HIROSHIMA MATH. J.
7 (1977), 427–447

Realizing Some Cyclic BP_*-modules and Applications to Stable Homotopy of Spheres

Shichirô OKA

(Received January 10, 1977)

Introduction

Let $BP_*(\ )$ be the Brown-Peterson homology theory localized at a prime $p \geqq 5$. Its coefficient ring BP_* is the polynomial ring $Z_{(p)}[v_1, v_2, \dots]$ over the integers localized at p on Hazewinkel's generators v_i of degree $2(p^i - 1)$ ([2], [3], [4], [6]).

In the previous paper [14; Th. D, DII, D′, D′II], we constructed the spectra realizing cyclic BP_*-modules $BP_*/(p, v_1^j, v_2^{sp})$ at $p \geqq 5$ in the following three cases: $1 \leqq j \leqq p$, $s \geqq 1$, $(j, s) \neq (p, 1)$; $p+1 \leqq j \leqq 2p-2$, $p|s$; $p+1 \leqq j \leqq 2p$, $2p|s$. In this paper, we shall prove the following realizability theorems.

THEOREM 4.3. *For $p \geqq 5$ and $s \geqq 2$, there exist spectra L_s such that $BP_*(L_s) = BP_*/(p^2, v_1^p, v_2^{sp^2})$.*

THEOREM 4.4. *For $p \geqq 5$, $s \geqq 2$ and j with $p+1 \leqq j \leqq 2p$, there exist spectra $Y_{s,j}$ such that $BP_*(Y_{s,j}) = BP_*/(p, v_1^j, v_2^{sp^2})$.*

Each L_s is an 8-cell complex and we define the element $\beta_{sp^2/(p,2)}$ in $\pi_*(S)$, the stable homotopy group of spheres, by the attaching map of the 5th cell at the 4th cell in L_s, and similarly we define $\beta_{sp^2/(j)} \in \pi_*(S)$ from $Y_{s,j}$ (for the details, see Definitions 5.1–5.2). Then using methods developed by H. R. Miller, D. C. Ravenel, W. S. Wilson and others ([7], [8], [9]), we see that the elements $\beta_{sp^2/(p,2)}$ and $\beta_{sp^2/(j)}$ of the same name in $H^2 BP_* = \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$ [8] survive nontrivially to E_∞ term in the Adams-Novikov spectral sequence and support the homotopy elements of the above.

THEOREM 5.3. *For $p \geqq 5$, $s \geqq 2$, the elements $\beta_{sp^2/(p,2)}$ in $\pi_{(sp^3+sp^2-p)q-2}(S)$ $(q = 2(p-1))$ are nontrivial of order p^2 and indecomposable. Hence the group $\pi_{(sp^3+sp^2-p)q-2}(S)$ contains a summand isomorphic to Z/p^2Z.*

THEOREM 5.4. *For $p \geqq 5$, $s \geqq 2$, $p+1 \leqq j \leqq 2p$, the elements $\beta_{sp^2/(j)}$ in $\pi_{(sp^3+sp^2-j)q-2}(S)$ $(q = 2(p-1))$ are indecomposable and generate cyclic summands of order p.*

The known elements in $\pi_*(S)$ of order p^2 are the elements in $\mathrm{Im}\, J$ [1] and the three elements ϕ, μ [12] and ϕ_2 [11]. None of them is of degree even. Theorem 5.3 shows that Coker J contains infinitely many elements of order p^2 and of degree even. We shall also construct at the end of this paper the elements ϕ_t in Coker J of order p^2 and of degree odd, for infinitely many $t \geqq 1$ and all $p \geqq 5$, as a generalization of the known elements $\phi=\phi_1$ and ϕ_2 (Theorem 5.5).

In §§ 1–3, we shall study the spectrum K realizing $BP_*/(p, v_1^p)$ and the algebra $\mathscr{A}_*(K)=\sum_k \mathscr{A}_k(K)$, $\mathscr{A}_k(K)=[\Sigma^k K, K]$, consisting of stable self-maps (Σ denotes the suspension). K has a CW-decomposition $S^0 \cup e^1 \cup e^{pq+1} \cup e^{pq+2}$, $q=2(p-1)$, and the smash product $K \wedge K$ is homotopy equivalent to the wedge $K \vee \Sigma K \vee \Sigma^{pq+1} K \vee \Sigma^{pq+2} K$ (see Remark 1.6 below). Moreover K is a commutative and associative ring spectrum (Theorems 1.10 and 2.1), and the projection to the first factor of the above decomposition is the multiplication μ_1 on K. These facts are useful to study the structure of the algebra $\mathscr{A}_*(K)$. Define linear maps $\theta\colon \mathscr{A}_k(K) \to \mathscr{A}_{k+1}(K)$ and $\psi\colon \mathscr{A}_k(K) \to \mathscr{A}_{k+pq+1}(K)$ by the compositions

$$\theta(f)\colon \Sigma^{k+1} K = \Sigma^k(\Sigma K) \subset \Sigma^k K \wedge K \xrightarrow{1 \wedge f} K \wedge K \xrightarrow{\mu_1} K,$$

$$\psi(f)\colon \Sigma^{k+pq+1} K = \Sigma^k(\Sigma^{pq+1} K) \subset \Sigma^k K \wedge K \xrightarrow{1 \wedge f} K \wedge K \xrightarrow{\mu_1} K$$

for $f \in \mathscr{A}_k(K)$. Then, for any $f \in \mathscr{A}_*(K)$, the element $1_K \wedge f$ is described, via the above decomposition, with 16 elements in $\mathscr{A}_*(K)$, which are written in terms of θ and ψ (Proposition 3.3).

Let $\delta \in \mathscr{A}_{-1}(K)$ (Lemma 1.7) and $\delta' \in \mathscr{A}_{-pq-1}(K)$ (Definition 1.9) be the generators such that $\theta(\delta)=\psi(\delta')=-1_K$ and $\psi(\delta)=\theta(\delta')=0$ (Lemma 3.2), and put $\mathscr{C}_*(K)=\operatorname{Ker}\theta \cap \operatorname{Ker}\psi$. Simple characterizations of elements in $\mathscr{C}_*(K)$ will be given in Corollary 3.4. We shall prove in § 3 the following results on the structure of $\mathscr{A}_*(K)$.

THEOREM 3.6. (i) *$\mathscr{A}_*(K)=\mathscr{C}_*(K) \otimes E(\delta, \delta')=E(\delta, \delta') \otimes \mathscr{C}_*(K)$, where E denotes the exterior algebra over Z/pZ.*

(ii) *$\mathscr{A}_*(K)$ has the two differentials θ and ψ of above which are derivative and commute to each other, i.e., $\theta^2=0$, $\psi^2=0$, $\theta\psi=-\psi\theta$ and for $d=\theta, \psi$*

$$d(fg) = (-1)^l d(f)g + f d(g), \qquad f \in \mathscr{A}_k(K), \quad g \in \mathscr{A}_l(K).$$

THEOREM 3.7. *The subalgebra $\mathscr{C}_*(K)$ is commutative, and for any $f \in \mathscr{C}_*(K)$, the commutators $[f, \delta]$ and $[f, \delta']$ are the elements in $\mathscr{C}_*(K)$.*

We constructed the element in $\mathscr{A}_*(K)$ realizing the multiplication by v_2^{sp} for $s \geqq 2$ [14; Th. CII]. We shall reconstruct this element so that it lies in $\mathscr{C}_*(K)$ (Lemma 4.2) and deduce in § 4 the above realizability theorems from Theorem 3.7.

§1. Spectrum K

In this paper, we shall work in the stable homotopy category of CW-spectra. We denote by S and M the sphere spectrum and the mod p Moore spectrum, respectively. Here *p denotes a fixed prime with $p \geqq 5$.* Denote the cofibering for M by

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{\pi} \Sigma S,$$

where Σ denotes the suspension functor.

We shall use the same notations as in [15], for example, $[X, Y]_k = [\Sigma^k X, Y]$ is the additive group of homotopy classes of maps $\Sigma^k X \to Y$, and if X and Y are *M-module spectra*[1], $[X, Y]_k^M$ is the subgroup of $[X, Y]_k$ of all *M-maps*[2]. We shall abbreviate $[X, X]_k$ to $\mathscr{A}_k(X)$ and $[X, X]_k^M$ to $\mathscr{B}_k(X)$. By the composition product, $\mathscr{A}_*(X) = \sum_k \mathscr{A}_k(X)$ is a graded ring and $\mathscr{B}_*(X) = \sum_k \mathscr{B}_k(X)$ is its subring.

We shall put $q = 2(p-1)$. Let $\alpha \in \mathscr{A}_q(M) = Z/pZ$ be a generator and denote by K the mapping cone of the element $\alpha^p \in \mathscr{A}_{pq}(M)$, so we have a cofibering

$$(1.1) \qquad \Sigma^{pq} M \xrightarrow{\alpha^p} M \xrightarrow{i'} K \xrightarrow{\pi'} \Sigma^{pq+1} M.$$

Since α is the M-map, K is an M-module spectrum by [15; Th. 4.3]. Noting that $\mathscr{A}_1(K) = \mathscr{A}_2(K) = 0$ and using [15; Th. 1.3, Prop. 5.4, Th. 4.3], we have

PROPOSITION 1.1. *K is an associative M-module spectrum having the unique M-action $m = m_K \colon M \wedge K \to K$ and the unique right inverse $n = n_K \colon \Sigma K \to M \wedge K$ of $\pi \wedge 1_K$ associated to m_K, i.e., $m_K n_K = 0$, $(i \wedge 1_K) m_K + n_K(\pi \wedge 1_K) = 1_{M \wedge K}$. The maps i' and π' in* (1.1) *are the M-maps.*

LEMMA 1.2. $\alpha^p \wedge 1_K = 0$ *in* $\mathscr{A}_{pq}(M \wedge K)$.

PROOF. The element $\pi \alpha^p i \in \pi_{pq-1}(S)$ is divisible by p ([17], [13; §4]) and 1_K is of order p [15; Prop. 1.1]. So $(\pi \alpha^p i) \wedge 1_K = 0$. Since $\mathscr{A}_{pq}(K) = 0$ and $\mathscr{A}_{pq+1}(K) = 0$, we have $m(\alpha^p \wedge 1_K) = 0$ and $(\alpha^p \wedge 1_K) n = 0$. Hence $\alpha^p \wedge 1_K = n(\pi \wedge 1_K)(\alpha^p \wedge 1_K)(i \wedge 1_K) m = 0$.

NOTATION. For M-module spectra (X, m_X) and (Y, m_Y), the smash product $X \wedge Y$ has the M-actions $m_X \wedge 1_Y$ and $(1_X \wedge m_Y)(T \wedge 1_Y)$, $T \colon M \wedge X \to X \wedge M$ being

---

1) By an M-module spectrum, we mean a CW-spectrum X equipped with a left inverse $m_X \colon M \wedge X \longrightarrow X$ of $i \wedge 1_X \colon X = S \wedge X \longrightarrow M \wedge X$; m_X being called an M-action on X.

2) A map $f \colon X \longrightarrow Y$ between M-module spectra is called an M-map, if f is compatible with the M-actions on X and Y, i.e., $m_Y(1_M \wedge f) = f m_X$.

the switching map, defined from the ones on X and on Y, cf. [15; (1.6)]. We shall write the former M-module spectrum as $\mathring{X} \wedge Y$ and the latter as $X \wedge \mathring{Y}$. Similarly, we use the notations $\mathring{X} \wedge Y \wedge Z$, $X \wedge \mathring{Y} \wedge Z$, etc.

PROPOSITION 1.3. *There exist elements*

$$m' \in [\mathring{K} \wedge K, \mathring{M} \wedge K]_0^M \cap [K \wedge \mathring{K}, M \wedge \mathring{K}]_0^M,$$

$$n' \in [\mathring{M} \wedge K, \mathring{K} \wedge K]_{pq+1}^M \cap [M \wedge \mathring{K}, K \wedge \mathring{K}]_{pq+1}^M$$

such that

$$m'(i' \wedge 1_K) = 1_{M \wedge K},\ (\pi' \wedge 1_K)n' = 1_{M \wedge K},\ m'n' = 0,$$

$$(i' \wedge 1_K)m' + n'(\pi' \wedge 1_K) = 1_{K \wedge K}.$$

PROOF. By (1.1) and Proposition 1.1,

$$\Sigma^{pq}\mathring{M} \wedge K \xrightarrow{\alpha^p \wedge 1} \mathring{M} \wedge K \xrightarrow{i' \wedge 1} \mathring{K} \wedge K \xrightarrow{\pi' \wedge 1} \Sigma^{pq+1}\mathring{M} \wedge K$$

is a cofibering of M-module spectra and M-maps. Applying [15; Th. 4.5] to this sequence and using Lemma 1.2, we obtain elements $m' \in [\mathring{K} \wedge K, \mathring{M} \wedge K]_0^M$ and $n' \in [\mathring{M} \wedge K, \mathring{K} \wedge K]_{pq+1}^M$ satisfying the desired equalities. Since $\alpha^p \wedge 1$, $i' \wedge 1$ and $\pi' \wedge 1$ are also M-maps with respect to $M \wedge \mathring{K}$ and $K \wedge \mathring{K}$, it follows from [15; Lemma 4.6] that these m' and n' are the M-maps with respect to $M \wedge \mathring{K}$ and $K \wedge \mathring{K}$.

DEFINITION 1.4.

$$\mu_1 = m_K m' : K \wedge K \longrightarrow K, \qquad \nu_1 = i'i \wedge 1_K : K \longrightarrow K \wedge K,$$

$$\mu_2 = (\pi \wedge 1_K)m' : K \wedge K \longrightarrow \Sigma K, \qquad \nu_2 = (i' \wedge 1_K)n_K : \Sigma K \longrightarrow K \wedge K,$$

$$\mu_3 = m_K(\pi' \wedge 1_K) : K \wedge K \longrightarrow \Sigma^{pq+1}K, \qquad \nu_3 = n'(i \wedge 1_K) : \Sigma^{pq+1}K \longrightarrow K \wedge K,$$

$$\mu_4 = \pi\pi' \wedge 1_K : K \wedge K \longrightarrow \Sigma^{pq+2}K, \qquad \nu_4 = n'n_K : \Sigma^{pq+2}K \longrightarrow K \wedge K.$$

The above two propositions show immediately the following

COROLLARY 1.5. $\mu_i\nu_i = 1_K$, $\mu_i\nu_j = 0$ *for* $i \neq j$, *and* $\nu_1\mu_1 + \nu_2\mu_2 + \nu_3\mu_3 + \nu_4\mu_4 = 1_{K \wedge K}$.

REMARK 1.6. These relations give a decomposition

$$K \wedge K = K \vee \Sigma K \vee \Sigma^{pq+1}K \vee \Sigma^{pq+2}K.$$

Hence, *the ring* $\mathscr{A}_*(K \wedge K)$ *is isomorphic to a subring of* (4, 4)-*matrices on*

$\mathscr{A}_*(K)$ by sending f to a matrix $(\mu_i f v_j)$.

We introduced in [15; § 2] (cf. [18]) the (additive) homomorphism

$$\theta = \theta_{m_X, m_Y}\colon [X, Y]_k \longrightarrow [X, Y]_{k+1}$$

for M-module spectra (X, m_X) and (Y, m_Y). θ has the following properties [15; Th. 2.3, Prop. 2.5, (2.2), (3.1)]:

(1.2) $\theta(fg) = (-1)^k\theta(f)g + f\theta(g)$ *for* $f \in [Y, Z]_l$, $g \in [X, Y]_k$;

(1.3) $\theta(f) = 0$ *if and only if f is an M-map*;

(1.4) $\theta^2(f) = 0$ *for* $f \in [X, Y]_k$ *if X and Y are associative, in particular* $\theta^2 = 0$ *on* $\mathscr{A}_*(M)$ *and* $\mathscr{A}_*(K)$;

(1.5) $\theta_1(f \wedge g) = \theta(f) \wedge g$ *and* $\theta_2(f \wedge g) = f \wedge \theta(g)$ *for* $f \in [X, Y]_k$, $g \in [X', Y']_l$, *where θ_1 and θ_2 are the operations θ on* $[\mathring{X} \wedge X', \mathring{Y} \wedge Y']_*$ *and on* $[X \wedge \mathring{X}', Y \wedge \mathring{Y}']_*$, *respectively*;

(1.6) $\theta(\delta_M) = -1_M$ *in* $\mathscr{A}_0(M)$, *where* $\delta_M = i\pi$ *is a generator of* $\mathscr{A}_{-1}(M) = Z/pZ$.

The element α^p commutes with δ_M. Hence the following lemma and proposition are direct consequences of [15; Prop. 7.3, Th. 7.5].

LEMMA 1.7. *There exists an element* $\delta = \delta_K \in \mathscr{A}_{-1}(K)$ *such that* $\delta^2 = 0$, $\theta(\delta) = -1_K$ *and* $\delta i' = i'\delta_M$.

PROPOSITION 1.8. $\mathscr{A}_*(K) = \mathscr{B}_*(K) \otimes E(\delta) = E(\delta) \otimes \mathscr{B}_*(K)$.

DEFINITION 1.9. We put $\delta' = i'\pi' \in \mathscr{A}_{-pq-1}(K)$. This satisfies $\theta(\delta') = 0$, $(\delta')^2 = 0$ and $\delta'\delta = -\delta\delta'$.

The following result determines the matrix corresponding to the switching map of $K \wedge K$.

THEOREM 1.10. *Let* $T\colon K \wedge K \to K \wedge K$ *be the map switching the factors. Then,*

$$\begin{aligned}
\mu_1 T &= \mu_1, & Tv_1 &= v_1 + v_2\delta + v_3\delta' + v_4\delta\delta',\\
\mu_2 T &= -\mu_2 + \delta\mu_1, & Tv_2 &= -v_2 + v_4\delta',\\
\mu_3 T &= -\mu_3 + \delta'\mu_1, & Tv_3 &= -v_3 - v_4\delta,\\
\mu_4 T &= \mu_4 - \delta\mu_3 + \delta'\mu_2 + \delta\delta'\mu_1, & Tv_4 &= v_4.
\end{aligned}$$

In other words, T corresponds to the lower triangular matrix

$$\begin{pmatrix} 1 & & & \\ \delta & -1 & & \\ \delta' & 0 & -1 & \\ \delta\delta' & \delta' & -\delta & 1 \end{pmatrix}.$$

REMARK. The first equality in the above theorem means that K *is a commutative ring spectrum with multiplication* μ_1 *and unit* $i'i$. By $(A_{1,1})$ of Theorem 2.1 in the below, *it is also associative.*

Theorem 1.10 is an easy restatement of the following

LEMMA 1.11. (i) $(\pi' \wedge 1_K)T(i' \wedge 1_K) = (i \wedge 1_K)\delta' m_K + n_K\delta'(\pi \wedge 1_K) + n_K\delta\delta' m_K$.

(ii) $m'T(i' \wedge 1_K) = (i \wedge 1_K)m_K - n_K(\pi \wedge 1_K) + n_K\delta m_K$.

(iii) $(\pi' \wedge 1_K)Tn' = -(i \wedge 1_K)m_K + n_K(\pi \wedge 1_K) - n_K\delta m_K$.

(iv) $m'Tn' = 0$.

PROOF. (i) By [5; Th. 7.10] ([18; Lemma 1.3]), the switching map T_M: $M \wedge M \to M \wedge M$ satisfies the equality

(1.7) $T_M = (i \wedge 1_M)m_M - n_M(\pi \wedge 1_M) + n_M\delta_M m_M$,

where m_M is the multiplication (M-action) on M and n_M is its dual. Since i' and π' are the M-maps, we have

(1.8) $m_K(1_M \wedge i') = i'm_M$, $n_K i' = (1_M \wedge i')n_M$, $\pi' m_K = m_M(1_M \wedge \pi')$, $(1_M \wedge \pi')n_K = -n_M\pi'$.

Then,

$$\begin{aligned}(\pi' \wedge 1)T(i' \wedge 1) &= (1_M \wedge i')T_M(1_M \wedge \pi') \\ &= (1_M \wedge i')(i \wedge 1_M)m_M(1_M \wedge \pi') \\ &\quad - (1_M \wedge i')n_M(\pi \wedge 1_M)(1_M \wedge \pi') \\ &\quad + (1_M \wedge i')n_M\delta_M m_M(1_M \wedge \pi') \\ &= (i \wedge 1)i'\pi' m_K + n_K i'\pi'(\pi \wedge 1) + n_K i'\delta_M\pi' m_K \\ &= (i \wedge 1)\delta' m_K + n_K\delta'(\pi \wedge 1) + n_K\delta\delta' m_K,\end{aligned}$$

by (1.7), (1.8) and Lemma 1.7.

(ii) By (1.7) and (1.8), $m'T(i' \wedge 1)(1_M \wedge i') = m'(i' \wedge i')T_M = (1_M \wedge i')T_M = ((i \wedge 1)m_K - n_K(\pi \wedge 1) + n_K\delta m_K)(1_M \wedge i')$. Since $(1_M \wedge i')^*$ is injective in degree 0, (ii) is obtained.

(iii) Similarly, $(1_M \wedge \pi')(\pi' \wedge 1)Tn' = (1_M \wedge \pi')(-(i \wedge 1)m_K + n_K(\pi \wedge 1) - n_K\delta m_K)$ and $(1_M \wedge \pi')_*$ is injective in degree 0.

(iv) Since T lies in $[\mathring{K} \wedge K, K \wedge \mathring{K}]_0^M \cap [K \wedge \mathring{K}, \mathring{K} \wedge K]_0^M$, $m'Tn'$ lies in $[\mathring{M} \wedge K, M \wedge \mathring{K}]_{pq+1}^M \cap [M \wedge \mathring{K}, \mathring{M} \wedge K]_{pq+1}^M$, which is trivial by easy calculations.

§2. Associativity

The purpose of this section is to prove the following associative formulas.

THEOREM 2.1. (i) (*Associativity of μ_i*)

$(\mathrm{A}_{i,j})$ $\mu_i(1_K \wedge \mu_j) = (-1)^{\deg\mu_i \deg\mu_j}\mu_j(\mu_i \wedge 1_K)$

for $i = j \neq 2$, $i = 4$, $(i, j) = (2, 1)$ *or* $(i, j) = (3, 1)$.

$(\mathrm{A}_{2,2})$ $\mu_2(1_K \wedge \mu_2) = -\mu_2(\mu_2 \wedge 1_K)$ *if* $p \geqq 7$,

and there is an element $\xi \in Z/pZ\{(\alpha_1\beta_1^3 \wedge 1_K)\delta'\}$ *such that*

$$\mu_2(1_K \wedge \mu_2) = -\mu_2(\mu_2 \wedge 1_K) + \xi\mu_3(\mu_3 \wedge 1_K) \quad \textit{if} \quad p = 5.$$

$(\mathrm{A}_{i,j})$ $\mu_i(1_K \wedge \mu_j) = (-1)^j\mu_j(\mu_i \wedge 1_K) + (-1)^{i+\deg\mu_j}\mu_i(\mu_j \wedge 1_K)$

for $(i, j) = (1, 2), (1, 3), (2, 4)$ *or* $(3, 4)$.

$(\mathrm{A}_{1,4})$ $\mu_1(1_K \wedge \mu_4) = \mu_4(\mu_1 \wedge 1_K) + \mu_1(\mu_4 \wedge 1_K) + \mu_3(\mu_2 \wedge 1_K) - \mu_2(\mu_3 \wedge 1_K)$.

$(\mathrm{A}_{2,3})$ $\mu_2(1_K \wedge \mu_3) = -\mu_3(\mu_2 \wedge 1_K) - \mu_1(\mu_4 \wedge 1_K)$.

$(\mathrm{A}_{3,2})$ $\mu_3(1_K \wedge \mu_2) = \mu_2(\mu_3 \wedge 1_K) - \mu_1(\mu_4 \wedge 1_K)$.

(ii) (*Associativity of ν_i*)

$(\mathrm{A}'_{i,j})$ $(1_K \wedge \nu_j)\nu_i = (-1)^{\deg\nu_i \deg\nu_j}(\nu_i \wedge 1_K)\nu_j$

for $i = j \neq 3$, $i = 1$, $(i, j) = (2, 4)$ *or* $(i, j) = (3, 4)$.

$(\mathrm{A}'_{3,3})$ $(1_K \wedge \nu_3)\nu_3 = -(\nu_3 \wedge 1_K)\nu_3$ *if* $p \geqq 7$,

$$(1_K \wedge \nu_3)\nu_3 = -(\nu_3 \wedge 1_K)\nu_3 - (\nu_2 \wedge 1_K)\nu_2\xi \quad \textit{if} \quad p = 5,$$

where ξ *is the same as in* $(\mathrm{A}_{2,2})$.

$(\mathrm{A}'_{i,j})$ $(1_K \wedge \nu_j)\nu_i = -(-1)^j(\nu_i \wedge 1_K)\nu_j - (-1)^{j+\deg\nu_j}(\nu_j \wedge 1_K)\nu_i$

$$for\quad (i, j) = (2, 1), (3, 1), (4, 2)\quad or\quad (4, 3).$$

($A'_{2,3}$) $(1_K \wedge v_3)v_2 = -(v_2 \wedge 1_K)v_3 + (v_1 \wedge 1_K)v_4$.

($A'_{3,2}$) $(1_K \wedge v_2)v_3 = (v_3 \wedge 1_K)v_2 + (v_1 \wedge 1_K)v_4$.

($A'_{4,1}$) $(1_K \wedge v_1)v_4 = (v_1 \wedge 1_K)v_4 + (v_4 \wedge 1_K)v_1 + (v_3 \wedge 1_K)v_2 - (v_2 \wedge 1_K)v_3$.

LEMMA 2.2. *Let θ_1 and θ_2 be the operations θ with respect to $\mathring{K} \wedge K$ and $K \wedge \mathring{K}$, respectively. Then*

(i) $\theta_1(\mu_1) = 0,\ \theta_1(\mu_2) = -\mu_1,\ \theta_1(\mu_3) = 0,\ \theta_1(\mu_4) = \mu_3,$

$\theta_1(v_1) = v_2,\ \theta_1(v_2) = 0,\ \theta_1(v_3) = v_4,\ \theta_1(v_4) = 0;$

(ii) $\theta_2(\mu_i) = 0,\ \theta_2(v_i) = 0$ *for* $i = 1, 2, 3, 4$.

PROOF. By Proposition 1.3 and (1.3), $\theta_i(m')=0$ and $\theta_i(n')=0$. By Proposition 1.1, (1.5) and (1.3), $\theta_i(i' \wedge 1)=0$ and $\theta_i(\pi' \wedge 1)=0$. By [15; Lemma 5.1], $\theta_1(i \wedge 1)=n_K$ and $\theta_1(\pi \wedge 1)=-m_K$. By [15; Prop. 5.4] and Proposition 1.1, $\theta_1(m_K)=0$ and $\theta_1(n_K)=0$. We have easily $\theta_2(i \wedge 1)=\theta_2(\pi \wedge 1)=\theta_2(m_K)=\theta_2(n_K)=0$. From these values of θ_i, using (1.2) we obtain the lemma.

LEMMA 2.3. *Let θ'_1 and θ'_2 be the operations θ with respect to $\mathring{K} \wedge K \wedge K$ and $K \wedge \mathring{K} \wedge K$, respectively, and θ_1 and θ_2 be as above. Then*

(i) $\theta'_1(\mu_i(1 \wedge \mu_j)) = (-1)^{\deg \mu_j}\theta_1(\mu_i)(1 \wedge \mu_j),$

$\theta'_1(\mu_i(\mu_j \wedge 1)) = (-1)^{\deg \mu_j}\theta_1(\mu_i)(\mu_j \wedge 1) + \mu_i(\theta_1(\mu_j) \wedge 1);$

(ii) $\theta'_2(\mu_i(1 \wedge \mu_j)) = \mu_i(1 \wedge \theta_1(\mu_j)),$

$\theta'_2(\mu_i(\mu_j \wedge 1)) = (-1)^{\deg \mu_j}\theta_1(\mu_i)(\mu_j \wedge 1);$

(iii) $\theta'_1((1 \wedge v_j)v_i) = (1 \wedge v_j)\theta_1(v_i),$

$\theta'_1((v_j \wedge 1)v_i) = (-1)^{\deg v_i}(\theta_1(v_j) \wedge 1)v_i + (v_j \wedge 1)\theta_1(v_i);$

(iv) $\theta'_2((1 \wedge v_j)v_i) = (-1)^{\deg v_i}(1 \wedge \theta_1(v_j))v_i,$

$\theta'_2((v_j \wedge 1)v_i) = (v_j \wedge 1)\theta_1(v_i).$

PROOF. By (1.5), $\theta'_1(1 \wedge \mu_j)=\theta_1(1) \wedge \mu_j=0$ and $\theta'_1(1 \wedge v_j)=0$. Hence (i) and (iii) follow easily from (1.3) and Lemma 2.2 (i). The elements $\mu_i(1 \wedge \mu_j)$ and $\mu_i(\mu_j \wedge 1)$ pass through $K \wedge K$, and the θ'_2-images of these elements do not depend on M-actions on the intermediate spectrum $K \wedge K$. Considering the M-action $K \wedge \mathring{K}$, we have $\theta'_2(\mu_i(1 \wedge \mu_j))=\mu_i(1 \wedge \theta_1(\mu_j)) \pm \theta_2(\mu_i)(1 \wedge \mu_j)=\mu_i(1 \wedge \theta_1(\mu_j))$ by (1.2), (1.5) and Lemma 2.2 (ii). Also, considering the M-action $\mathring{K} \wedge K$, we have $\theta'_2(\mu_i(\mu_j \wedge 1)) = (-1)^{\deg \mu_j}\theta_1(\mu_i)(\mu_j \wedge 1)$, and (ii) is obtained. (iv) is similar to (ii).

PROOF OF THEOREM 2.1. Let $(i, j)=(2, 4)$, $(4, 2)$ or $(4, 4)$. Then, by Lemmas 2.2–2.3, $(A_{i,j})$ implies $(A_{i-1,j})$, $(A_{i,j-1})$ and $(A_{i-1,j-1})$ by operating θ'_1, θ'_2 and $\theta'_1\theta'_2$ to $(A_{i,j})$, respectively. So, we prove $(A_{i,j})$.

Since $\mu_4 = \pi\pi' \wedge 1$, $\mu_4(1 \wedge \mu_j) = \mu_j(\pi\pi' \wedge 1 \wedge 1) = \mu_j(\mu_4 \wedge 1)$, in particular $(A_{4,2})$ and $(A_{4,4})$ follow. Similarly, $(1 \wedge \mu_4)(v_1 \wedge 1) = v_1\mu_4$ and so $\mu_2(1 \wedge \mu_4)(v_1 \wedge 1) = 0$. Since $1 \wedge \mu_4 = (\mu_4 \wedge 1)(T \wedge 1)$, we have $\mu_2(1 \wedge \mu_4)(v_k \wedge 1) = \mu_2((\mu_4 T v_k) \wedge 1) = \mu_2(\delta' \wedge 1)$ for $k=2$, $= -\mu_2(\delta \wedge 1)$ for $k=3$, and $= \mu_2$ for $k=4$, by Theorem 1.10. By definition, $\mu_2(\delta' \wedge 1) = (\pi \wedge 1)m'(i' \wedge 1)(\pi' \wedge 1) = \pi\pi' \wedge 1 = \mu_4$. To prove $\mu_2(\delta \wedge 1) = 0$, we prepare the following

LEMMA 2.4. $m'(1_K \wedge \delta)n' = 0$.

Then we have $\mu_2(1 \wedge \delta) = (\pi \wedge 1)m'(1 \wedge \delta)(i' \wedge 1)m' = (\pi \wedge 1)(1_M \wedge \delta)m' = -\delta(\pi \wedge 1)m' = -\delta\mu_2$ and similarly $\mu_1(1 \wedge \delta) = m(1_M \wedge \delta)m' = (\delta m - \pi \wedge 1)m' = \delta\mu_1 - \mu_2$, by Propositions 1.3, 1.1 and Lemma 1.7. Hence $\mu_2(\delta \wedge 1) = \mu_2 T(1 \wedge \delta)T = 0$ by Theorem 1.10. Therefore $\mu_2(1 \wedge \mu_4) = \sum_k \mu_2(1 \wedge \mu_4)(v_k \wedge 1)(\mu_k \wedge 1) = \mu_4(\mu_2 \wedge 1) + \mu_2(\mu_4 \wedge 1)$ and $(A_{2,4})$ follows. Thus, we have obtained $(A_{i,j})$ except for $(i, j) = (1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 2)$.

By using Lemma 2.3 (iii), (iv) instead of (i), (ii), we can similarly obtain $(A'_{i,j})$ except for $(i, j) = (3, 3)$, $(3, 4)$, $(4, 3)$ and $(4, 4)$.

We next consider $(A_{2,2})$. We have $\mu_2(1 \wedge \mu_2)(i' \wedge 1 \wedge 1) = \mu_2(i' \wedge 1)(1_M \wedge \mu_2) = (\pi \wedge 1)(1_M \wedge \mu_2) = -\mu_2(\pi \wedge 1 \wedge 1) = -\mu_2(\mu_2 \wedge 1)(i' \wedge 1 \wedge 1)$, and hence $\mu_2(1 \wedge \mu_2) = -\mu_2(\mu_2 \wedge 1) + \xi_1(\mu_3 \wedge 1) + \xi_2(\mu_4 \wedge 1)$ for some $\xi_1 \in [K \wedge \mathring{K}, K]^M_{pq-1}$ and $\xi_2 \in [K \wedge \mathring{K}, K]^M_{pq}$, by [15; Th. 4.5]. Using exact sequences derived from (1.1), we can compute $\mathscr{B}_k(K)$ for small k from the results on $\mathscr{B}_*(M)$ [13], and we obtain the following results:

$$\mathscr{B}_{pq-1}(K) = Z/pZ, \ \mathscr{B}_{pq}(K) = 0, \ \mathscr{B}_{pq+1}(K) = 0,$$

$$\mathscr{B}_{2pq}(K) = \begin{cases} Z/pZ\{\bar{\beta}\} & \text{for } p \geqq 7 \\ Z/pZ\{\bar{\beta}\} + Z/pZ\{(\alpha_1\beta_1^3 \wedge 1_K)\delta'\} & \text{for } p = 5, \end{cases}$$

$$\mathscr{B}_{2pq+1}(K) = \begin{cases} 0 & \text{for } p \geqq 7 \\ Z/pZ\{i'\eta\pi'\} & \text{for } p = 5, \end{cases}$$

$$\mathscr{B}_{2pq+2}(K) = 0,$$

where $\bar{\beta}$ satisfies $\pi'\bar{\beta}i' = \beta_{(1)} \in \mathscr{B}_{pq-1}(M)$, $\alpha_1 = \pi\alpha i \in \pi_{q-1}(S)$, $\beta_1 = \pi\beta_{(1)}i \in \pi_{pq-2}(S)$ and $\eta = \alpha(\delta_M\beta_{(1)})^3$ ([13], [19]).

From these results, $\xi_1 = \xi_3\mu_1 + x\bar{\beta}\mu_3 + \xi\mu_3 + \xi_4\mu_4$ and $\xi_2 = \xi_5\mu_4$ for some ξ_3

$\in \mathscr{B}_{pq-1}(K)$, $x \in Z/pZ$, $\xi \in \mathscr{B}_{2pq}(K)/\{\bar{\beta}\}$, ξ_4, $\xi_5 \in \mathscr{B}_{2pq+1}(K)$ (ξ, ξ_4, $\xi_5=0$ if $p \geqq 7$). By $(A'_{3,1})$ and $(A'_{1,3})$, $(\nu_3 \wedge 1)\nu_1 = (1 \wedge \nu_1)\nu_3 - (1 \wedge \nu_3)\nu_1$, and so $\xi_3 = \mu_2(1 \wedge \mu_2)(\nu_3 \wedge 1)\nu_1 = 0$. The functional P^p-operation for $\beta_{(1)}$ is nontrivial [19], and hence $x \neq 0$ implies $P^p \neq 0$ on $H^*(K \wedge K \wedge K; Z/pZ)$. But $P^i=0$ on $H^*(K; Z/pZ)$ for $i \geqq 1$ and the Cartan formula shows $P^p=0$ on $H^*(K \wedge K \wedge K; Z/pZ)$, so x must be trivial. Thus we have

$(A_{2,2})'$ $\quad \mu_2(1 \wedge \mu_2) = -\mu_2(\mu_2 \wedge 1) + \xi\mu_3(\mu_3 \wedge 1) + \xi_4\mu_4(\mu_3 \wedge 1) + \xi_5\mu_3(\mu_4 \wedge 1).$

By considering θ'_1, θ'_2 and $\theta'_1\theta'_2$-images of $(A_{2,2})'$, we also have

$(A_{2,1})'$ $\quad \mu_2(1 \wedge \mu_1) = \mu_1(\mu_2 \wedge 1) + \xi_4\mu_3(\mu_3 \wedge 1),$

$(A_{1,2})'$ $\quad \mu_1(1 \wedge \mu_2) = -\mu_1(\mu_2 \wedge 1) + \mu_2(\mu_1 \wedge 1) + (\xi_5 - \xi_4)\mu_3(\mu_3 \wedge 1),$

and the associativity $(A_{1,1})$ of μ_1. In case $p \geqq 7$, ξ, ξ_4 and ξ_5 are trivial, so $(A_{2,2})$, $(A_{1,2})$ and $(A_{2,1})$ are obtained too.

In a similar manner to the above discussion on $(A_{2,2})$, we obtain $(A'_{3,3})$, $(A'_{3,4})$, $(A'_{4,3})$ and $(A'_{4,4})$ in case $p \geqq 7$, and a weak form of $(A'_{3,3})$

$$(1 \wedge \nu_3)\nu_3 = -(\nu_3 \wedge 1)\nu_3 + (\nu_2 \wedge 1)\nu_2\xi' + (\nu_2 \wedge 1)\nu_1\xi'_4 + (\nu_1 \wedge 1)\nu_2\xi'_5$$

in case $p=5$. By $(A_{4,3})$, $(A_{3,3})$ and $(A_{2,2})'$, $\mu_2(\mu_2 \wedge 1) = -\mu_2(1 \wedge \mu_2) - \xi\mu_3(1 \wedge \mu_3) + \xi_4\mu_3(1 \wedge \mu_4) + (\xi_5 - \xi_4)\mu_4(1 \wedge \mu_3)$ and so $\xi' = \mu_2(\mu_2 \wedge 1)(1 \wedge \nu_3)\nu_3 = -\xi$. By $(A_{3,3})$ and $(A_{2,1})'$, $\mu_1(\mu_2 \wedge 1) = \mu_2(1 \wedge \mu_1) + \xi_4\mu_3(1 \wedge \mu_3)$, and so $\xi'_4 = \mu_2(\mu_1 \wedge 1)(1 \wedge \nu_3)\nu_3 = \xi_4$. Similarly we have $\xi'_5 = \xi_5$ from $(A_{3,3})$, $(A_{2,1})'$ and $(A_{1,2})'$. We have therefore obtained, in case $p=5$,

$(A'_{3,3})'$ $\quad (1 \wedge \nu_3)\nu_3 = -(\nu_3 \wedge 1)\nu_3 - (\nu_2 \wedge 1)\nu_2\xi + (\nu_2 \wedge 1)\nu_1\xi_4 + (\nu_1 \wedge 1)\nu_2\xi_5,$

$(A'_{3,4})'$ $\quad (1 \wedge \nu_4)\nu_3 = (\nu_3 \wedge 1)\nu_4 + (\nu_2 \wedge 1)\nu_2\xi_4,$

$(A'_{4,3})'$ $\quad (1 \wedge \nu_3)\nu_4 = (\nu_4 \wedge 1)\nu_3 - (\nu_3 \wedge 1)\nu_4 + (\nu_2 \wedge 1)\nu_2(\xi_5 - \xi_4),$

and $(A'_{4,4})$.

The proof of $\xi_4 = \xi_5 = 0$ in case $p=5$ is delayed to the end of this section.

Proof of Lemma 2.4. Since $\mathscr{A}_{pq}(K) = \mathscr{A}_{pq+1}(K) = 0$, we can put $m'(1 \wedge \delta)n' = n_K f m_K$ for $f \in \mathscr{A}_{pq-1}(K)$. Then $(1_M \wedge \delta)nfm - nfm(1_M \wedge \delta) = m'(1 \wedge \delta)(i' \wedge 1)m'(1 \wedge \delta)n' + m'(1 \wedge \delta)n'(\pi' \wedge 1)(1 \wedge \delta)n' = m'(1 \wedge \delta^2)n' = 0$. Compositing m from the left and using $\theta(\delta) = -1_K$, we have $fm=0$. Therefore $m'(1 \wedge \delta)n' = nfm = 0$.

The rest of this section is devoted to show $\xi_4 = \xi_5 = 0$. Let W be the mapping cone of $\alpha^2 \in \mathscr{B}_{2q}(M)$, $q=2(p-1)$, and denote the cofibering for W by

(2.1) $$\Sigma^{2q}M \xrightarrow{\alpha^2} M \xrightarrow{i_W} W \xrightarrow{\pi_W} \Sigma^{2q+1}M.$$

Since $\mathscr{A}_1(W)=\mathscr{A}_2(W)=0$, W is an M-module spectrum having the unique associative M-action m_W and its dual n_W. Also, by [15; Th. 4.3], i_W and π_W are the M-maps. Let L be the mapping cone of $\alpha_2=\pi\alpha^2 i\in\pi_{2q-1}(S)$ and

(2.2) $$\Sigma^{2q-1}S \xrightarrow{\alpha_2} S \xrightarrow{i_L} L \xrightarrow{\pi_L} \Sigma^{2q}S$$

be the cofibering for L. By easy calculations, $\mathscr{A}_{2q}(W)=0$, $\mathscr{A}_{2q+1}(W)=0$, and hence $\alpha^2\wedge 1_W=n_W(\alpha_2\wedge 1_W)m_W$. Since $W\wedge W$ is the mapping cone of $\alpha^2\wedge 1_W$, $W\wedge W$ is homotopy equivalent to $W\vee(\Sigma L\wedge W)\vee\Sigma^{2q+2}W$ with the inclusions $i_1\colon W\to W\wedge W$, $i_2\colon \Sigma L\wedge W\to W\wedge W$ and $i_3\colon \Sigma^{2q+2}W\to W\wedge W$ and with their left inverses $p_1\colon W\wedge W\to W$, $p_2\colon W\wedge W\to\Sigma L\wedge W$ and $p_3\colon W\wedge W\to\Sigma^{2q+2}W$ such that $i_1=i_W i\wedge 1_W$, $i_2(i_L\wedge 1_W)=(i_W\wedge 1_W)n_W$, $(\pi_L\wedge 1_W)p_2=m_W(\pi_W\wedge 1_W)$ and $p_3=\pi\pi_W\wedge 1_W$. Putting $\mu_W=p_1$, $\mu'_W=p_2$, $\nu_W=i_3$ and $\nu'_W=i_2$, we have easily the fol lowing

PROPOSITION 2.5. *There are elements*

$$\mu_W\in[W\wedge W, W]_0,\quad \mu'_W\in[W\wedge W, L\wedge W]_{-1},$$

$$\nu_W\in[W, W\wedge W]_{-2q-2},\quad \nu'_W\in[L\wedge W, W\wedge W]_1,$$

which satisfy the following relations:

(i) $\mu_W(i_W i\wedge 1_W)=1_W$, $\mu'_W\nu'_W=1_{L\wedge W}$, $(\pi\pi_W\wedge 1_W)\nu_W=1_W$,

$\mu_W\nu'_W=0$, $\mu_W\nu_W=0$, $\mu'_W(i_W i\wedge 1_W)=0$, $\mu'_W\nu_W=0$, $(\pi\pi_W\wedge 1_W)\nu'_W=0$;

(ii) $(i_W i\wedge 1_W)\mu_W+\nu'_W\mu'_W+\nu_W(\pi\pi_W\wedge 1_W)=1_{W\wedge W}$;

(iii) $\mu_W(i_W\wedge 1_W)=m_W$, $(\pi_W\wedge 1_W)\nu_W=n_W$,

$\mu'_W(i_W\wedge 1_W)=(i_L\wedge 1_W)(\pi\wedge 1_W)$, $(\pi_W\wedge 1_W)\nu'_W=(i\wedge 1_W)(\pi_L\wedge 1_W)$,

$(\pi_L\wedge 1_W)\mu'_W=m_W(\pi_W\wedge 1_W)$, $\nu'_W(i_L\wedge 1_W)=(i_W\wedge 1_W)n_W$.

REMARK. From this proposition, we see that *W is a ring spectrum with multiplication μ_W and unit $i_W i$.* We can also prove that *μ_W is commutative and associative.* We notice that in case $p=3$ this proposition and the commutativity of μ_W also hold but the associativity does not (cf. [15; Th. 6.3]). In a forthcoming paper, we shall prove that *the mapping cone X_j of α^j is a ring spectrum for $p\geqq 3$ and $j\geqq 1$ except for $(p, j)=(3, 1)$, i.e., the spectrum $V(1)$ at $p=3$* (cf. [18]).

LEMMA 2.6. (i) $\mu_W\in[\mathring{W}\wedge W, W]_0^M\cap[W\wedge\mathring{W}, W]_0^M$.

(ii) $\mu'_W\in[W\wedge\mathring{W}, L\wedge\mathring{W}]_{-1}^M$, *and* $\theta(\mu'_W)=-(i_L\wedge 1_W)\mu_W$ *for* θ *on* $[\mathring{W}\wedge W,$

$L \wedge \mathring{W}]_{-1}$.

PROOF. Let θ_1 and θ_2 be the θ's with respect to $\mathring{W} \wedge W$ and $W \wedge \mathring{W}$, respectively. Using exact sequences derived from (2.1)–(2.2), we have

$$[W \wedge W, W]_1 = 0 \quad \text{and} \quad [W \wedge W, L \wedge W]_0 = Z/pZ\{(i_L \wedge 1_W)\mu_W\}$$

from the known results on $\mathscr{A}_*(M)$ ([13], [19]). Hence $\theta_i(\mu_W)=0$ and $\theta_i(\mu'_W) = x_i(i_L \wedge 1_W)\mu_W$, $x_i \in Z/pZ$, $i=1, 2$. By considering θ_i-images of the third equality in (iii) of Proposition 2.5, we have $x_1 = -1$ and $x_2 = 0$ as desired.

By [15; Th. 4.4], there exists the M-map

$$(2.3) \qquad \rho: K \longrightarrow W \quad \text{such that} \quad \rho i' = i_W \quad \text{and} \quad \pi_W \rho = \alpha^{p-2}\pi'.$$

LEMMA 2.7. *There hold the relations*

$$\mu_W(\rho \wedge \rho) = \rho\mu_1 \quad \text{and} \quad \mu'_W(\rho \wedge \rho) = (i_L \wedge 1_W)\rho\mu_2.$$

PROOF. By Lemma 2.2 (ii), the group $[K \wedge \mathring{K}, W]^M_*$ is determined from $[K, W]^M_*$ via the decomposition of Remark 1.6. From the results on $\mathscr{B}_*(M)$, we have $[K, W]^M_0 = Z/pZ\{\rho\}$, $[K, W]^M_k = 0$ for $k=1$, $pq+1$ and for $k=pq+2$, $p \geqq 7$, and $[K, W]^M_{pq+2} = Z/pZ\{i_W\zeta\pi'\}$ for $p=5$, where $\zeta = \alpha_1\beta_1^2 \wedge 1_M \in \mathscr{B}_*(M)$. Since $\theta_1(\rho\mu_1)=0$ and $\theta_1(i_W\zeta\pi'\mu_4) \neq 0$ by Lemma 2.2 (i), we obtain

$$[\mathring{K} \wedge K, W]^M_0 \cap [K \wedge \mathring{K}, W]^M_0 = Z/pZ\{\rho\mu_1\}.$$

By Lemma 2.6, the element $\mu_W(\rho \wedge \rho)$ lies in this group. Since $\mu_W(\rho \wedge \rho)(i' \wedge 1_K) = m_W(1_M \wedge \rho) = \rho\mu_1(i' \wedge 1_K)$, the first equality is obtained.

By computing $[K \wedge \mathring{K}, L \wedge \mathring{W}]^M_{-1}$, the second one is similarly obtained.

LEMMA 2.8. *There hold the associative formulas*:

(i) $\mu'_W(1_W \wedge \mu_W) = (1_L \wedge \mu_W)(\mu'_W \wedge 1_W)$;

(ii) $(1_L \wedge \mu_W)(T_{W,L} \wedge 1_W)(1_W \wedge \mu'_W) = -(1_L \wedge \mu_W)(\mu'_W \wedge 1_W) + \mu'_W(\mu_W \wedge 1_W)$,

where $T_{W,L}$: $W \wedge L \to L \wedge W$ is the switching map.

PROOF. We abbreviate 1_W, μ_W and μ'_W to 1, μ and μ'. By Proposition 2.5 (iii), we have

$$\begin{aligned} \mu'(1 \wedge \mu)(i_W \wedge 1 \wedge 1) &= (i_L \wedge 1)\mu(\pi \wedge 1 \wedge 1) \\ &= (1_L \wedge \mu)(\mu' \wedge 1)(i_W \wedge 1 \wedge 1), \end{aligned}$$

$$(1_L \wedge \mu)(T_{W,L} \wedge 1)(1 \wedge \mu')(i_W \wedge 1 \wedge 1) = (1_L \wedge m_W)(T_{M,L} \wedge 1)(1_M \wedge \mu'),$$

$$(\mu'(\mu \wedge 1) - (1_L \wedge \mu)(\mu' \wedge 1))(i_W \wedge 1 \wedge 1)$$
$$= \mu'(m_W \wedge 1) - (i_L \wedge 1)\mu(\pi \wedge 1 \wedge 1),$$

where $T_{M,L}$: $M \wedge L \rightarrow L \wedge M$ is the switching map. By Lemma 2.6 (ii), $-(i_L \wedge 1)\mu = \theta_1(\mu') = (1_L \wedge m_W)(T_{M,L} \wedge 1)(1_M \wedge \mu')(n_W \wedge 1)$ and hence

$$(1_L \wedge m_W)(T_{M,L} \wedge 1)(1_M \wedge \mu') = \mu'(m_W \wedge 1) - (i_L \wedge 1)\mu(\pi \wedge 1 \wedge 1).$$

Thus, we see that the desired relations hold if $(i_W \wedge 1 \wedge 1)^*$ is injective on $[W \wedge W \wedge W, L \wedge W]_{-1}$.

From the results on $\mathscr{A}_*(M)$, we have $\mathscr{A}_0(W) = Z/pZ\{1\}$, $\mathscr{A}_{2q-1}(W) = Z/pZ\{\alpha_2 \wedge 1\}$ and $\mathscr{A}_k(W) = 0$ for $k = 1, 2, 2q, 2q+1, 2q+2, 2q+3, 4q+1, 4q+2, 4q+3$. Therefore $[M \wedge W \wedge W, L \wedge W]_{2q} = 0$, and so $(i_W \wedge 1 \wedge 1)^*$ is injective as desired.

Proof of Theorem 2.1 (continued). To accomplish the theorem, it suffices to show $\xi_4 = \xi_5 = 0$ in case $p = 5$. By Lemma 2.7, we have

$$(i_L \wedge 1)\rho\mu_2(1 \wedge \mu_1) = \mu'(1 \wedge \mu)(\rho \wedge \rho \wedge \rho),$$
$$(i_L \wedge 1)\rho\mu_1(\mu_2 \wedge 1) = (1_L \wedge \mu)(\mu' \wedge 1)(\rho \wedge \rho \wedge \rho),$$
$$(i_L \wedge 1)\rho\mu_1(1 \wedge \mu_2) = (1_L \wedge \mu)(T_{W,L} \wedge 1)(1 \wedge \mu')(\rho \wedge \rho \wedge \rho),$$
$$(i_L \wedge 1)\rho\mu_2(\mu_1 \wedge 1) = \mu'(\mu \wedge 1)(\rho \wedge \rho \wedge \rho),$$

where $1 = 1_W$ or 1_K, $\mu = \mu_W$ and $\mu' = \mu'_W$. By Lemma 2.8 (i) and $(A_{2,1})'$, we have $(i_L \wedge 1)\rho\xi_4\mu_3(\mu_3 \wedge 1) = 0$, and by Lemma 2.8 (ii) and $(A_{1,2})'$, $(i_L \wedge 1)\rho(\xi_5 - \xi_4)\mu_3(\mu_3 \wedge 1) = 0$. Hence, $(i_L \wedge 1)\rho\xi_j = 0$ for $j = 4, 5$.

Since $[K, W]^M_{2pq+1} = 0$ for $p \geqq 7$, $= Z/pZ\{i_W\eta\pi'\}$ for $p = 5$, where $\eta = \alpha(\delta_M\beta_{(1)})^3$, and since $[K, W]^M_{(2p-2)q+2} = 0$, we see that $((i_L \wedge 1)\rho)_*$ is injective on $\mathscr{B}_{2pq+1}(K)$. Therefore $\xi_4 = \xi_5 = 0$, and the theorem holds entirely.

§3. Algebra $\mathscr{A}_*(K)$

Definition. We define a linear map

$$\psi: \mathscr{A}_k(K) \longrightarrow \mathscr{A}_{k+pq+1}(K)$$

by the formula $\psi(f) = \mu_1(1_K \wedge f)\nu_3$.

Lemma 3.1. (i) $\psi\psi = 0$.
(ii) $\psi\theta = -\theta\psi$.

Proof. (i) By $(A'_{3,3})$ of Theorem 2.1, $(\mu_1 \wedge 1)(1 \wedge \nu_3)\nu_3 = 0$. Hence, by $(A_{1,1})$ of Theorem 2.1, $\psi\psi(f) = \mu_1(1 \wedge \mu_1)(1 \wedge 1 \wedge f)(1 \wedge \nu_3)\nu_3 = \mu_1(\mu_1 \wedge 1)(1 \wedge 1$

$\wedge f)(1\wedge \nu_3)\nu_3=\mu_1(1\wedge f)(\mu_1\wedge 1)(1\wedge \nu_3)\nu_3=0$ for $f\in\mathscr{A}_k(K)$.

(ii) By (1.5), $\theta(1\wedge f)=1\wedge\theta(f)$ for θ on $[K\wedge\mathring{K}, K\wedge\mathring{K}]^*$. Then, $\theta\psi(f)=\theta(\mu_1(1\wedge f)\nu_3)=(-1)^{\deg\nu_3}\mu_1(1\wedge\theta(f))\nu_3=-\psi\theta(f)$ by (1.2) and Lemma 2.2(ii).

LEMMA 3.2. $\psi(\delta)=0$, $\psi(\delta')=-1_K$.

PROOF. The first equality is immediate from Lemma 2.4. By Theorem 1.10, $\psi(\delta')=\mu_1T(i'\wedge 1)(\pi'\wedge 1)T\nu_3=-1_K$.

PROPOSITION 3.3. *Let $f\in\mathscr{A}_k(K)$. Then*

$$\mu_i(1\wedge f)\nu_j=0 \qquad for \quad i>j;$$

$$\mu_i(1\wedge f)\nu_i=\begin{cases} f & for \quad i=1,4 \\ (-1)^k f & for \quad i=2,3;\end{cases}$$

$$\mu_1(1\wedge f)\nu_2=\theta(f),\quad \mu_3(1\wedge f)\nu_4=(-1)^k\theta(f);$$

$$\mu_2(1\wedge f)\nu_3=0,\quad \mu_2(1\wedge f)\nu_4=-(-1)^k\psi(f);$$

$$\mu_1(1\wedge f)\nu_4=\theta\psi(f)=-\psi\theta(f).$$

In other words, $1_K\wedge f$ corresponds to a triangular matrix

$$\begin{pmatrix} f & \theta(f) & \psi(f) & \theta\psi(f) \\ & (-1)^k f & 0 & -(-1)^k\psi(f) \\ & & (-1)^k f & (-1)^k\theta(f) \\ & & & f \end{pmatrix}.$$

PROOF. We put $\psi_{ij}(f)=\mu_i(1_K\wedge f)\nu_j$, in particular $\psi_{13}(f)=\psi(f)$. From the relations

$$m'(1\wedge f)(i'\wedge 1)=(-1)^k(\pi'\wedge 1)(1\wedge f)n'=1_M\wedge f,$$

$$m(1_M\wedge f)(i\wedge 1)=(-1)^k(\pi\wedge 1)(1_M\wedge f)n=f,$$

$$(\pi'\wedge 1)(1\wedge f)(i'\wedge 1)=0,\quad (\pi\wedge 1)(1_M\wedge f)(i\wedge 1)=0,$$

we see easily that $\psi_{ij}(f)=0$ for $i>j$, $\psi_{ii}(f)=(-1)^{k\deg\mu_i}f$ and $\psi_{12}(f)=(-1)^k\psi_{34}(f)=\theta(f)$.

The homomorphism ψ_{23} satisfies

$$(*)\qquad \psi_{23}(gh)=(-1)^{\deg g}g\psi_{23}(h)+(-1)^{\deg h}\psi_{23}(g)h \qquad for \quad g, h\in\mathscr{A}_*(K),$$

because $\psi_{23}(gh)=\sum_i\psi_{2i}(g)\psi_{i3}(h)=\psi_{22}(g)\psi_{23}(h)+\psi_{23}(g)\psi_{33}(h)$. By Proposition 1.8 and [15; Th. 7.5], any element f can be written as $f=\theta(g)+\theta(h)\delta$ for some

$g, h \in \mathscr{A}_*(K)$. Then $\psi_{23}\theta(g) = \psi_{23}\psi_{12}(g) = \mu_1(\mu_2 \wedge 1)(1 \wedge 1 \wedge g)((v_1 \wedge 1)v_4 + (v_3 \wedge 1)v_2) = 0$ by Theorem 2.1 ($A_{2,1}$) and ($A'_{3,2}$). By Lemma 2.4, $\psi_{23}(\delta) = 0$. Hence $\psi_{23}(f) = 0$ by (*).

Considering the M-action $\mathring{K} \wedge K$, we have $\theta\psi(f) = \mu_1(1 \wedge f)\theta_1(v_3) \pm \theta_1(\mu_1)(1 \wedge f)v_3 = \psi_{14}(f)$ by (1.2), (1.5) and Lemma 2.2 (i). Similarly we have $0 = \theta\psi_{23}(f) = \psi_{24}(f) + (-1)^k\psi_{13}(f)$, so $\psi_{24}(f) = -(-1)^k\psi(f)$. Thus, the proposition is proved.

We shall introduce a subalgebra of $\mathscr{B}_*(K)$.

Definition. $\mathscr{C}_k(K) = \mathscr{B}_k(K) \cap \operatorname{Ker}\psi$, $\mathscr{C}_*(K) = \sum_k \mathscr{C}_k(K)$.

From the above proposition, we see that $f \in \mathscr{C}_k(K)$ if and only if $1_K \wedge f$ corresponds to a diagonal matrix, and hence

Corollary 3.4. *Let $f \in \mathscr{A}_k(K)$. The following statements are equivalent to each other.*

(i) *f lies in $\mathscr{C}_k(K)$.*

(ii) $\mu_1(1_K \wedge f) = f\mu_1$.

(iii) $\mu_2(1_K \wedge f) = (-1)^k f\mu_2$ *and* $\mu_3(1_K \wedge f) = (-1)^k f\mu_3$.

(iv) $(1_K \wedge f)v_2 = (-1)^k v_2 f$ *and* $(1_K \wedge f)v_3 = (-1)^k v_3 f$.

(v) $(1_K \wedge f)v_4 = v_4 f$.

Remark 3.5. For $f \in \mathscr{C}_k(K)$, the element $f \wedge 1_K$ is not a diagonal matrix, in fact, $f \wedge 1_K$ corresponds to the triangular matrix

$$\begin{pmatrix} f & & & \\ [\delta, f] & (-1)^k f & & \\ [\delta', f] & 0 & (-1)^k f & \\ [\delta, [\delta', f]] & [f, \delta'] & -[f, \delta] & f \end{pmatrix},$$

where [,] denotes the commutator: $[f, g] = fg - (-1)^{\deg f \deg g} gf$. Also, the elements $\delta \wedge 1_K$ and $\delta' \wedge 1_K$ correspond to

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

respectively. By Theorem 3.6 (i) below, the matrix corresponding to $f \wedge 1_K$ for any $f \in \mathscr{A}_k(K)$ is computed from the above matrices.

THEOREM 3.6. (i) $\mathscr{A}_*(K)=\mathscr{C}_*(K)\otimes E(\delta, \delta')=E(\delta, \delta')\otimes\mathscr{C}_*(K)$.

(ii) $\mathscr{A}_*(K)$ *has the two differentials* θ *and* ψ *of above which are derivative, i.e., for* $d=\theta, \psi$, *there hold* $d^2=0$ *and*

$$d(fg) = (-1)^l d(f)g + fd(g), \quad f \in \mathscr{A}_k(K), \ g \in \mathscr{A}_l(K).$$

Furthermore, $\theta\psi=-\psi\theta$.

PROOF. (i) By Proposition 1.8, it suffices to show $\mathscr{B}_*(K)=\mathscr{C}_*(K)\otimes E(\delta')$ $=E(\delta')\otimes\mathscr{C}_*(K)$, which follows from Lemmas 3.1 (i), 3.2 and Definition 1.9 in the same way as [15; Th. 7.5].

(ii) By (1.2), (1.4) and Lemma 3.1, it suffices to show that ψ is derivative. We have $\psi(fg)=\sum_i\psi_{1i}(f)\psi_{i3}(g)$, where ψ_{ij} are the same as in the proof of Proposition 3.3, and hence $\psi(fg)=(-1)^l\psi(f)g+f\psi(g)$ by Proposition 3.3.

THEOREM 3.7. *The subalgebra* $\mathscr{C}_*(K)$ *is commutative and*

$$[\mathscr{C}_*(K), \delta] \subset \mathscr{C}_*(K), \quad [\mathscr{C}_*(K), \delta'] \subset \mathscr{C}_*(K),$$

where $[A, f]$ *denotes the subgroup generated by commutators* $[a, f]$ *for* $a \in A$.

PROOF. Let $f\in\mathscr{C}_k(K)$, $g\in\mathscr{C}_l(K)$ and put $h=f\delta'\in\mathscr{B}_{k-pq-1}(K)$. Then $\mu_1(h\wedge 1)v_3=\mu_1T(1\wedge h)Tv_3=-\mu_1(1\wedge h)v_3-\mu_1(1\wedge h)v_4\delta=-\psi(h)-(\theta\psi(h))\delta=f$ by Theorem 1.10, Proposition 3.3 and Theorem 3.6 (ii). Then, $\mu_1(h\wedge g)v_3$ $=\mu_1(h\wedge 1)(1\wedge g)v_3=\sum\mu_1(h\wedge 1)v_i\mu_i(1\wedge g)v_3=\mu_1(h\wedge 1)v_3\psi_{33}(g)=(-1)^l fg$, and similarly $\mu_1(h\wedge g)v_3=(-1)^{(k-1)l}\mu_1(1\wedge g)(h\wedge 1)v_3=(-1)^{(k-1)l}gf$. Therefore $gf=(-1)^{kl}fg$ as desired. Since $\theta(\delta)=\psi(\delta')=-1_K$, $\psi(\delta)=\theta(\delta')=0$ and θ and ψ are derivative, we have $\theta[f, \delta]=\psi[f, \delta]=0$, $\theta[f, \delta']=\psi[f, \delta']=0$ for any $f\in\mathscr{C}_*(K)$.

COROLLARY 3.8. *Let* f *be any element in* $\mathscr{C}_*(K)$ *of even degree. Then* f^p *commutes with any element in* $\mathscr{A}_*(K)$.

PROOF. By the second half of Theorem 3.7, $f\delta-\delta f$ and $f\delta'-\delta' f$ are in $\mathscr{C}_*(K)$, and hence $(f\delta-\delta f)f=f(f\delta-\delta f)$ and $(f\delta'-\delta' f)f=f(f\delta'-\delta' f)$ by the commutativity of $\mathscr{C}_*(K)$. By the induction, we have $f^k\delta-\delta f^k=k(f^k\delta-f^{k-1}\delta f)$ and $f^k\delta'-\delta' f^k=k(f^k\delta'-f^{k-1}\delta' f)$ for $k\geqq 1$. In particular, $f^p\delta-\delta f^p$ $=0$ and $f^p\delta'-\delta' f^p=0$. Therefore, f^p commutes with any element in $\mathscr{A}_*(K)$, by Theorems 3.6–3.7.

PROPOSITION 3.9. *The following homomorphisms are isomorphic*:

$$i'^*\colon \mathscr{C}_k(K) \longrightarrow [M, K]^M_k,$$

$$\pi'_*\colon \mathscr{C}_k(K) \longrightarrow [K, M]^M_{k-pq-1}.$$

PROOF. For $f \in [M, K]^M_*$, we have $i'^*\psi(f\pi') = -f$ by easy calculations using Theorem 1.10. Hence i'^* is a split epimorphism and $-\psi\pi'^*$ is its right inverse. Next, let $f \in \mathscr{C}_k(K) \cap \operatorname{Ker} i'^*$. Then $f = g\pi'$ for some $g \in [M, K]^M_{k+pq+1}$ by [15; Th. 4.5]. Since i'^* is onto, $g = hi'$ for some $h \in \mathscr{C}_{k+pq+1}(K)$. Then $0 = \psi(f) = \psi(h\delta') = -h$ and $f = 0$. Hence i'^* is isomorphic. The second half is similar.

§4. Realizing BP_*-modules

Let X_j be the mapping cone of $\alpha^j \in \mathscr{B}_{jq}(M)$, $j \geqq 1$ ($X_p = K$ and $X_2 = W$ in §2), and

(4.1) $$\Sigma^{jq}M \xrightarrow{\alpha^j} M \xrightarrow{i_j} X_j \xrightarrow{\pi_j} \Sigma^{jq+1} M$$

be the cofibering for X_j ($i_p = i'$, $\pi_p = \pi'$ in (1.1), $i_2 = i_W$, $\pi_2 = \pi_W$ in (2.1)). By [15; Th. 4.3], X_j is the M-module spectrum and i_j and π_j are the M-maps. By [15; Th. 4.4], there exist the M-maps $\lambda = \lambda_j\colon \Sigma^q X_{j-1} \to X_j$ and $\rho = \rho_j\colon X_j \to X_{j-1}$ such that

(4.2) $$\lambda i_{j-1} = i_j\alpha, \quad \pi_j\lambda = \pi_{j-1}; \quad \rho i_j = i_{j-1}, \quad \pi_{j-1}\rho = \alpha\pi_j.$$

($\lambda = A$ and $\rho = B$ in [14], and the element ρ in (2.3) is equal to $\rho_3 \cdots \rho_p$).

Let M' be the mod p^2 Moore spectrum $S^0 \cup_{p^2} e^1$. It is homotopy equivalent to the mapping cone of δ_M, and so there is a cofibering

(4.3) $$M \xrightarrow{\lambda_M} M' \xrightarrow{\rho_M} M.$$

Since $\alpha^p\delta_M = \delta_M\alpha^p$, there exists $\alpha'\colon \Sigma^{pq}M' \to M'$ such that $\alpha'\lambda_M = \lambda_M\alpha^p$ and $\rho_M\alpha' = \alpha^p\rho_M$ [13; §4]. The mapping cone K' of α' is homotopy equivalent to the one of $\delta = \delta_K$. We therefore have the following two cofiberings:

(4.4) (i) $\Sigma^{-1}K \xrightarrow{\delta} K \xrightarrow{\lambda_K} K'$;

(ii) $M' \xrightarrow{i''} K' \xrightarrow{\pi''} \Sigma^{pq+1}M'$.

Notice that all spectra and maps in (4.3)–(4.4) are M'-module spectra and M'-maps.

Now, we shall consider the Brown-Peterson homology of the above spectra and maps. It is clear that

$$BP_*(M) = BP_*/(p), \quad BP_*(M') = BP_*/(p^2).$$

By L. Smith [16],

$$\alpha_* = v_1 : BP_*/(p) \longrightarrow BP_*/(p), \quad \alpha'_* = v_1^p : BP_*/(p^2) \longrightarrow BP_*/(p^2)$$

for a suitable choice of α'. From (4.1)–(4.4), we have immediately

LEMMA 4.1. (i) $BP_*(X_j)=BP_*/(p, v_1^j)$, $BP_*(K')=BP_*/(p^2, v_1^p)$.
(ii) $(i_j)_*$ *and* i''_* *are surjective*, $(\pi_j)_*=0$ *and* $\pi''_*=0$.
(iii) $\lambda_*=v_1$ *and* ρ_* *is surjective.*
(iv) $(\lambda_K)_*=p$.

The following lemma is an improvement of [14; Th. DII].

LEMMA 4.2. *For* $s\geqq 2$, *there exist elements* $f_s\in\mathscr{C}_{sp(p+1)q}(K)$ *such that* $(f_s)_*=v_2^{sp}$.

PROOF. By [14; Th. C, D], there is the M-map

$$R_{p-1}: \Sigma^{p(p+1)p}X_{p-1} \longrightarrow X_{p-1}$$

such that $(R_{p-1})_*=v_2^p$. By the relation $(*)$ in the proof of [14; Th. CII] and by [15; Th. 4.5], there are M-maps

$$g_s: \Sigma^{sp(p+1)q}K \longrightarrow K, \qquad s\geqq 2,$$

such that $g_s\lambda=\lambda(R_{p-1})^s$. Write $g_s=h_s+h'_s\delta'$ for h_s, $h'_s\in\mathscr{C}_*(K)$. Then $(h_s\lambda)_* =(g_s\lambda)_*=v_1v_2^{sp}$ by Lemma 4.1, and hence $(h_s)_*\equiv v_2^{sp} \bmod (v_1^{p-1})\cdot BP_*/(p, v_1^p) =B$. In degree $2p(p+1)q$, $B=0$ and $(h_2)_*=v_2^{2p}$. In degree $3p(p+1)q$, B is generated by $v_1^{p-1}v_2^{p-1}v_3$. Put $(h_3)_*=v_2^{3p}+av_1^{p-1}v_2^{p-1}v_3$. Then the ideal $(p, v_1^p, v_2^{3p}+av_1^{p-1}v_2^{p-1}v_3)$ is invariant under the coaction of BP_*BP, and hence we see that a must be trivial ([10], cf. [3; §7]). Hence we can take $f_{2s}=(h_2)^s$ and $f_{2s+1}=(h_2)^{s-1}h_3$.

THEOREM 4.3. *For* $p\geqq 5$, $s\geqq 2$, *there exist* M'*-maps*

$$F_s: \Sigma^{sp^2(p+1)q}K' \longrightarrow K'$$

which induce the multiplications by $v_2^{sp^2}$, *and hence the mapping cone* L_s *of* F_s *satisfies* $BP_*(L_s)=BP_*/(p^2, v_1^p, v_2^{sp^2})$.

PROOF. By Lemma 4.2 and Corollary 3.8, $(f_s)^p\delta=\delta(f_s)^p$. Hence, by (4.4) (i) and [15; Th. 4.5], there are M'-maps g_s such that $g_s\lambda_K=\lambda_K(f_s)^p$. By Lemmas 4.1 (iv) and 4.2, $(g_s)_*\equiv v_2^{sp^2} \bmod (p)\cdot BP_*/(p^2, v_1^p)$. For $s=2, 3$, if $(p^2, v_1^p, v_2^{sp^2}+px)$, $\deg x=sp^2(p+1)q$, is invariant, then x is a multiple of $v_2^{sp^2}$. Hence $(g_s)_*=(1+a_sp)v_2^{sp^2}$, $a_s\in Z/pZ$, for $s=2, 3$, and we can take $F_s=(1-a_sp)g_s$ for $s=2, 3$, $F_{2s}=(F_2)^s$ and $F_{2s+1}=(F_2)^{s-1}F_3$.

REMARK. R. S. Zahler [20] showed that the ideal (p^2, v_1^p, v_2^t) is invariant if and only if $p^2|t$. Hence, $BP_*/(p^2, v_1^p, v_2^t)$, $p\geqq 5$, $t\neq p^2$, *is realizable if and only if* $p^2|t$. *We do not know the realizability of* $BP_*/(p^2, v_1^p, v_2^{p^2})$.

THEOREM 4.4. *For* $p\geqq 5$, $s\geqq 2$, $p+1\leqq j\leqq 2p$, *there exist maps*

$$G_{s,j}\colon \Sigma^{sp^2(p+1)q}X_j \longrightarrow X_j$$

such that $(G_{s,j})_*=v_2^{sp^2}$, *and hence the mapping cone* $Y_{s,j}$ *of* $G_{s,j}$ *satisfies* $BP_*(Y_{s,j}) =BP_*/(p, v_1^j, v_2^{sp^2})$.

PROOF. By Lemma 4.2 and Corollary 3.8, $(f_s)^p\delta'=\delta'(f_s)^p$. In the same way as [14; Th. C′], we can construct maps $g_{s,j}\in\mathscr{A}_{sp^2(p+1)q}(X_j)$ such that $g_{s,j}\lambda =\lambda g_{s,j-1}$ and $g_{s,p+1}\lambda=\lambda(f_s)^p$. Similar discussions on the invariance of $(p, v_1^j, v_2^{sp^2}+v_1^{j-1}x)$ as in the proofs of Lemma 4.2 and Theorem 4.3 imply $(g_{s,j})_*=v_2^{sp^2}$ for $s=2$, 3 by replacing $g_{s,p+2}$ suitably. Then, we can take $G_{2s,j}=(g_{2,j})^s$ and $G_{2s+1,j}=(g_{2,j})^{s-1}g_{3,j}$.

§5. Constructing homotopy elements

DEFINITION 5.1. For $p\geqq 5$, $s\geqq 2$, we define elements $\beta_{sp^2/(p,2)}$ in $\pi_*(S)$ by

$$\beta_{sp^2/(p,2)} = \bar{\pi}\pi''F_s i''\bar{i},$$

where $\bar{i}\colon S\to M'$ is the inclusion and $\bar{\pi}\colon M'\to\Sigma S$ is the projection. Each $\beta_{sp^2/(p,2)}$ is of degree $(sp^3+sp^2-p)q-2$ and satisfies $p^2\beta_{sp^2/(p,2)}=0$.

DEFINITION 5.2. For $p\geqq 5$, $s\geqq 2$, $p+1\leqq j\leqq 2p$, define $\beta_{sp^2/(j)}\in\pi_*(S)$ by

$$\beta_{sp^2/(j)} = \pi\pi_j G_{s,j} i_j i.$$

The degree of $\beta_{sp^2/(j)}$ is $(sp^3+sp^2-j)q-2$ and there holds $p\beta_{sp^2/(j)}=0$.

We shall consider the Adams-Novikov spectral sequence for BP:

$$E_2(X) = H^*(BP_*(X)) \Longrightarrow \pi_*(X)_{(p)},$$

where $H^*M=\mathrm{Ext}^*_{BP_*BP}(BP_*, M)$ for a BP_*BP-comodule M. The following is useful to prove the nontriviality of the elements of Definitions 5.1–5.2.

THEOREM ([7; Th. 1.7], [9; Lemma 2.10]). *Let* $W\to X\to Y\xrightarrow{h}\Sigma W$ *be a cofiber sequence of finite CW-spectra such that* $h_*=0$ *in BP-homology. Denote by* $\delta\colon H^tBP_*(Y)\to H^{t+1}BP_*(W)$ *the connecting homomorphism associated to the short exact sequence* $0\to BP_*(W)\to BP_*(X)\to BP_*(Y)\to 0$. *If* $x\in E_2(Y)$ *converges to an element* $\alpha\in\pi_*(Y)_{(p)}$, *then* $\delta(x)\in E_2(W)$ *converges to* $h_*(\alpha)\in \pi_*(W)_{(p)}$.

Let $\delta_1: H^tBP_*/(p^2, v_1^p) \to H^{t+1}BP_*/(p^2)$ and $\delta_2: H^tBP_*/(p^2) \to H^{t+1}BP_*$ be the connecting homomorphisms associated to the exact sequences

$$0 \longrightarrow BP_*/(p^2) \xrightarrow{v_1^p} BP_*/(p^2) \longrightarrow BP_*/(p^2, v_1^p) \longrightarrow 0,$$

$$0 \longrightarrow BP_* \xrightarrow{p^2} BP_* \longrightarrow BP_*/(p^2) \longrightarrow 0.$$

Since $(F_s i''\bar{i})_* = v_2^{sp^2} \in H^0BP_*/(p^2, v_1^p)$, the element $v_2^{sp^2} \in E_2^{0,*}(K')$ converges to $F_s i''\bar{i} \in \pi_*(K')_{(p)}$. Since $(\pi'')_* = 0$ and $\bar{\pi}_* = 0$, the above theorem shows that $\delta_1(v_2^{sp^2}) \in H^1BP_*/(p^2) = E_2^{1,*}(M')$ converges to $\pi'' F_s i''\bar{i} \in \pi_*(M')_{(p)}$ and $\delta_2\delta_1(v_2^{sp^2}) \in H^2BP_* = E_2^{2,*}(S)$ converges to $\beta_{sp^2/(p,2)} \in \pi_*(S)_{(p)}$.

Recently, H. R. Miller, D. C. Ravenel and W. S. Wilson ([8], [9]) have completely determined H^2BP_*. In particular, $\delta_2\delta_1(v_2^{sp^2}) = \beta_{sp^2/(p,2)}$ is nontrivial and generates a summand Z/p^2Z. Since any element in $E_2^{2,*}(S)$ can not be hit by a differential, $\delta_2\delta_1(v_2^{sp^2})$ survives nontrivially to E_∞, and hence $\beta_{sp^2/(p,2)} \neq 0$ in $\pi_*(S)$. Since $\beta_{sp^2/(p,2)}$ in H^2BP_* is indecomposable, $\beta_{sp^2/(p,2)}$ in $\pi_*(S)$ generates a summand Z/p^2Z and is indecomposable. Thus, we have obtained

THEOREM 5.3. *The elements $\beta_{sp^2/(p,2)}$, $s \geqq 2$, of Definition 5.1 are indecomposable and generate cyclic summands of order p^2 in $\pi_{(sp^3+sp^2-p)q-2}(S)$.*

In the same way as above, we also obtain

THEOREM 5.4. *The elements $\beta_{sp^2/(j)}$, $s \geqq 2$, $p+1 \leqq j \leqq 2p$, of Definition 5.2 are indecomposable and generate cyclic summands of order p in $\pi_{(sp^3+sp^2-j)q-2}(S)$.*

At the end of this section, we notice that the results of H. R. Miller, D. C. Ravenel and W. S. Wilson on H^3BP_* imply the existence of infinitely many elements in $\pi_*(S)$ of order p^2 and of degree $\equiv -3 \bmod q$. In [14; Cor. 7.6], we have proved the relation $\alpha_1\beta_{tp} = p\phi_t$ in $\pi_{(tp^2+tp)q-3}(S)$ for $p \geqq 5$ and $t \geqq 1$. ϕ_t is of order p^2 if $\alpha_1\beta_{tp} \neq 0$. Hence, by [9; Th. 2.13]

THEOREM 5.5. *Let $p \geqq 5$, $n \geqq 0$, $p \nmid s \geqq 1$, and assume that $s \not\equiv -1 \bmod p$, $s \equiv -1 \bmod p^{n+3}$ or $s = p-1$. Then, the element ϕ_{sp^n} in $\pi_{(sp^{n+2}+sp^{n+1})q-3}(S)$ is nontrivial of order p^2.*

Note added in proof. In the previous paper [15], Theorem 4.5 is incorrect, and we have used this theorem in the proofs of Prop. 1.3, Th. 2.1, Prop. 3.9, Lemma 4.2 and Th. 4.3. But these results can be proved without this erroneous theorem. The details will be seen in the correction of [15]. We would like to appreciate Professor Z. Yosimura who kindly pointed out the error of the proof of [15; Th. 4.5, Lemma 4.6].

References

[1] J. F. Adams, *On the groups J(X)-IV*, Topology **5** (1966), 21–71.

[2] ———, *Quillen's work on formal group laws and complex cobordism*, University of Chicago lecture notes series, 1970.
[3] ———, *Localisation and completion*, Univ. of Chicago, Lecture note in Math. 1975.
[4] S. Araki, *Typical formal groups in complex cobordism and K-theory*, Lecture Notes in Math. Kyoto Univ. Kinokuniya, 1973.
[5] ——— and H. Toda, *Multiplicative structures in* mod q *cohomology theories II*, Osaka J. Math. **3** (1966), 81–120.
[6] M. Hazewinkel, *Constructing formal groups I. Over* $Z_{(p)}$ *algebras*, Report of the Econometric Institute #7119, Netherlands School of Economics, 1971.
[7] D. C. Johnson, H. R. Miller, W. S. Wilson and R. S. Zahler, *Boundary homomorphisms in the generalized Adams spectral sequence and the nontriviality of infinitely many* γ_t *in stable homotopy*, Conference on homotopy theory (Northwestern Univ., Evanston, Ill., 1974), edited by D. M. Davis, pp. 47–59, Mathematical Notes and Symposia, Vol. 1, Soc. Mat. Mex., Mexico City, 1975.
[8] H. R. Miller, D. C. Ravenel and W. S. Wilson, *Novikov's* Ext^2 *and the nontriviality of the gamma family*, Bull. Amer. Math. Soc. **81** (1975), 1073–1075.
[9] ———, *Periodic phenomena in the Adams-Novikov spectral sequence*, preprint.
[10] H. R. Miller and W. S. Wilson, *On Novikov's* Ext^1 *modulo an invariant prime ideal*, Topology **15** (1976), 131–141.
[11] O. Nakamura and S. Oka, *Some differentials in the* mod p *Adams spectral sequence* $(p\geqq 5)$, Hiroshima Math. J. **6** (1976), 305–330. ———, *Corrections to "Some differentials in the* mod p *Adams spectral sequence* $(p\geqq 5)$", this journal 655–656.
[12] S. Oka, *The stable homotopy groups of spheres II, III*, Hiroshima Math. J. **2** (1972), 99–161; **5** (1975), 407–438.
[13] ———, *On the stable homotopy ring of Moore spaces*, Hiroshima Math. J. **4** (1974), 629–678.
[14] ———, *A new family in the stable homotopy groups of spheres I, II*, Hiroshima Math. J. **5** (1975), 87–114; **6** (1976), 331–342.
[15] ———, *Module spectra over the Moore spectrum*, Hiroshima Math. J. **7** (1977), 93–118.
[16] L. Smith, *On realizing complex bordism modules*, Amer. J. Math. **92** (1970), 793–856.
[17] H. Toda, *p-Primary components of homotopy groups IV*, Mem. Coll. Sci. Univ. Kyoto, **32** (1959), 297–332.
[18] ———, *Algebra of stable homotopy of* Z_p*-spaces and applications*, J. Math. Kyoto Univ. **11** (1971), 197–251.
[19] N. Yamamoto, *Algebra of stable homotopy of Moore space*, J. Math. Osaka City Univ. **14** (1963), 45–67.
[20] R. S. Zahler, *Nonrealizability of some cyclic complex bordism modules*, Proc. Amer. Math. Soc. **47** (1975), 218–222.

Department of Mathematics,
Faculty of Science,
Hiroshima University)*

*) The present address of the author is as follows: Department of Mathematics, Faculty of Science, Kyushu University 33, Fukuoka, Japan.