(3)}\delta+\frac{1}{3}\delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(3)}.$$

$$\beta_{(1)}\bar{\varepsilon}=\bar{\varepsilon}\beta_{(1)} = -\frac{1}{2}(\alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(2)}\delta-\delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(2)}), \tag{6.18}$$

$$\bar{\varepsilon}\delta\beta_{(1)} = \beta_{(1)}\delta\bar{\varepsilon} - \delta\alpha(\delta\beta_{(1)})^{p-1}\beta_{(2)}\delta\,.$$

$$(6.19) \qquad \beta_{(2)}\bar{\varepsilon} = \bar{\varepsilon}\beta_{(2)} = -\frac{1}{3}(\alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(3)}\delta - \delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(3)})\,,$$

$$\bar{\varepsilon}\delta\beta_{(2)} = -\beta_{(2)}\delta\bar{\varepsilon} = -\frac{1}{3}\delta\alpha(\delta\beta_{(1)})^{p-1}\delta\beta_{(3)}\delta\,.$$

PROOF. We first prove (6.18). Set $\xi = \alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(2)}$ and $t = (p^2+p+1)q - 3$. As is seen in the proof of Proposition 6.8, $\mathscr{A}_t(M_p) \cap \operatorname{Ker} D$ is generated by $\xi\delta - \delta\xi$. So we can put $\beta_{(1)}\bar{\varepsilon} = x(\xi\delta - \delta\xi)$. By using (5.16) (ii), (5.5), (5.2) (i) and (5.4), we have

$$\beta_{(1)}\bar{\varepsilon} \in \langle \beta_{(1)}, (\beta_{(1)}\delta)^{p-1}\beta_{(1)}, \alpha\delta - \delta\alpha\rangle(\alpha\delta - \delta\alpha) = -A(\alpha\delta - \delta\alpha)\,,$$

where $A = \langle \beta_{(1)}, \beta_{(1)}, \delta\alpha(\delta\beta_{(1)})^{p-1}\rangle$, which is a coset of the subgroup $Z_p\{\eta\delta\}$ of $\mathscr{A}_{t-q+1}(M_p) = Z_p\{\eta\delta, \delta\eta, \varepsilon\alpha^{p-2}\delta\alpha, \alpha^{p^2+p-1}\delta\alpha\delta\}$, $\eta = (\beta_{(1)}\delta)^{p-1}\beta_{(2)}$. Since $\alpha\eta = 0$, αA consists of a single element, and we have

$$\alpha A = -\langle \alpha, \beta_{(1)}, \beta_{(1)}\rangle\delta\alpha(\delta\beta_{(1)})^{p-1}\,.$$

According to N. Yamamoto [14; Prop. 7.3], $\langle \alpha, \beta_{(1)}, \beta_{(1)}\rangle$ contains $-(1/2)\beta_{(2)}$, and so

$$\alpha A = (1/2)\beta_{(2)}\delta\alpha(\delta\beta_{(1)})^{p-1} = (1/2)\xi.$$

Considering the kernel of $\alpha_*\colon \mathscr{A}_{t-q+1}(M_p) \to \mathscr{A}_{t+1}(M_p)$, we see

$$A \equiv (1/2)\delta\eta \bmod Z_p\{\eta\delta, \varepsilon\alpha^{p-2}\delta\alpha\}\,.$$

Hence $\beta_{(1)}\bar{\varepsilon} \equiv -A(\alpha\delta - \delta\alpha) \equiv (1/2)\delta\xi \bmod Z_p\{\xi\delta\}$, and we obtain $x = -1/2$. This shows the first equality of (6.18). The second follows immediately from (5.7).

Next we prove (6.17). We have

$$\begin{aligned}
\varepsilon\delta\beta_{(2)} &\in \langle \varepsilon\delta\beta_{(1)}, \alpha, \beta_{(1)}\rangle \qquad \text{by (5.2) (ii)}\\
&= \langle \xi\delta, \alpha, \beta_{(1)}\rangle \qquad \text{by (6.16)}\\
&\supset (\beta_{(1)}\delta)^{p-1}\langle \alpha\delta\beta_{(2)}\delta, \alpha, \beta_{(1)}\rangle\\
&\ni (2/3)(\beta_{(1)}\delta)^{p-1}\alpha\delta\beta_{(3)}\delta = (2/3)\xi'\delta \quad \text{by Lemma 6.10,}
\end{aligned}$$

where $\xi = \alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(2)}$ and $\xi' = \alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(3)}$. As the indeterminacy of $\langle \xi\delta, \alpha, \beta_{(1)}\rangle$ is generated by $\delta\xi'$, we can put

$$\varepsilon\delta\beta_{(2)} = (2/3)\xi'\delta + x\delta\xi'.$$

Applying D to this and using (6.15), we obtain $x = 1/3$, and the second of (6.17) is proved. The first follows immediately from (5.7).

Finally we prove (6.19). Similarly as above, we have

$$\begin{aligned}\bar{\varepsilon}\beta_{(2)} &\in \langle \bar{\varepsilon}\beta_{(1)}, \alpha, \beta_{(1)}\rangle \qquad \text{by (5.2) (ii)} \\ &\subset -(1/2)\langle \xi\delta, \alpha, \beta_{(1)}\rangle + (1/2)\langle \delta\xi, \alpha, \beta_{(1)}\rangle \qquad \text{by (6.18)} \\ &\ni -(1/3)\xi'\delta + (1/2)\delta\xi' \quad \text{by Lemma 6.10 and (5.2) (ii),}\end{aligned}$$

and so $\bar{\varepsilon}\beta_{(2)} \equiv -(1/3)\xi'\delta \bmod Z_p\{\delta\xi'\}$. Since $D(\bar{\varepsilon}\beta_{(2)})=0$ and $D(\xi')=0$, we obtain $\beta_{(2)}\bar{\varepsilon}=\bar{\varepsilon}\beta_{(2)}=-(1/3)(\xi'\delta-\delta\xi')$. By (6.13), we can put $\bar{\varepsilon}\delta\beta_{(2)}=x\delta\xi'\delta$ and $\beta_{(2)}\delta\bar{\varepsilon}=y\delta\xi'\delta$. Then $\bar{\varepsilon}\beta_{(2)}=D(\bar{\varepsilon}\delta\beta_{(2)})=x(\xi'\delta-\delta\xi')$ and $x=-(1/3)$. Similarly, $y=1/3$, and (6.19) is proved. q.e.d.

PROPOSITION 6.12. *There exists a relation*

$$\beta_{(1)}\bar{\varphi} = -\bar{\varphi}\beta_{(1)} = \alpha\delta\beta_{(2)}\delta\beta_{(p-1)}, \tag{6.20}$$

where the second equality holds up to non zero coefficient.

PROOF. By (6.8) (iii), $\beta_{(1)}\bar{\varphi}=-\bar{\varphi}\beta_{(1)}$. By (6.13), we can put $\beta_{(1)}\bar{\varphi}=x\alpha\delta\beta_{(2)}\delta\beta_{(p-1)}$. Then $-\beta_1\varphi=\pi_*i^*\beta_{(1)}\bar{\varphi}=x\alpha_1\beta_2\beta_{p-1}$. Hence we have $x\neq 0$ by the relation (14.2) of [6]. q.e.d.

Now, we prove Theorem 0.1 in the introduction.

PROOF OF THEOREM 0.1. The additive structure is determined in Proposition 6.4. All relations (i)–(xi) in Theorem 0.1 are already obtained in previous discussions, that is, (i) is (4.2)–(4.3), (ii) is (5.4) and (5.2)(i), (iii) is (5.6) and (6.12), (iv) is (5.5), (v) is (5.13)–(5.14), (vi) is (5.16), (5.18) and (6.4), (vii) is (5.17) and (5.19)–(5.20), (viii) is (6.11), (ix) is (6.15)–(6.17), (x) is (6.18)–(6.19), and (xi) is (6.9), (6.14) and (6.20).

Multiplying α to (viii) from the right, and using (i) and (ii), we obtain the relation (6.10). Multiplying $\delta\alpha$ to (6.10) from the right and using (i) we have (6.7). Hence (i), (ii), (iv), (vii) and (viii) imply all relations in Corollary 6.6. Similarly, we can see by tedious and easy calculations that any relation is implied from (i)–(xi). q.e.d.

REMARK. The element $\beta_{(p)} \in \mathscr{A}_{(p^2+p-1)q-1}(M_p)$ of (5.1) does not appear in Theorem 0.1, since our results (6.1) give no information about the element β_p. But we see easily that

$$\beta_{(p)} = \varepsilon\alpha^{p-2} = \alpha^{p-2}\varepsilon,$$

where the first equality holds up to non zero coefficient. This is a slight generalization of the statement [13; (5.4) (ii)] for $t=1$.

§ 7. The ring structure of $\mathscr{A}_*(M_{p^r})$, $r \geqq 2$.

We start from the discussion on the ring $\mathscr{A}_*(M_{p^2})$. We recall the elements

$$\delta_2 = i_2\pi_2 \in \mathscr{A}_{-1}(M_{p^2}),\ \lambda \in \{M_p, M_{p^2}\}_0,\ \rho \in \{M_{p^2}, M_p\}_0$$

in (1.9) and (3.1).

The element $\delta_1 = i_1\pi_1 \in \mathscr{A}_{-1}(M_p)$ is simply denoted by δ. The element δ_2 is of order p^2, and λ, ρ and δ are of order p. These satisfy

(7.1) $$\lambda\delta = 0, \quad \delta\rho = 0, \quad \rho\lambda = 0,$$

(7.2) $$\rho\delta_2\lambda = \delta, \quad \lambda\rho = p\cdot 1_M \qquad (M = M_{p^2}).$$

We define some elements of $\mathscr{A}_*(M_{p^2})$ as follows:

(7.3) (i) $$\xi_k = \lambda\alpha^k\rho \in \mathscr{A}_{kq}(M_{p^2}),$$

(ii) $$\beta'_{(s)} = \lambda\beta_{(s)}\rho \in \mathscr{A}_{(sp+s-1)q-1}(M_{p^2}),$$

(iii) $$\bar{\varepsilon}' = \lambda\bar{\varepsilon}\rho \in \mathscr{A}_{(p^2+1)q-2}(M_{p^2}),$$

(iv) $$\varepsilon'_{(i)} = \lambda\varepsilon\alpha^{i-1}\rho \in \mathscr{A}_{(p^2+i)q-1}(M_{p^2}).$$

Then, Lemma 3.2 (iv) implies $[\alpha_k]_2 = \xi_k$ for $k \not\equiv 0 \bmod p$ by (4.6), $[\beta_s]_2 = \beta'_{(s)}$ for $1 \leqq s \leqq p+1$, $s \neq p$ by (5.9), $[\varepsilon']_2 = \bar{\varepsilon}'$ by (5.16) (i), and $[\varepsilon_i]_2 = \varepsilon'_{(i)}$ by (6.2).

By (7.1)–(7.2), for $\xi, \eta \in \mathscr{A}_*(M_p)$, $\lambda\xi\rho = 0$ if $\xi = \xi'\delta$ or $\delta\xi'$, and $\lambda(\xi\delta\eta)\rho = (\lambda\xi\rho)\delta_2(\lambda\eta\rho)$. Then, we have

(7.3) (v) $$(\beta'_{(1)}\delta_2)^r\beta'_{(s)} = \lambda(\beta_{(1)}\delta)^r\beta_{(s)}\rho = [(\beta_1)^r\beta_s]_2,$$

(vi) $$(\beta'_{(1)}\delta_2)^r\beta'_{(2)}\delta_2\beta'_{(p-1)} = \lambda(\beta_{(1)}\delta)^r\beta_{(2)}\delta\beta_{(p-1)}\rho = [(\beta_1)^r\beta_2\beta_{p-1}]_2$$

(vii) $$\xi_1(\delta_2\beta'_{(1)})^r\delta_2\beta'_{(s)} = \lambda\alpha(\delta\beta_{(1)})^r\delta\beta_{(s)}\rho = [\alpha_1(\beta_1)^r\beta_s]_2,$$

(viii) $$\xi_1(\delta_2\beta'_{(1)})^r\delta_2\beta'_{(2)}\delta_2\beta'_{(p-1)} = \lambda\alpha(\delta\beta_{(1)})^r\delta\beta_{(2)}\delta\beta_{(p-1)}\rho$$
$$= [\alpha_1(\beta_1)^r\beta_2\beta_{p-1}]_2,$$

(ix) $$(\beta'_{(1)}\delta_2)^r\bar{\varepsilon}' = \lambda(\beta_{(1)}\delta)^r\bar{\varepsilon}\rho = [(\beta_1)^r\varepsilon']_2,$$

(x) $$\varepsilon'_{(i)}\delta_2\xi_1 = \lambda\varepsilon\alpha^{i-1}\delta\alpha\rho = [\varepsilon_i\alpha_1]_2.$$

Consider the submodule $\mathscr{B}_* = \lambda_*\rho^*\mathscr{A}_*(M_p)$ of $\mathscr{A}_*(M_{p^2})$. Then the following lemma is proved immediately from Theorem 0.1, (7.1)–(7.3), Proposition 2.2

and (1.12).

LEMMA 7.1. *$\mathscr{B}_*$ is a Z_p-vector space, and its basis is given by the elements* (7.3) (i)–(x) *for degree$<(p^2+3p+1)q-6$. The following relations hold*:

$$\xi\eta=0 \qquad \text{for any } \xi,\ \eta\in\mathscr{B}_*;$$

$$D(\xi)=0 \qquad \text{for any } \xi\in\mathscr{B}_*;$$

$$\xi\delta_2\eta=(-1)^{(\deg\xi+1)(\deg\eta+1)}\eta\delta_2\xi \qquad \text{for any } \xi,\ \eta\in\mathscr{B}_*.$$

For the last elements $\xi\delta_2\eta$, we have the following

LEMMA 7.2. *The following equalities hold.*

$$\xi_s\delta_2\beta'_{(t)}=\beta'_{(t)}\delta_2\xi_s=0 \qquad \text{for } s\geqq 2.$$

$$\beta'_{(s)}\delta_2\beta'_{(t)}=\begin{cases}(st/(s+t-1))\beta'_{(1)}\delta_2\beta'_{(s+t-1)} & \text{for } s+t\not\equiv 0,\ 1 \bmod p,\\ (st/(s+t-2))\beta'_{(2)}\delta_2\beta'_{(s+t-2)} & \text{for } s+t\not\equiv 0,\ 2 \bmod p.\end{cases}$$

$$\beta'_{(s)}\delta_2\beta'_{(p-s)}=s^2\beta'_{(1)}\delta_2\beta'_{(p-1)}.$$

$$\xi_s\delta_2\bar{\varepsilon}'=\bar{\varepsilon}'\delta_2\xi_s=0.$$

$$\xi_s\delta_2\varepsilon'_{(i)}=\varepsilon'_{(i)}\delta_2\xi_s=\begin{cases}s\varepsilon'_{(i+s)}\delta_2\xi_1 & \text{for } i+s\leqq p-3 \text{ and for } i+s=p-1,\\ 0 & \text{otherwise.}\end{cases}$$

$$\beta'_{(2)}\delta_2\bar{\varepsilon}'=\bar{\varepsilon}'\delta_2\beta'_{(2)}=0, \quad \beta'_{(s)}\delta_2\varepsilon'_{(i)}=\varepsilon'_{(i)}\delta_2\beta'_{(s)}=0 \qquad \text{for } s=1,2.$$

PROOF. For any $\xi\in\mathscr{A}_*(M_p)$, denote by ξ' the element $\lambda\xi\rho$. Then, by (7.1)–(7.2),

$$\xi'\delta_2\eta'=(\xi_1\delta\eta_1)' \quad \text{if} \quad \xi\delta\eta\equiv\xi_1\delta\eta_1 \bmod \delta\mathscr{A}_*(M_p)+\mathscr{A}_*(M_p)\delta.$$

Then the lemma is an easy consequence of the relations (4.4), (5.2) (i), (5.6), (6.12), Corollary 6.6, (6.16), (6.17) and (6.19). q.e.d.

Next we recall the element

$$\alpha'=[\alpha'_p]\in\mathscr{A}_{pq}(M_{p^2})$$

in §4. This is of order p^2 and satisfies $D(\alpha')=0$. Also, $\alpha'^s=[\alpha'_{sp}]_2\in\mathscr{A}_{spq}(M_{p^2})$ for $s\not\equiv 0 \bmod p$. The following relations are proved in Proposition 4.2 and Theorem 4.3.

(7.4) $$(\delta_2)^2=0, \quad \delta_2\alpha'^2=-\alpha'^2\delta_2+2\alpha'\delta_2\alpha'.$$

(7.5) $\qquad \xi_s\xi_t = 0, \quad \xi_s\alpha'^t = \alpha'^t\xi_s = \xi_{tp+s}.$

(7.6) $\qquad \xi_s\delta_2\xi_t = 0$ *for* $s+t\not\equiv 0 \bmod p, \quad \xi_{sp-t}\delta_2\xi_t = tp(\alpha'^s\delta_2-\alpha'^{s-1}\delta_2\alpha'),$

$$\xi_s\delta_2\alpha'^t = \xi_s\alpha'^t\delta_2 = \xi_{tp+s}\delta_2, \quad \alpha'^t\delta_2\xi_s = \delta_2\alpha'^t\xi_s = \delta_2\xi_{tp+s}.$$

LEMMA 7.3. *The following relations hold.*

$$\alpha'\beta'_{(s)} = \beta'_{(s)}\alpha' = 0, \quad \alpha'\delta_2\beta'_{(t)} = \beta'_{(t)}\delta_2\alpha' = 0.$$

$$\alpha'\bar{\varepsilon}' = \bar{\varepsilon}'\alpha' = 0, \quad \alpha'\delta_2\bar{\varepsilon}' = \bar{\varepsilon}'\delta_2\alpha' = 0.$$

$$\alpha'\varepsilon'_{(i)} = \varepsilon'_{(i)}\alpha' = 0, \quad \alpha'\delta_2\varepsilon'_{(i)} = \varepsilon'_{(i)}\delta_2\alpha' = 0.$$

PROOF. Let $\xi=\beta_{(s)}$, $\bar{\varepsilon}$ or $\varepsilon\alpha^{i-1}$, and set $\xi'=\lambda\xi\rho$. By (4.8) and (B)–(C) in the proof of Theorem 4.3, the element α' satisfies

$$\lambda\alpha^p = \alpha'\lambda, \quad \alpha^p\rho = \rho\alpha',$$

$$\lambda\alpha^{p-1}\delta\alpha = (\delta_2\alpha'-\alpha'\delta_2)\lambda, \alpha^{p-1}\delta\alpha\rho = \rho(\delta_2\alpha'-\alpha'\delta_2).$$

By (5.2) (i) and Corollary 6.6, we have $\xi\alpha^p=\alpha^p\xi=0$ and $\xi\alpha^{p-1}\delta\alpha=\alpha^{p-1}\delta\alpha\xi=0$. Hence, $\alpha'\xi'=\lambda\alpha^p\xi\rho=0$, $\xi'\alpha'=\lambda\xi\alpha^p\rho=0$, $\alpha'\delta_2\xi'=(\alpha'\delta_2-\delta_2\alpha')\xi'=-\lambda\alpha^{p-1}\delta\alpha\xi\rho=0$, and $\xi'\delta_2\alpha'=\xi'(\delta_2\alpha'-\alpha'\delta_2)=\lambda\xi\alpha^{p-1}\delta\alpha\rho=0$. q.e.d.

To describe the ring structure of $\mathscr{A}_*(M_{p^2})$, we finally introduce an element φ' by

(7.7) $$\varphi' = [\varphi]_2 \in \mathscr{A}_{(p^2+p)q-2}(M_{p^2}).$$

This is of order p^2 and satisfies

(7.7)′ $$D(\varphi') = 0, \qquad \pi_2\varphi' i_2 = \varphi.$$

We have also $\langle\varphi\rangle_2=\varphi'\delta_2-\delta_2\varphi'$ by (3.5), and hence

(7.8) $$\varphi'\lambda = \lambda\varepsilon\alpha^{p-2}\delta\alpha, \quad \rho\varphi' = \varepsilon\alpha^{p-2}\delta\alpha\rho,$$

$$(\varphi'\delta_2-\delta_2\varphi')\lambda = \lambda\bar{\varphi}, \qquad \rho(\varphi'\delta_2-\delta_2\varphi') = \bar{\varphi}\rho,$$

by Lemmas 3.1–3.2, (6.6) and (6.8) (i).

LEMMA 7.4. *The following relations are satisfied.*

$$\xi_s\delta_2\varepsilon'_{(p-1-s)} = \varepsilon'_{(p-1-s)}\delta_2\xi_s = sp\varphi', \quad \lambda\bar{\varphi}\rho = p(\varphi'\delta_2-\delta_2\varphi'),$$

$$\xi_s\varphi' = \varphi'\xi_s = 0, \quad \xi_s\delta_2\varphi' = \varphi'\delta_2\xi_s = 0,$$

$$\beta'_{(s)}\varphi' = \varphi'\beta'_{(s)} = 0, \quad \alpha'\varphi' = \varphi'\alpha',$$

$$\beta'_{(1)}\delta_2\varphi' = \varphi'\delta_2\beta'_{(1)} = \xi_1\delta_2\beta'_{(2)}\delta_2\beta'_{(p-1)},$$

where the last equality holds up to non zero coefficient.

PROOF. By Corollary 6.6, (7.8) and (7.2), $\xi_s\delta_2\varepsilon'_{(p-1-s)} = \varepsilon'_{(p-1-s)}\delta_2\xi_s = s\lambda\varepsilon\alpha^{p-2}\delta\alpha\rho = sp\varphi'$ and $\lambda\bar{\varphi}\rho = (\varphi'\delta_2 - \delta_2\varphi')\lambda\rho = p(\varphi'\delta_2 - \delta_2\varphi')$. Since $\xi\varepsilon\alpha^{p-2}\delta\alpha = \varepsilon\alpha^{p-2}\delta\alpha\xi = 0$ for $\xi = \alpha^s$ and $\beta_{(s)}$, $s \geqq 1$, by Corollary 6.6 and (5.2) (i), it follows from (7.8) that $\xi^s\varphi' = \varphi'\xi_s = 0$ and $\beta_{(s)}\varphi' = \varphi'\beta_{(s)} = 0$. By (1.11), $\alpha'\varphi' = \varphi'\alpha'$. By (7.8) and (6.14), we have $\xi_s\delta_2\varphi' = -\xi_s(\varphi'\delta_2 - \delta_2\varphi') = -\lambda\alpha^s\bar{\varphi}\rho = 0$ and also $\varphi'\delta_2\xi_s = 0$. Similarly, the other equalities follow from (6.20). q.e.d.

Now we prove Theorem 0.2 in the introduction.

PROOF OF THEOREM 0.2. By using (6.1) and (7.3), the assertion on the additive structure is an easy consequence of Theorem 3.5. By Lemmas 7.1–7.4 and (7.4)–(7.6) the relations are proved except the following

$$\xi_1(\delta_2\beta'_{(1)})^p = 0, \quad (\beta'_{(1)}\delta_2)^p\beta'_{(2)} = 0, \quad \alpha'\varphi' = 0.$$

The first two relations are obvious by (5.13)–(5.14).

The element $\alpha'\varphi' = \varphi'\alpha'$ belongs to $\mathscr{A}_{(p^2+2p)q-2}(M_{p^2}) \cap \operatorname{Ker} D$ which is equal to $Z_p\{\beta'_{(p+1)}\delta_2 + \delta_2\beta'_{(p+1)}\}$. Set $k = (p^2+2p)q-2$. We have $\alpha'\varphi'\lambda = \lambda\alpha^p\varepsilon\alpha^{p-2}\delta\alpha = 0$, and hence $\alpha'\varphi'$ belongs to

$$\mathscr{A}_k(M_{p^2}) \cap \operatorname{Ker} D \cap \operatorname{Ker} \lambda^*.$$

Applying the result on $\mathscr{A}_k(M_p)$ to the exact sequence (3.2) for $X = M_p$, $r = s = 1$, we see that $\lambda\beta_{(p+1)}\delta \neq 0$ in $\{M_p, M_{p^2}\}_k$. So, we have $\lambda^*(\beta'_{(p+1)}\delta_2 + \delta_2\beta'_{(p+1)}) = \lambda\beta_{(p+1)}\rho\delta_2\lambda = \lambda\beta_{(p+1)}\delta \neq 0$. Hence $\mathscr{A}_k(M_{p^2}) \cap \operatorname{Ker} D \cap \operatorname{Ker} \lambda^* = 0$ and $\alpha'\varphi' = 0$.

We can check by easy and tedious calculations that all relations are exhausted by the previous ones. q.e.d.

In the rest of this section, we discuss the ring structure of $\mathscr{A}_*(M_{p^r})$ for $r \geqq 3$. We define

$$\xi_s = \lambda^{r-1}\alpha^s\rho^{r-1} \in \mathscr{A}_{sq}(M_{p^r}) \qquad \textit{for} \;\; s \not\equiv 0 \bmod p,$$

$$\xi_{sp} = \lambda^{r-2}\alpha'^s\rho^{r-2} \in \mathscr{A}_{spq}(M_{p^r}) \qquad \textit{for} \;\; s \not\equiv 0 \bmod p,$$

$$\xi_{p^2} = \lambda^{r-3}\alpha''\rho^{r-3} \in \mathscr{A}_{p^2q}(M_{p^r}) \qquad (\xi_{p^2} = \alpha'' \;\; \textit{if} \;\; r = 3),$$

$$\beta'_{(s)} = \lambda^{r-1}\beta_{(s)}\rho^{r-1} \in \mathscr{A}_{(sp+s-1)q-1}(M_{p^r}),$$

$$\bar{\varepsilon}' = \lambda^{r-1}\bar{\varepsilon}\rho^{r-1} \in \mathscr{A}_{(p^2+1)q-2}(M_{p^r}),$$

$$\varepsilon'_{(i)} = \lambda^{r-1}\varepsilon\alpha^{i-1}\rho^{r-1} \in \mathscr{A}_{(p^2+i)q-1}(M_{p^r}) \qquad for\ 1 \leqq i \leqq p-1\,,$$

$$\varphi'' = \lambda^{r-2}\varphi'\rho^{r-2} \in \mathscr{A}_{(p^2+p)q-2}(M_{p^r}),$$

where $\alpha''=[\alpha''_{p^2}]_3 \in \mathscr{A}_{p^2q}(M_{p^3})$ is the element in Theorem 4.4.

THEOREM 7.5. *Let p be a prime$\geqq 5$ and $r\geqq 3$. Then, the group $\mathscr{A}_k(M_{p^r})$ for $k<(p^2+3p+1)q-6$ is the direct sum of cyclic groups generated by the following elements*:

$$\delta_r = i_r\pi_r,\ 1_M \qquad\qquad of\ order\ p^r;$$

$$\xi_{p^2},\ \delta_r\xi_{p^2},\ \xi_{p^2}\delta_r,\ \delta_r\xi_{p^2}\delta_r \qquad of\ order\ p^3;$$

$$\delta_r^a\xi_{sp}\delta_r^b \quad (s \not\equiv 0 \bmod p,\ 1 \leqq s \leqq p+3), \quad \delta_r^a\varphi''\delta_r^b \quad of\ order\ p^2;$$

$$\delta_r^a\xi_s\delta_r^b \quad (s \not\equiv 0 \bmod p,\ 1 \leqq s < p^2+3p)\,,$$

$$\delta_r^a(\beta'_{(1)}\delta_r)^{s-1}\beta'_{(1)}\delta_r^b \quad (1 \leqq s \leqq p+3)\,,$$

$$\delta_r^a(\beta'_{(1)}\delta_r)^{s}\beta'_{(t)}\delta_r^b \quad (0 \leqq s < p,\ 2 \leqq t \leqq p+1,\ t \neq p,\ s+t \leqq p+2)\,,$$

$$\delta_r^a(\beta'_{(1)}\delta_r)^{s}\beta'_{(2)}\delta_r\beta'_{(p-1)}\delta_r^b \quad (s = 0,\ 1)\,,$$

$$\delta_r^a\xi_1\delta_r(\beta'_{(1)}\delta_r)^{s-1}\beta'_{(1)}\delta_r^b \quad (1 \leqq s < p)\,,$$

$$\delta_r^a\xi_1\delta_r(\beta'_{(1)}\delta_r)^{s}\beta'_{(t)}\delta_r^b \quad (0 \leqq s<p,\ 2 \leqq t \leqq p+1,\ t \neq p,$$
$$s+t \leqq p+2,\ (s,t) \neq (1,\ p+1))\,,$$

$$\delta_r^a\xi_1\delta_r(\beta'_{(1)}\delta_r)^{s}\beta'_{(2)}\delta_r\beta'_{(p-1)}\delta_r^b \quad (s = 0,\ 1)\,,$$

$$\delta_r^a(\beta'_{(1)}\delta_r)^{s}\bar{\varepsilon}'\delta_r^b \quad (0 \leqq s \leqq 3)\,,$$

$$\delta_r^a\varepsilon'_{(i)}\delta_r^b \quad (1 \leqq i \leqq p-1)\,,$$

$$\delta_r^a\varepsilon'_{(i)}\delta_r\xi_1\delta_r^b \quad (1 \leqq i \leqq p-3) \qquad of\ order\ p;$$

where a, b=0 or 1.

The ring $\mathscr{A}_*(M_{p^r})$ is generated, within the limits of degree less than $(p^2+3p+1)q-6$, by the elements δ_r, $\xi_s(s\leqq p^2$ for $r=3$, $s\leqq p^2+3p$ for $r\geqq 4)$, $\beta'_{(s)}$ $(1\leqq s\leqq p+1,\ s\neq p)$, $\bar{\varepsilon}'$, $\varepsilon'_{(i)}(1\leqq i\leqq p-1)$ and φ'', with the following relations:

(i) $(\delta_r)^2 = 0$

(ii) $\eta\zeta = 0$ *for $\eta,\ \zeta \in \{\xi_s,\ \beta'_{(s)},\ \bar{\varepsilon}',\ \varepsilon'_{(i)},\ \varphi''\}$ except the case $(\eta,\zeta) = (\xi_s,\ \xi_t)$.*

(iii) *If $r=3$, then $\xi_{sp}\xi_{tp} = p\xi_{(s+t)p}$ for $s,\ t,\ s+t\not\equiv 0 \bmod p$,*

$\xi_{sp}\xi_{p^2-sp} = p^2\xi_{p^2}$ *for* $s \not\equiv 0 \bmod p$, $\xi_{p^2}\xi_s = \xi_s\xi_{p^2} = \xi_{p^2+s}$ *for* $s \not\equiv 0 \bmod p^2$, *and* $\xi_s\xi_t = 0$ *for other* s, t.

If $r=4$, *then* $\xi_{p^2}\xi_{sp} = \xi_{sp}\xi_{p^2} = p\xi_{p^2+sp}$ *for* $s \not\equiv 0 \bmod p$, *and* $\xi_s\xi_t = 0$ *for other* s, t.

If $r \geqq 5$, $\xi_s\xi_t = 0$.

(iv) $\eta\delta_r\zeta = \zeta\delta_r\eta = 0$ *for* $(\eta, \zeta) = (\xi_s, \beta'_{(t)})(s \geqq 2)$, $(\xi_s, \bar{\varepsilon}')$, (ξ_s, φ''), $(\beta'_{(2)}, \bar{\varepsilon}')$, $(\beta'_{(s)}, \varepsilon'_{(i)})$.

$\beta'_{(s)}\delta_r\xi_1 = \xi_1\delta_r\beta'_{(s)}$, $\bar{\varepsilon}'\delta_r\beta'_{(1)} = \beta'_{(1)}\delta_r\bar{\varepsilon}'$, $\beta'_{(1)}\delta_r\varphi'' = \varphi''\delta_r\beta'_{(1)}$ $= \xi_1\delta_r\beta'_{(2)}\delta_r\beta'_{(p-1)}$ *up to non zero coefficient.*

(v) *If* $r = 3$, *then*

$$\xi_s\delta_3\xi_{p^2-s} = sp^2(\xi_{p^2}\delta_3 - \delta_3\xi_{p^2}) \quad \text{for } s \not\equiv 0 \bmod p,$$

$$= tp\xi_{p^2}\delta_3 + (p-t)p\delta_3\xi_{p^2} \quad \text{for } s = tp \not\equiv 0 \bmod p^2,$$

$$\xi_{sp}\delta_3\xi_{tp} = (p/(s+t))(s\xi_{(s+t)p}\delta_3 + t\delta_3\xi_{(s+t)p}) \quad \text{for } s,\ t,\ s+t \not\equiv 0 \bmod p,$$

$$\xi_{p^2}\delta_3\xi_s = \delta_3\xi_{p^2+s} \quad \text{for } s \not\equiv 0 \bmod p,$$

$$= (1/(t+p))(p\xi_{p^2+tp}\delta_3 + t\delta_3\xi_{p^2+tp}) \quad \text{for } s = tp \not\equiv 0 \bmod p,$$

$$\xi_s\delta_3\xi_{p^2} = \xi_{p^2+s}\delta_3 \quad \text{for } s \not\equiv 0 \bmod p,$$

$$= (1/(t+p))(t\xi_{p^2+tp}\delta_3 + p\delta_3\xi_{p^2+tp}) \quad \text{for } s = tp \not\equiv 0 \bmod p^2,$$

$$\xi_s\delta_3\xi_t = 0 \quad \text{for other } s,\ t.$$

If $r = 4$, *then* $\xi_{sp}\delta_4\xi_{p^2-sp} = sp^2(\xi_{p^2}\delta_4 - \delta_4\xi_{p^2})$,

$$\xi_{sp}\delta_4\xi_{p^2} = (sp/(s+p))\xi_{p^2+sp}\delta_4 \quad \text{for } s \not\equiv 0 \bmod p,$$

$$\xi_{p^2}\delta_4\xi_{sp} = (sp/(s+p))\delta_4\xi_{p^2+sp} \quad \text{for } s \not\equiv 0 \bmod p,$$

and $\xi_s\delta_4\xi_t = 0$ *for other* s, t.

If $r \geqq 5$, *then* $\xi_s\delta_r\xi_t = 0$.

(vi)

$$\beta'_{(s)}\delta_r\beta'_{(t)} = \begin{cases} (st/(s+t-1))\beta'_{(1)}\delta_r\beta'_{(s+t-1)} & \text{for } s+t \neq p+1, \\ s(s-1)\beta'_{(2)}\delta_r\beta'_{(p-1)} & \text{for } s+t = p+1 \end{cases}$$

$$\xi_s\delta_r\varepsilon'_{(i)} = \varepsilon'_{(i)}\delta_r\xi_s = \begin{cases} s\varepsilon'_{(i+s)}\delta_r\xi_1 & \text{for } i+s \leqq p-3\,, \\ sp\varphi'' & \text{for } i+s = p-1\,, \\ 0 & \text{for } i+s = p-2 \text{ and for } i+s \geqq p. \end{cases}$$

(vii) $\xi_1(\delta_r\beta'_{(1)})^p = 0, \quad (\beta'_{(1)}\delta_r)^p\beta'_{(2)} = 0\,.$

PROOF. Except the relations (iii) and (v), the results coincide with the case $r=2$, and are proved in the same way as Theorem 0.2. The relations (iii) and (v) are easy restatements of Theorem 4.4. q.e.d.

§ 8. Some bracket formulae in G_*.

In this section we give some relations on the stable Toda bracket in G_*, and prove Lemmas 6.1 and 6.7. Here we assume $p\geqq 5$.

PROPOSITION 8.1. *Let $r\geqq 1$, $s\geqq 2$ and $r+s\leqq p+1$. Then,*

$$\langle(\beta_1)^p, \alpha_r, \alpha_s\rangle = \begin{cases} \pm rs\,\varepsilon_{r+s-2}\alpha_1 & \text{for } r+s \leqq p-1 \text{ and for } r+s = p+1, \\ 0 & \text{for } r+s = p. \end{cases}$$

$$\langle(\beta_1)^p, \alpha_1, \alpha'_p\rangle = \pm\varepsilon_{p-1}\alpha_1.$$

Here the brackets have trivial indeterminacies.

PROOF. Set $A=\langle(\beta_{(1)}\delta)^{p-1}\beta_{(1)},\ \alpha^r\delta-\delta\alpha^r,\ \alpha^s\delta-\delta\alpha^s\rangle$. Then by an easy calculation we see that A has an indeterminacy $Z_p\{s\alpha^{p^2+r+s-2}\delta\alpha\delta\}$ and that $\pi Ai=\pm\langle(\beta_1)^p,\ \alpha_r,\ \alpha_s\rangle$ mod zero. Since $\alpha^r\delta-\delta\alpha^r=-r(\alpha\delta-\delta\alpha)\alpha^{r-1}=-r\alpha^{r-1}(\alpha\delta-\delta\alpha)$ by (4.4), it follows that

$$A = rsB\alpha^{r+s-2} \qquad \text{for} \quad B = \langle(\beta_{(1)}\delta)^{p-1}\beta_{(1)}, \alpha\delta-\delta\alpha, \alpha\delta-\delta\alpha\rangle\,.$$

As is seen in the proof of Proposition 5.2, $B=\bar\varepsilon+Z_p\{\delta\gamma\delta,\ \alpha^{p^2}\delta\alpha\delta\}$, $\gamma=\alpha(\delta\beta_{(1)})^{p-2}\cdot\beta_{(2)}$. By (5.2) (i) and (5.4), $\alpha\delta\gamma\delta=\delta\gamma\delta\alpha=0$, and hence by (4.4) and (5.19) α commutes with any element of B. Also, the element $\alpha\delta-\delta\alpha$ commutes with any element by (1.12). Therefore,

$$\begin{aligned} A &= rs\alpha^{r+s-2}B = \pm rs\alpha^{r+s-3}\langle\alpha, (\beta_{(1)}\delta)^{p-1}\beta_{(1)}, \alpha\delta-\delta\alpha\rangle(\alpha\delta-\delta\alpha) \\ &= \pm rs(\alpha\delta-\delta\alpha)\alpha^{r+s-3}\langle\alpha, (\beta_{(1)}\delta)^{p-1}\beta_{(1)}, \alpha\delta-\delta\alpha\rangle\,. \end{aligned}$$

As is seen in the proof of Proposition 5.2, the last bracket contains the element $-\varepsilon$. So, we have $\pi Ai=rs\alpha_1\varepsilon_{r+s-2}=rs\varepsilon_{r+s-2}\alpha_1$ up to sign. Thus we obtain the first formula.

Next consider $A'=\langle(\beta_{(1)}\delta)^{p-1}\beta_{(1)},\ \alpha\delta-\delta\alpha,\ \alpha^p\delta-\alpha^{p-1}\delta\alpha\rangle$. Then, $(\alpha^p\delta-\alpha^{p-1}\delta\alpha)i=\langle\alpha'_p\rangle i=-i\alpha'_p$ and $\pi A'i=\pm\langle(\beta_1)^p,\ \alpha_1,\ \alpha'_p\rangle$. Similarly as above, we have $A'=\pm(\alpha\delta-\delta\alpha)\alpha^{p-2}\langle\alpha,\ (\beta_{(1)}\delta)^{p-1}\beta_{(1)},\ \alpha\delta-\delta\alpha\rangle \ni \pm\alpha\delta\alpha^{p-2}\varepsilon=\pm\varepsilon\alpha^{p-2}\delta\alpha$. Thus the second formula follows. q.e.d.

Now to prove Lemma 6.1, we introduce some results on the unstable groups $\pi_{n+k}(S^n)$ from [5; § 8] and [11]. Set $k=(p^2+p-1)q-2$, and let $S^\infty\colon \pi_{n+t}(S^n)\to G_t$ be the natural homomorphism. By Theorem 15.2 of [11],

(8.1) *There exists an element* $\varepsilon=\varepsilon_{p-1}(2p+3)\in\pi_{2p+3+k}(S^{2p+3})$ *of order p such that* $S^\infty\varepsilon=\varepsilon_{p-1}\in G_k$.

Following [11], we denote by Q_2^{2n-1} the space $\Omega(\Omega^2S^{2n+1},S^{2n-1})$ of paths in the double loop space Ω^2S^{2n+1} starting from the subspace S^{2n-1} and ending to the base point $*$. Denote by $H^{(2)}\colon \pi_{i+2}(S^{2n+1})\to\pi_{i-1}(Q_2^{2n-1})$ the homomorphism defined from the induced homomorphism from the inclusion $(\Omega^2S^{2n+1},*)\to(\Omega^2S^{2n+1},S^{2n-1})$. For $\gamma\in{}_pG_{i-2np+3}$, denote by $Q^n(\gamma)\in{}_p\pi_i(Q_2^{2n-1})$ the image of γ by the homomorphism $I'\colon {}_pG_{i-2np+3}\approx{}_p\pi_{i+2}(S^{2np-1})\to{}_p\pi_i(Q_2^{2n-1})$. (Cf. [5; pp. 331–332]).

LEMMA 6.1. $\langle\varepsilon_{p-1},p,\alpha_1\rangle=0 \mod$ *zero.*

PROOF. As is well known, $S^\infty\colon\pi_{n+q-1}(S^n)\to G_{q-1}$ is an isomorphism of p-components for $n\geqq 3$. Let $\alpha\in\pi_{2p+4}(S^7)$ be an element such that $p\alpha=0$ and $S^\infty\alpha=\alpha_1$. Then we consider the Toda bracket $\{\alpha,\ p\iota_{2p+4},\ S\varepsilon\}\subset\pi_{k+2p+5}(S^7)$, whose S^∞-image is equal to our bracket $\langle\varepsilon_{p-1},p,\alpha_1\rangle$ up to sign, by [10; (3.9), i)].

Next we calculate the groups ${}_p\pi_{k+4p-5}(S^{2p-3})$ and ${}_p\pi_{k+4p-3}(S^{2p-1})$. By [5; (8.4)], ${}_p\pi_{k+4p-8}(Q_2^{2p-5})=Z_p\{Q^{p-2}(\alpha_{p^2+2})\}$ and ${}_p\pi_{k+4p-6}(Q_2^{2p-3})=Z_p\{Q^{p-1}\cdot(\alpha_{p^2+1}),\ Q^{p-1}(\alpha_1\beta_1^{p-2}\beta_2)\}$. By the discussions in [5; pp. 332–333], we see that ${}_p\pi_{k+4p-5}(S^{2p-3})=Z_{p^2}\{\gamma_{p-2}\}$ and ${}_p\pi_{k+4p-3}(S^{2p-1})=Z_{p^2}\{\gamma_{p-1}\}+Z_p\{\beta\}$, where γ_{p-2} and γ_{p-1} are called *the unstable elements of second type* and satisfy $H^{(2)}\gamma_{p-2}=Q^{p-2}(\alpha_{p^2+2})$, $H^{(2)}\gamma_{p-1}=Q^{p-1}(\alpha_{p^2+1})$, $S^2\gamma_{p-2}=p\gamma_{p-1}$ and $S^4\gamma_{p-2}=0$, and the element β satisfies $H^{(2)}\beta=Q^{p-1}(\alpha_1\beta_1^{p-2}\beta_2)$. We have therefore $S^\infty{}_p\pi_{k+4p-5}(S^{2p-3})=0$. Since $S^{2p-10}\{\alpha,\ p\iota_{2p+4},\ S\varepsilon\}\subset{}_p\pi_{k+4p-5}(S^{2p-3})$, it follows that $S^\infty\{\alpha,\ p\iota_{2p+4},\ S\varepsilon\}=0$. q.e.d.

REMARK. Set $k=(p^2+p-1)q-2$ and $l=k+q$. The stable groups ${}_pG_k$ and ${}_pG_l$ are generated by ε_{p-1} and $\beta_1^{p-1}\beta_2$. For the element $\beta\in{}_p\pi_{l+2p-1}(S^{2p-1})$, we see further that $S^\infty\beta=\beta_1^{p-1}\beta_2$, i.e., the element $\beta_1^{p-1}\beta_2$ belongs to $S^\infty\pi_{l+2p-1}(S^{2p-1})$ and not to $S^\infty\pi_{l+2p-3}(S^{2p-3})$. This is true for the case $p=3$, but the above proof is negative for $p=3$ since (8.1) does not hold for $p=3$.

Finally to prove Lemma 6.7, we employ some results of [5] and [6]. Let

$K_k(n)$ be the space obtained from S^n by attaching cells of dimension greater than $n+k$ and killing the homotopy groups $\pi_{n+j}(S^n)$ for $j \geqq k$. Here n is a sufficiently large integer. Set $k=(p^2+p)q-3$. By (1.3) of [5] and Theorem 13.1 of [6], we have the following results on the cohomology group $H^*(K_k(n); Z_p)$:

$$\begin{aligned}(8.2)\quad & H^n = Z_p\{a_0\}, \quad H^{n+k+1} = Z_p\{f\}, \quad H^{n+k+2} = Z_p\{f', b\}, \\ & H^{n+k+3} = Z_p\{a, \Delta b\}, \quad H^{n+k+4} = Z_p\{a'\}, \quad H^{n+k+q-1} = Z_p\{c\}, \\ & H^{n+k+q} = Z_p\{\Delta c\}, \quad H^{n+k+q+1} = Z_p\{P^1 f\}, \quad H^{n+k+q+2} = Z_p\{\Delta P^1 f, P^1 f'\}, \\ & H^{n+k+q+3} = Z_p\{P^1 a, \Delta P^1 f', P^1 \Delta b\}, \quad H^i = 0 \textit{ otherwise for} \\ & 0 < i \leqq n+k+q+3,\end{aligned}$$

where $H^i = H^i(K_k(n); Z_p)$, $a=a_{p^2+p}$, $a'=a'_{p^2+p}$, $b=b_2^{p-1}$ and $c=c_1^p$ in Theorem 13.1 of [6], and Δ and P^1 denote the Bockstein operation and the reduced power operation.

LEMMA 6.7. $\langle \varphi, \alpha_1, p \rangle \ni 0.$

PROOF. From (8.2) we see easily that the elements a_0, f, $P^1 f$ and $\Delta P^1 f$ form a subcomplex of $K_k(n)$ up to mod p homotopy equivalence. In more detail, there exist a complex

$$L = S^n \cup e^{n+k+1} \cup e^{n+k+q+1} \cup e^{n+k+q+2}$$

and a map $F: L \to K_k(n)$ such that $H^*(L; Z_p)$ is spanned by $F^*(a_0)$, $F^*(f)$, $F^*(P^1 f)$ and $F^*(\Delta P^1 f)$. The $(n+k+1)$-skeleton of L is the complex $P_k^n(f)$ of Definition 2.1 of [5], and so it is the mapping cone of φ by the fact $\phi(\varphi)=f$ for the homomorphism ϕ of [5; (2.1)] and by Lemma 2.2 of [5]. Since $F^*(P^1 f) = P^1 F^*(f)$ and α_1 is detected by P^1, the $(n+k+q+1)$-skeleton of L/S^n is the mapping cone of α_1, and since $F^*(\Delta P^1 f) = \Delta P^1 F^*(f)$, $L/(S^n \cup e^{n+k+1})$ is the mapping cone of $p \in G_0$. So the existence of such L leads us to the lemma. q.e.d.

REMARK. By an argument similar to Lemma 6.7, we can also prove Lemma 6.1 without the assumption $p \geqq 5$.

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Department of Mathematics,
Faculty of Science,
Hiroshima University