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Self H-equivalences of H-spaces with applications to H-spaces of rank 2

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Introduction

The homotopy classification of spaces and maps is a subject of classical studies in algebraic topology. The group $\mathscr{E}(X)$ of self equivalences of a space X and the subgroup $\mathscr{E}_H(X)$ of self H-equivalences of an H-space X arose from such classification problem. For a based space X, $\mathscr{E}(X)$ is defined to be the set of all homotopy classes of homotopy equivalences of X to itself with group multiplication induced by the composition of maps; and it has been investigated by several authors including [2], [10], [19], [20] and [22], where calculating $\mathscr{E}(X)$ has been made with two exact sequences, originally due to Barcus-Barratt [2], given by either the skeletons or the Postnikov system of X. When X is an H-space, $\mathscr{E}_H(X)$ is defined to be the subgroup of $\mathscr{E}(X)$ consisting of H-maps, which has been studied in [13] and [24] for instance. But much less examples of calculation are known; in fact, when X is a finite 1-connected H-complex (H-space being a CW-complex), $\mathscr{E}_H(X)$ has determined only in case that X is of rank$\leqq 2$ with no torsion in homology.

This paper is divided into two parts. In Part I, we present an exact sequence for calculating $\mathscr{E}_H(X)$ of a 1-connected H-complex X in terms of its Postnikov system. The aim of Part II is the determination of $\mathscr{E}_H(G_{2,b})$ made use of the exact sequence given in Part I, where $G_{2,b}$ $(-2\leqq b\leqq 5)$ are of rank 2 with torsion in homology given by Mimura-Nishida-Toda [17].

Let X be a 1-connected H-complex, and consider the Postnikov system $\{X_n\}$ of X with obvious map f_n: $X\to X_n$ and usual fiber sequence

(1) $$\Omega X_{n-1} \xrightarrow{\Omega k} K(\pi_n, n) \xrightarrow{i_n} X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{k} K(\pi_n, n+1)$$

(Ω is the loop functor)

where $\pi_n(X)$ is sometimes abbreviated to π_n and the Postnikov invariant k^{n+1} to k. Then, the theorem of J. D. Stasheff [26, Th. 5] states that X_n is an H-space in such a way that all the structure maps f_n, k, p_n and i_n are H-maps; and we have proved in the previous paper [25, Th. 1.3] that

(2) *f_n induces a homomorphism $f_{n!}$: $\mathscr{E}_H(X)\to\mathscr{E}_H(X_n)$ which is monomorphic if $n\geqq\dim X$ and isomorphic if $n\geqq 2\dim X$.*

This motivates our study of relation between $\mathscr{E}_H(X_n)$ and $\mathscr{E}_H(X_{n-1})$ in order to give an exact sequence for the calculation of $\mathscr{E}_H(X)$.

For this purpose, we consider more generally the mapping track E_f and the usual fiber sequence

(3) $\Omega A \xrightarrow{\Omega f} \Omega B \xrightarrow{i} E_f \xrightarrow{p} A \xrightarrow{f} B$ of a given H-map f between H-complexes A and B,

where E_f is an H-space so that p and i are also H-maps (cf. [26, Th. 2]). Denote the homotopy set by [,] and consider the exact sequence and the induced map

$$(3') \qquad \begin{aligned} &[E_f, \Omega A] \xrightarrow{(\Omega f)_*} [E_f, \Omega B] \xrightarrow{i_*} [E_f, E_f] \xrightarrow{p_*} [E_f, A], \\ &[A, \Omega B] \xrightarrow{p^*} [E_f, \Omega B]. \end{aligned}$$

Then, by the theorems due to Y. Nomura [19] and J. W. Rutter [22], in case when

(3″) $\pi_i(A)=0$ unless $m \leqq i < n$, $\pi_j(B)=0$ unless $n < j \leqq m+n$, for some integers $n > m \geqq 2$,

the restriction of the exact sequence in (3′) to $\mathscr{E}(E_f)$ ($\subset [E_f, E_f]$) gives us the exact sequence

$$(4) \qquad [A, \Omega B] \xrightarrow{\kappa p^*} \mathscr{E}(E_f) \xrightarrow{(\varphi,\psi)} \mathscr{E}(A) \times \mathscr{E}(\Omega B) \ (\mathscr{E}(\Omega B) \cong \mathscr{E}(B),\ \kappa = 1 + i_*)$$

in Theorem 2.5 of groups and homomorphisms, where [, ΩB] is abelian as usual and φ and ψ are the homomorphisms induced by p and i, respectively. Restricting (4) to $\mathscr{E}_H(E_f)$ gives rise to an exact sequence for the computation of $\mathscr{E}_H(E_f)$ from $\mathscr{E}_H(A)$ and $\mathscr{E}_H(B)$, which is our main result in Part I and is stated as follows.

THEOREM I–1. *Let A and B be H-complexes satisfying* (3″). *Let $f: A \to B$ be an H-map and consider its mapping track E_f which is an H-space so that p and i in* (3) *are H-maps. Then there is an exact sequence*

$$(5) \qquad 0 \to \tilde{H}(f) \to \mathscr{E}_H(E_f) \to \tilde{G}(f) \to 1,$$

where the abelian group $\tilde{H}(f)$ and the group $\tilde{G}(f)$ are given as follows:

(5′) $\tilde{H}(f) = p^*(P(f))/\mathrm{Im}\,(\Omega f)_* \cap p^*(P(f))$, $\quad P(f) = (\kappa p^*)^{-1}(\mathscr{E}_H(E_f)) \subset [A, \Omega B]$,

where $(\Omega f)_*$, p^* are in (3′) *and κp^* is in* (4); *and $P(f)$ can be taken to be the subgroup $[A, \Omega B]_H$ consisting of all H-maps if the condition* (2.8.4) *stated below is satisfied.*

(5″) $\tilde{G}(f) = \{(h_1, h_2) \in \mathscr{E}_H(A) \times \mathscr{E}_H(B) \mid fh_1 = h_2 f$ *in* $[A, B]$ *with a secondary homotopy stated in* (2.7.2)$\}$.

The sequence (1) for a 1-connected H-complex X is considered as (3) for $A=X_{n-1}$, $B=K(\pi_n, n+1)$ and $f=k$ with (3″) for $m=2$, and the above results can be applied to obtain the following

THEOREM I–2. *Let X be a 1-connected H-complex and $\{X_n\}$ in* (1) *be its Postnikov system. Then there are exact sequences*

$$(6) \qquad 0 \to H_n \to \mathscr{E}(X_n) \to G_n \to 1, \quad 0 \to \tilde{H}_n \to \mathscr{E}_H(X_n) \to \tilde{G}_n \to 1,$$

where H_n, G_n, $\tilde{H}_n$ and $\tilde{G}_n$ are given as follows:

$$(6') \quad H_n = \mathrm{Im}\, p_n^*/\mathrm{Im}\,(\Omega k)_* \supset \tilde{H}_n = \tilde{H}(k) = p_n^*(P_n)/\mathrm{Im}\,(\Omega k)_* \cap p_n^*(P_n), \quad P_n = P(k),$$

where $H^n(X_{n-1}; \pi_n) \xrightarrow{p_n^*} H^n(X_n; \pi_n) \xleftarrow{(\Omega k)_*} [X_n, \Omega X_{n-1}]$ $(k=k^{n+1})$; *and P_n can be taken to be the subgroup $PH^n(X_{n-1}; \pi_n)$ consisting of all primitive elements if the condition* (3.7.5) *stated below is satisfied.*

$$(6'') \quad G_n = \{(h_1, h_2) \in \mathscr{E}(X_{n-1}) \times \mathrm{aut}\, \pi_n \mid h_1^* k = h_{2*} k \quad \text{in} \quad H^{n+1}(X_{n-1}; \pi_n)\}$$
$$\supset G_n \cap (\mathscr{E}_H(X_{n-1}) \times \mathrm{aut}\, \pi_n) \supset \tilde{G}_n = \tilde{G}(k);$$

and $G_n \cong \rho(G_n) \subset \mathscr{E}(X_{n-1})$ and $\tilde{G}_n \cong \rho(\tilde{G}_n) \subset \mathscr{E}_H(X_{n-1})$ by the projection onto the first factor if p_n^* is epimorphic.

In Part II, we consider a 1-connected H-complex of rank 2 with 2-torsion in homology, i.e.,

(7) $G_{2,b}$ $(-2 \leqq b \leqq 5)$ given in [17, Th. 5.1] (see §4 for the definition).

The group $\mathscr{E}(G_{2,b})$ is investigated in the previous paper [18] collaborated with M. Mimura by studying the exact sequences on the skeletons of $G_{2,b}$ due to Barcus-Barratt [2]. By using some results obtained there, we can show that the groups $\tilde{H}_n$ in (6′) and $\tilde{G}_n$ in (6″) with $X=G_{2,b}$ satisfy

$$(8) \quad \tilde{H}_n = 0 \quad \textit{and} \quad \tilde{G}_n \cong \rho(\tilde{G}_n) \subset \mathscr{E}_H(X_{n-1}) \qquad \textit{for} \quad 4 \leqq n \leqq 14 = \dim G_{2,b}.$$

Notice that $X_3 = K(Z, 3)$ and $\mathscr{E}_H(X_3) = Z_2$ in case $X = G_{2,b}$. Then, by the exactness of (6) and (2), we have the following

PROPOSITION 5.6. *Let f_3: $G_{2,b} \to K(Z, 3)$ be the map killing the homotopy groups except π_3, and $f_{3!}$: $\mathscr{E}_H(G_{2,b}) \to \mathscr{E}_H(K(Z, 3)) = Z_2$ be the induced homomorphism in* (2). *Then, $f_{3!}$ is monomorphic, and hence $\mathscr{E}_H(G_{2,b})$ is trivial or equal to Z_2.*

Furthermore, we notice that

(9) *$G_{2,b}$ is an H-space so that the inclusion $S^3 \subset G_{2,b}$ is an H-map with*

respect to the usual multiplication on S^3; and we can prove the following main result in Part II:

THEOREM II. *Let* $G_{2,b}$ *be the H-space in* (9). *Then the group* $\mathscr{E}_H(G_{2,b})$ *is trivial, i.e., any homotopy equivalent H-map of* $G_{2,b}$ *to itself is homotopic to the identity map.*

In case when a 1-connected H-complex X of rank 2 is 2-torsion free in homology, Hilton-Roitberg [8] and A. Zabrodsky [31] proved that

(10) *X is $S^3 \times S^3$, $SU(3)$, E_k $(k=0, 1, 3, 4, 5)$ or $S^7 \times S^7$, up to homotopy type,*

where E_k is the principal S^3-bundle over S^7 with classifying map $k\omega \in \pi_7(BS^3) = \pi_6(S^3) = Z_{12}$ (ω: a generator). We notice that the group $\mathscr{E}_H(X)$ of such an H-complex X with canonical multiplication is determined as follows:

(11) ([24], [25] and K. Maruyama [11]) $\mathscr{E}_H(SU(3)) = Z_2$, $\mathscr{E}_H(E_k) = 1$,

$$\mathscr{E}_H(S^\ell \times S^\ell) = \{a = (a_{ij}) \in GL(2, Z) \mid a_{ij} \equiv (1 + (-1)^{i+j} \det a)/2 \mod k_\ell\} \ (\ell = 3, 7),$$

where $k_3 = 24$ and $k_7 = 240$. Furthermore, we remark that $\mathscr{E}_H(E_k) = 1$ is valid for any multiplication on E_k by [24] and Maruyama-Oka [13], but K. Maruyama [12] has proved recently that there is a multiplication on $SU(3)$ with $\mathscr{E}_H(SU(3)) = 1$.

Part I consists of §§1–3. In §1, we attempt functorial treatments of $\mathscr{E}(X)$ and of $\mathscr{E}_H(X)$. In §2, we recall the exact sequence (4) together with the results on $\mathrm{Ker}(\kappa p^*)$ and $\mathrm{Im}(\varphi, \psi)$ in Theorem 2.5. We prove Theorem I–1 in Theorem 2.8, and notice any multiplication on E_f in Remark 2.9. In §3, we give some corollaries to Theorems 2.5 and 2.8, and prove Theorem I–2 in Corollary 3.7. Part II consists of §§4–7. In §4, we recall the definition and the properties of $G_{2,b}$ given in [17], and prepare some results on p_n^* and $PH^n(X_{n-1}; \pi_n)$ in (6′) with $X = G_{2,b}$. In §5, we prove (8) in Lemmas 5.4–5 under Assertion 5.3, and Theorem II in Theorem 5.8 by using Proposition 5.6 and the fact that $\pi_6(S^3) = Z_{12}$ is generated by the obstruction to homotopy commutativity of the usual multiplication on S^3. Finally in §§6–7, we prove Assertion 5.3 by using the exact sequence of homotopy sets induced by the fibering in (1) with $X = G_{2,b}$ and by studying several related homotopy sets in detail.

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Part I. Self H-equivalences of the mapping track of an H-map

§1. Preliminaries on self (H-)equivalences

In this paper, all (topological) spaces are 1-connected spaces with base points $*$ and have the homotopy types of CW-complexes, and all (continuous) maps and homotopies preserve $*$. For a space or CW-complex X, the lower or upper indexing X_n or X^n is used to denote the n-stage of the Postnikov system $\{X_n\}$ of X or the n-skeleton of X, respectively, unless otherwise stated. For any spaces X and Y, we denote the set of homotopy classes of maps of X to Y by $[X, Y]$ as usual, and often use the same symbol to refer to a map and its homotopy class.

A given map $g: X \to X'$ (resp. $h: Y \to Y'$) induces the map

$$g^*: [X', Y] \to [X, Y] \text{ with } g^*f = fg \text{ (resp. } h_*: [X, Y] \to [X, Y'] \text{ with } h_*f = hf)$$

between the homotopy sets by composing g (resp. h). A cofibering (resp. fibering) induces the Puppe (resp. homotopy) exact sequence and we have the following by the standard homotopy theory (cf., e.g., [4]), where we say that a map $g: X \to X'$ is n-connected if

$$g_*: \pi_i(X) \to \pi_i(X') \text{ is isomorphic for } i < n \text{ and epimorphic for } i = n.$$

(1.1.1) *If $g: X \to X'$ is n-connected, then $g^*: [X', Y] \to [X, Y]$ is bijective when $\pi_i(Y)=0$ for $i \geqq n$, and is injective when $\pi_i(Y)=0$ for $i > n$.*

(1.1.2) *If X is $(n-1)$-connected and $\pi_i(Y)=0$ for $i \geqq n$, then $[X, Y]=0$.*

(1.1.3) *If $h: Y \to Y'$ is n-connected and X is a finite dimensional CW-complex, then $h_*: [X, Y] \to [X, Y']$ is bijective when* $\dim X < n$, *and surjective when* $\dim X \leqq n$.

Furthermore, we notice the following facts on the connectivity:

(1.1.4) *If X and Y are m- and n-connected, respectively, then $X \times Y$ is* $\min\{m, n\}$*-connected and the smash product $X \wedge Y = X \times Y / X \vee Y$ is $(m+n+1)$-connected.*

(1.1.5) *$g: X \to X'$ is n-connected, if and only if the homotopy fiber (mapping track) of g is $(n-1)$-connected, or equivalently, the homotopy cofiber (mapping cone) of g is n-connected.*

(1.1.6) *For a CW-complex X and its n-skeleton X^n, the inclusion $j_n: X^n \subset X$ is n-connected.*

(1.1.7) *If $g: X \to X'$ and $h: Y \to Y'$ are k- and ℓ-connected, respectively, then $g \times h: X \times Y \to X' \times Y'$ is* $\min\{k, \ell\}$*-connected. If X, X', Y and Y' are*

m-, m'-, n- and n'-connected, respectively, in addition, then $g \wedge h\colon X \wedge Y \to X' \wedge Y'$ *is* $\max\{\min\{m+\ell+1, n'+k+1\}, \min\{m'+k+1, n+\ell+1\}\}$*-connected.*

For any space X, we denote the subset of $[X, X]$ consisting of all classes of self equivalences of X (homotopy equivalences of X to itself) by

$$\mathscr{E}(X)\ (\subset [X, X]),$$

which forms a group under the composition of maps. To study this group, we use the induced homomorphisms given in the following

LEMMA 1.2. *Let $f\colon X \to Y$ be a map, and consider the induced maps*

$$[X, X] \xrightarrow{f_*} [X, Y] \xleftarrow{f^*} [Y, Y].$$

(i) *If f^* is bijective, then $f^{*-1}f_*$ defines the homomorphism*

(1.2.1) $f_!\colon \mathscr{E}(X) \to \mathscr{E}(Y)$ *determined by* $(f_!(h))f = fh$ *in* $[X, Y]$ *for* $h \in \mathscr{E}(X)$.

(ii) *If f_* is bijective, then $f_*^{-1}f^*$ defines the homomorphism*

(1.2.2) $f^!\colon \mathscr{E}(Y) \to \mathscr{E}(X)$ *determined by* $f(f^!(g)) = gf$ *in* $[X, Y]$ *for* $g \in \mathscr{E}(Y)$.

PROOF. If f^* is bijective, then for $h \in [X, X]$, $h' = f^{*-1}(f_*h) \in [Y, Y]$ is determined uniquely by the condition $h'f = fh$ in $[X, Y]$. Thus $f^{*-1}f_*$ preserves the identity map and the composition of maps, and we see (i). Similarly, we can prove (ii). q.e.d.

For a given space X, we consider the n-stage X_n in the Postnikov system $\{X_n\}$ of X, i.e.,

(1.3.1) X_n is a space with $\pi_i(X_n) = 0$ for $i > n$, and there is an $(n+1)$-connected map $f_n\colon X \to X_n$,

or,

(1.3.2) up to homotopy type, X_n is a space obtained by attaching i-cells with $i \geqq n+2$ to X so that X_n and the inclusion map $f_n\colon X \subset X_n$ satisfy (1.3.1).

Then, $f_n^*\colon [X_n, X_n] \to [X, X_n]$ is bijective by (1.1.1) and f_n induces the homomorphism

(1.3.3) $\qquad f_{n!}\colon \mathscr{E}(X) \to \mathscr{E}(X_n)$ of (1.2.1) for $f = f_n$.

When X is a CW-complex having no $(n+1)$-cells, we have the following duality between $\mathscr{E}(X_n)$ and $\mathscr{E}(X^n)$ of the n-skeleton X^n of X:

PROPOSITION 1.4. *Let X be a CW-complex, and X^n be its n-skeleton.*

(i) *If X has no $(n+1)$-cells, then the inclusion $j_n\colon X^n \subset X$ and the composition $f_n j_n\colon X^n \to X_n$ induce the homomorphisms of* (1.2.2) *in the commutative diagram*

$$\begin{tikzcd} \mathscr{E}(X) \arrow[r, "f_{n!}"] \arrow[d, equal] & \mathscr{E}(X_n) \arrow[d, "\cong"', "(f_n j_n)^!"] \\ \mathscr{E}(X) \arrow[r, "j_n^!"] & \mathscr{E}(X^n) \end{tikzcd}$$

where $f_{n!}$ is the one in (1.3.3), *and $(f_n j_n)^!$ is an isomorphism.*

(ii) (cf. [23, Lemma 7.1]) *If X is a finite dimensional CW-complex, then $f_{n!}$ is an isomorphism for $n \geqq \dim X$.*

Proof. (i) If $X^{n+1}=X^n$, then the induced maps in the commutative diagram

$$\begin{tikzcd} {[X^n, X^n]} \arrow[r, "(f_n j_n)_*"] \arrow[d, "j_{n*}"] & {[X^n, X_n]} \arrow[d, equal] & {[X_n, X_n]} \arrow[l, "(f_n j_n)^*"'] \arrow[d, "f_n^*"] \\ {[X^n, X]} \arrow[r, "f_{n*}"] & {[X^n, X_n]} & {[X, X_n]} \arrow[l, "j_n^*"'] \end{tikzcd}$$

are all bijective. In fact, j_n is $(n+1)$-connected by (1.1.6) since $X^{n+1}=X^n$, and so is f_n by (1.3.1). Thus j_{n*} and f_{n*} are bijective by (1.1.3), and so are j_n^* and f_n^* by (1.1.1) and (1.3.1).

Therefore, the induced homomorphisms $j_n^!$ and $(f_n j_n)^!$ are defined by the above lemma, and so is also $(f_n j_n)_!\colon \mathscr{E}(X^n) \to \mathscr{E}(X_n)$ which is the inverse of $(f_n j_n)^!$. The commutativity of the diagram in (i) is seen by the definitions (1.2.1–2).

(ii) is an immediate corollary of (i). q.e.d.

We now consider H-spaces. We use the notation $\sim$ for '*homotopic*' as usual, and the ones

$$\Delta\colon X \to X \times X, \quad \nabla\colon X \vee X \to X \quad \text{and} \quad \pi\colon X \times Y \to X \times Y/X \vee Y = X \wedge Y$$

always to denote the *diagonal*, *folding* and *collapsing maps*, respectively.

A space X is an *H-space* if there is a map $m\colon X \times X \to X$, called a *multiplication*, such that $m|X \vee X \sim \nabla\colon X \vee X \to X$. When a CW-complex X is an H-space, we call it an *H-complex* whose multiplication m can be taken (up to homotopy) to be $m|X \vee X = \nabla$. For example, we have the following:

(1.5.1) *If $\pi_i(A)=0$ unless $n<i\leqq 2n$ for some $n\geqq 1$, then A is an H-space with unique multiplication* (*up to homotopy*).

In fact, $A \simeq A'$ ($\simeq$ means '*homotopy equivalent*') for some CW-complex A' and there is uniquely an extension $m': A'\times A'\to A'$ of ∇ by the obstruction theory.

We notice the following (1.5.2–6) where $X=(X, m)$ is a given H-space:

(1.5.2) ([9, Th. 1.1]) $[A, X]$ *for any* A *forms a loop with sum* $+_m$ *and identity* $0=*$, *where*

(1.5.3) $g+_m h=m(g\times h)\triangle: A\xrightarrow{\triangle} A\times A\xrightarrow{g\times h} X\times X\xrightarrow{m} X$ *for* $g, h: A\to X$;

i.e., for any g, g', *there are uniquely* h, h' *so that* $g+_m h=g'=h'+_m g$ *and* $h=0=h'$ *if* $g=g'$.

(1.5.4) ([21, Satz 6]) *For* $A\supset B$, *assume that* $B\xrightarrow{i} A\xrightarrow{q} A/B$ (i: *the inclusion*, q: *the collapsing map) is a cofibering, and consider the Puppe exact sequence* $[A/B, X]\xrightarrow{q^*}[A, X]\xrightarrow{i^*}[B, X]$. *Then, for any* $g, g': A\to X$ *with* $g|B\sim g'|B$: $B\to X$, *there is a separation element*

$d=d(g, g')\in[A/B, X]$ *such that* $g+_m q^*d=g'$ *in* $[A, X]$, *which is unique if* q^* *is injective.*

In fact, taking $h\in[A, X]$ in (1.5.2), we see that $i^*h=0$ and $h\in \operatorname{Im} q^*$. Especially,

(1.5.5) $Y\vee Y\to Y\times Y\xrightarrow{\pi} Y\wedge Y$ *is a cofibering and* $\pi^*:[Y\wedge Y, X]\to[Y\times Y, X]$ *is injective*; *and*

(1.5.6) *for any multiplications* m' *and* m'' *on* X, *the separation element* $d(m', m'')\in[X\wedge X, X]$ *is defined so that* $m'\sim m''$ *if and only if* $d(m', m'')=0$ *or* $d(m, m')=d(m, m'')$ *in* $[X\wedge X, X]$.

For H-spaces $X=(X, m_X)$ and $Y=(Y, m_Y)$, a map $f: X\to Y$ is an *H-map* if $fm_X\sim m_Y(f\times f): X\times X\to Y$; and we denote the subset of $[X, Y]$ consisting of all classes of H-maps by $[X, Y]_H$ ($\subset[X, Y]$). Then, since $fm_X|X\vee X\sim f\nabla=\nabla(f\vee f)\sim m_Y(f\times f)|X\vee X: X\vee X\to Y$ for $f: X\to Y$,

(1.5.7) *we have the map* $\phi: [X, Y]\to[X\wedge X, Y]$ *with* $[X, Y]_H=\operatorname{Ker}\phi$ *given by*

$\phi(f)=d(m_Y(f\times f), fm_X)\in[X\wedge X, Y]$, *the separation element in* (1.5.4) (cf. (1.5.5)), *for* $f\in[X, Y]$.

By the results due to I. M. James [9, Cor. 4.4 and §3], we notice the following:

(1.5.8) *Let* (X, m_X) *and* (Y, m_Y) *be H-complexes with* $m_X|X\vee X=\nabla$ *and* $m_Y|Y\vee Y=\nabla$. *Then, for any H-map* $f: X\to Y$, *we can take a homotopy* F: $X\times X\times I\to Y$ rel $X\vee X$ *of* fm_X *to* $m_Y(f\times f)$.

For any H-space $X=(X, m)$, we denote the subgroup of $\mathscr{E}(X)$ consisting of all classes of self H-equivalences of X (homotopy equivalent H-maps of (X, m) to itself) by

$$\mathscr{E}_H(X)\ (=\mathscr{E}_H(X, m)) = \mathscr{E}(X) \cap [X, X]_H\ (\subset \mathscr{E}(X)).$$

As a sufficient condition for $\mathscr{E}_H(X)=\mathscr{E}(X)$, we see the following by (1.5.7), (1.1.4) and (1.1.2):

(1.5.9) ([24, Prop. 2.7]) *If* $[X \wedge X, X]=0$, *e.g., if* X *is* A *given in* (1.5.1), *then* $\mathscr{E}_H(X)=\mathscr{E}(X)$.

On the induced homomorphisms given in Lemma 1.2, we have the following

LEMMA 1.6. *Let* X *and* Y *be* H*-spaces and* $f: X\to Y$ *be an* H*-map. If* f^* (*resp.* f_*) *in Lemma* 1.2 *is bijective and*

$$(f\times f)^*: [Y\times Y, Y] \to [X\times X, Y] \quad (\textit{resp.}\ f_*: [X\times X, X]\to [X\times X, Y])$$

is injective, then the restriction of the induced homomorphism

$$f_!: \mathscr{E}(X) \to \mathscr{E}(Y) \textit{ in } (1.2.1) \quad (\textit{resp.}\ f^!: \mathscr{E}(Y)\to \mathscr{E}(X) \textit{ in } (1.2.2))$$

defines the homomorphism

(1.6.1) $f_! = f_! \mid \mathscr{E}_H(X): \mathscr{E}_H(X) \to \mathscr{E}_H(Y)$ $\quad$ (*resp.* $f^! = f^! \mid \mathscr{E}_H(Y): \mathscr{E}_H(Y) \to \mathscr{E}_H(X)$).

PROOF. Assume that $h: (X, m_X)\to(X, m_X)$ is an H-map. Then, by the assumption that $f: (X, m_X)\to(Y, m_Y)$ is an H-map and the definition of $h'=f_!(h)$ in (1.2.1), we see easily that $h'm_Y(f\times f)=m_Y(h'\times h')(f\times f)$ in $[X\times X, Y]$. Thus $h'm_Y=m_Y(h'\times h')$ in $[Y\times Y, Y]$ since $(f\times f)^*$ is injective, and h' is an H-map. The remaining half can be proved similarly. q.e.d.

When $X=(X, m)$ is an H-space, $m: X\times X\to X$ can be extended to a multiplication $m_n: X_n\times X_n\to X_n$ uniquely (up to homotopy) for X_n in (1.3.2) by the obstruction theory. Thus

(1.7.1) *the* n*-stage* X_n *in the Postnikov system of an* H*-space* X *given in* (1.3.1) *is an* H*-space with unique multiplication* m_n *so that* $f_n: X\to X_n$ *in* (1.3.1) *is an* H*-map.*

Furthermore, $(f_n\times f_n)^*: [X_n\times X_n, X_n]\to[X\times X, X_n]$ is bijective by (1.3.1), (1.1.4) and (1.1.1). Thus the H-map f_n in (1.7.1) induces the homomorphism

(1.7.2) $f_{n!}: \mathscr{E}_H(X)\to\mathscr{E}_H(X_n)$ of (1.6.1) for $f=f_n$, which is the restriction of $f_{n!}$ in (1.3.3).

We have proved in [25, Th. 1.3] the following

(1.7.3) *If X is a finite dimensional H-complex, then* $f_{n!}$: $\mathscr{E}_H(X) \to \mathscr{E}_H(X_n)$ *in* (1.7.2) *is monomorphic for* $n \geqq \dim X$, *and isomorphic for* $n \geqq 2 \dim X$.

By this result, the group $\mathscr{E}_H(X)$ is determined by $\mathscr{E}_H(X_n)$ for large n, and the latter will be investigated inductively by using the fibering $X_n \to X_{n-1}$ with fiber $K(\pi_n(X), n)$.

§ 2. Self (H-)equivalences of the mapping track

The group $\mathscr{E}(E_f)$ of self equivalences of the mapping track E_f of $f: A \to B$ is investigated by Y. Nomura [19] and J. W. Rutter [22]. In this section, we study the group $\mathscr{E}_H(E_f)$ of self H-equivalences of E_f which is an H-space when f is an H-map as is seen in (2.1.4).

Throughout this section, we assume that

(2.1.1) $A = (A, m_1)$ and $B = (B, m_2)$ are given H-complexes with $m_1 | A \vee A = \nabla$ and $m_2 | B \vee B = \nabla$, and $f: A \to B$ is a given H-map with a homotopy $F: A \times A \times I \to B$ rel $A \vee A$ of fm_1 to $m_2(f \times f)$ (cf. (1.5.8)).

Then, by using the path space $PB = \{\ell: I \to B \mid \ell(1) = *\}$ and the loop functor Ω, we have

(2.1.2) the mapping track $E_f = \{(a, \ell) \mid a \in A,\ \ell \in PB,\ f(a) = \ell(0)\}$ $(\subset A \times PB)$ of f, and

(2.1.3) the fiber sequence $\Omega A \xrightarrow{\Omega f} \Omega B \xrightarrow{i} E_f \xrightarrow{p} A \xrightarrow{f} B$ (p: the projection, i: the inclusion); and

(2.1.4) (J. D. Stasheff [26, Th. 2]) *E_f is an H-space so that p and i in* (2.1.3) *are H-maps, where the multiplication m on E_f is defined by using F in* (2.1.1) *and* m_2: $PB \times PB \to PB$ ($m_2(\ell, \ell') = m_2(\ell \times \ell')\Delta$) *as follows*:

$$m((a, \ell), (a', \ell')) = (m_1(a, a'), \ell'');$$
$$\ell''(t/2) = F(a, a', t)\ (0 \leqq t \leqq 1), \ = m_2(\ell, \ell')\,(t-1)\ (1 \leqq t \leqq 2).$$

Hereafter, we are concerned with this H-space $E_f = (E_f, m)$, (cf. also Remark 2.9). Then,

(2.1.5) *the loop action* μ: $E_f \times \Omega B \to E_f$ *is an H-map, and* $\mu = m(1 \times i)$ *in* $[E_f \times \Omega B, E_f]$, *where*

$$\mu((a, \ell), \ell') = (a, \mu(\ell, \ell')), \quad \mu(\ell, \ell')(t/2) = \ell(t)\ (0 \leqq t \leqq 1), = \ell'(t-1)(1 \leqq t \leqq 2);$$

because the loop action μ: $PB \times \Omega B \to PB$ is homotopic to $m_2 | PB \times \Omega B$ as usual.

In [19] and [22], the group $\mathscr{E}(E_f)$ is studied by considering the map

(2.2.1) $\kappa\colon [E_f, \Omega B] \to [E_f, E_f]$ defined by $\kappa(\alpha) = \mu(1 \times \alpha)\triangle$ for $\alpha \in [E_f, \Omega B]$,

where 1 denotes the identity map and μ is the loop action in (2.1.5). Then, we have

(2.2.2) $\kappa(\alpha+\beta) = \kappa(\beta)\kappa(\alpha)$ *for* $\alpha \in [E_f, \Omega B]$ *and* $\beta \in \mathrm{Im}\,(p^*\colon [A, \Omega B] \to [E_f, \Omega B])$,

($+$ is $+_\mu$ in (1.5.2) of the loop multiplication μ on ΩB), by the following equalities in the homotopy sets:

$$\begin{aligned}\kappa(\alpha+\beta) &= \mu(1\times\mu)(1\times\alpha\times\beta)(1\times\triangle)\triangle\\ &= \mu(\mu\times 1)(1\times\alpha\times\beta)(\triangle\times 1)\triangle = \mu(\kappa(\alpha)\times\beta)\triangle,\end{aligned}$$

$$\kappa(\beta)\bar{\alpha} = \mu(1\times\beta)\triangle\bar{\alpha} = \mu(\bar{\alpha}\times\beta\bar{\alpha})\triangle,\ \beta' p\bar{\alpha} = \beta' p\mu(1\times\alpha)\triangle = \beta' p$$
$$(\bar{\alpha} = \kappa(\alpha),\ \beta' \in [A, \Omega B]).$$

Now, we notice that $[\ , \Omega B]$ is the abelian group as usual by $+ = +_\mu = +_{m_2}$ in our case, and consider

(2.2.3) $[X, \Omega B] \xrightarrow{\phi} [X \wedge X, \Omega B] \xrightarrow{\pi^*} [X \times X, \Omega B]$ in (1.5.7) and (1.5.5) for any H-space $X = (X, m)$, where π^* is monomorphic and ϕ is the homomorphism with $\mathrm{Ker}\,\phi = [X, \Omega B]_H$ given by

$$\alpha m = m_2(\alpha\times\alpha) + \pi^*\phi(\alpha) \qquad \text{for} \quad \alpha \in [X, \Omega B] \text{ (cf. (1.5.4))},$$
$$\text{or} \quad \pi^*\phi = m^* - p_1^* - p_2^* \ (p_i\text{: the } i\text{-th projection}).$$

LEMMA 2.3. (i) $\kappa\colon [E_f, \Omega B] \to [E_f, E_f]$ *in* (2.2.1) *is given by* $i_*\colon [E_f, \Omega B] \to [E_f, E_f]$ *as follows*:

(2.3.1) $\kappa(\alpha) = 1 + i_*\alpha$ *for* $\alpha \in [E_f, \Omega B]$, *where* $+$ *is* $+_m$ *on* $[\ , E_f]$ *given in* (1.5.2).

(ii) *If* $\alpha \in [E_f, \Omega B]_H$, *then* $\kappa(\alpha) \in [E_f, E_f]_H$.

(iii) *In the sequence* $[E_f, \Omega A] \xrightarrow{(\Omega f)_*} [E_f, \Omega B] \xrightarrow{\phi} [E_f \wedge E_f, \Omega B] \xrightarrow{i_*} [E_f \wedge E_f, E_f]$,

(2.3.2) *assume that a subset* $Q \subset [E_f, \Omega B]$ *satisfies* $\phi(Q) \cap \mathrm{Ker}\ i_* \subset \mathrm{Im}\,(\phi(\Omega f)_*)$.

Then, for any $\alpha \in Q$ *with* $\kappa(\alpha) \in [E_f, E_f]_H$, *there is* $\alpha' \in [E_f, \Omega B]_H$ *such that* $\kappa(\alpha') = \kappa(\alpha)$.

PROOF. (i) We have $\kappa(\alpha) = \mu(1\times\alpha)\triangle = m(1\times i\alpha)\triangle = 1 + i_*\alpha$ by (2.2.1) and the equality in (2.1.5).

(ii) Noticing that μ is an H-map by (2.1.5), we have similarly the following in $[E_f \times E_f, E_f]$:

$$(2.3.3) \quad \kappa(\alpha)m = \mu(1\times\alpha)\triangle m = m(1\times i\alpha)(m\times m)\triangle = m + i\alpha m,$$

$$m(\kappa(\alpha)\times\kappa(\alpha)) = m(\mu\times\mu)(1\times\alpha\times 1\times\alpha)(\triangle\times\triangle)$$
$$= \mu(m\times m_2)(1\times 1\times\alpha\times\alpha)\triangle = m + im_2(\alpha\times\alpha).$$

Therefore, if α is an H-map, then these are equal to each other and $\kappa(\alpha)$ is an H-map.

(iii) Let $\alpha\in Q$ and assume that $\kappa(\alpha)$ is an H-map. Then $i\alpha m = im_2(\alpha\times\alpha)$ in $[E_f\times E_f, E_f]$ by (2.3.3) and (1.5.2). Thus $\pi^* i_*\phi(\alpha) = i_*\pi^*\phi(\alpha) = 0$ and $i_*\phi(\alpha) = 0$ by (2.2.3). Therefore,

$\phi(\alpha) = \phi((\Omega f)_*\beta)$ for some $\beta\in[E_f, \Omega A]$, by the assumption (2.3.2).

Put $\alpha' = \alpha - (\Omega f)_*\beta$. Then $\phi(\alpha') = 0$, and $\alpha'\in[E_f, \Omega B]_H$ by (2.2.3). Further $\kappa(\alpha) = 1 + i_*(\alpha' + (\Omega f)_*\beta) = 1 + (i_*\alpha' + i_*(\Omega f)_*\beta) = \kappa(\alpha')$ by (i), since i is an H-map by (2.1.4) and $i(\Omega f)\sim *$. q.e.d.

In the rest of this section, we assume that the homotopy groups of A and B in (2.1.1) satisfy

(2.4.1) $\pi_i(A) = 0$ unless $m\leqq i < n$, $\pi_j(B) = 0$ unless $n < j \leqq m+n$, for some integers $n > m \geqq 2$.

We consider the cofiber sequence in the upper line of the homotopy commutative diagram

$$(2.4.2) \quad \begin{array}{ccccccc} \Omega B & \xrightarrow{i} & E_f & \xrightarrow{j} & C_i & \xrightarrow{k} & S\Omega B \\ \| & & \| & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle e} \\ \Omega B & \xrightarrow{i} & E_f & \xrightarrow{p} & A & \xrightarrow{f} & B \end{array} \left(\begin{array}{l} C_i = E_f \cup_i C\Omega B\text{: the mapping cone of } i, \\ S\Omega B = C_i/E_f\text{: the suspension of } \Omega B, \\ j\text{: the inclusion, } k\text{: the collapsing map} \end{array}\right),$$

where the lower line is the fiber sequence (2.1.3), q is the map with $q(C\Omega B) = *$ and $qj = p$, and e is the evaluation map. Then, under the assumption (2.4.1), we notice the following:

(2.4.3) *p, j, q and e in (2.4.2) are n-, n-, $(m+n)$- and $(2n+1)$-connected, respectively.*

This is seen for p clearly, for q since $p_*\colon H_i(E_f, \Omega B)\to\tilde{H}_i(A)$ is isomorphic if $i < m+n$ and epimorphic if $i = m+n$, hence for j, and for e since the fiber of e is the join $\Omega B * \Omega B$ being $2n$-connected ([3, Prop. 3.2] and [14, Lemma 2.3]).

Now, consider the following commutative diagram of the induced maps:

$$\begin{tikzcd}
{[A, \Omega A]} \arrow[r, "(\Omega f)_*"] \arrow[d, "p^*", "\cong"'] & {[A, \Omega B]} \arrow[d, "p^*"] & {[C_i, \Omega B]} \arrow[l, "q^*"', "\cong"] \arrow[dl, "j^*"] & {[A, A]} \arrow[d, "p^*", "\cong"'] \\
{[E_f, \Omega A]} \arrow[r, "(\Omega f)_*"] & {[E_f, \Omega B]} \arrow[r, "i_*"'] \arrow[d, "i^*"] & {[E_f, E_f]} \arrow[r, "p_*"] \arrow[d, "i^*"] & {[E_f, A]} \\
{[B, B]} \arrow[r, "e^*=\Omega", "\cong"'] & {[S\Omega B, B]=[\Omega B, \Omega B]} \arrow[r, "i_*", "\cong"'] & {[\Omega B, E_f],}
\end{tikzcd} \tag{2.4.4}$$

where the middle horizontal sequence (resp. $\xrightarrow{j^*}\xrightarrow{i^*}$) is the homotopy (resp. Puppe) exact sequence of the fiber sequence (2.1.3) (resp. cofibering in (2.4.2)). Then under (2.4.1), we see that

(2.4.5) *the maps indicated by $\cong$ are all bijective, and the vertical sequence is exact.*

In fact, the two p^*'s, q^* and e^* are bijective by (2.4.1), (2.4.3) and (1.1.1), and so the latter half holds. The lower i_* is bijective, since it is in the homotopy exact sequence with $[\Omega B, \Omega A]=0=[\Omega B, A]$ by (1.1.2).

Therefore, Lemma 1.2 shows that the restrictions of $p^{*-1}p_*$ and $\Omega^{-1}i_*^{-1}i^*$ induce the homomorphisms

$$\begin{aligned} &\phi=p_!: \mathscr{E}(E_f)\to\mathscr{E}(A) \text{ determined by } \phi(h)p=ph \text{ in } [E_f, A], \text{ and} \\ &\psi=\Omega^{-1}i^!: \mathscr{E}(E_f)\to\mathscr{E}(B) \text{ determined by } i\Omega(\psi(h))=hi \text{ in } [\Omega B, E_f], \end{aligned} \tag{2.4.6}$$

respectively. Furthermore, by Y. Nomura [19, Th. 2.1, 2.9] and J. W. Rutter [22, Th. 3.1], we have the following

THEOREM 2.5. *Assume that H-complexes A and B satisfy* (2.4.1). *Then the group $\mathscr{E}(E_f)$ of the mapping track E_f in* (2.1.3) *of an H-map $f: A\to B$ is in the short exact sequence*

$$0 \longrightarrow H(f) \xrightarrow{\kappa} \mathscr{E}(E_f) \xrightarrow{(\varphi,\psi)} G(f) \longrightarrow 1, \tag{2.5.1}$$

where

$$\begin{aligned} &H(f)=\mathrm{Im}\,(p^*: [A, \Omega B]\to[E_f, \Omega B])/\mathrm{Im}\,((\Omega f)_*: [E_f, \Omega A]\to[E_f, \Omega B]), \\ &G(f)=\{(h_1, h_2) \mid h_1\in\mathscr{E}(A), h_2\in\mathscr{E}(B), fh_1=h_2f \text{ in } [A, B]\}\ (\subset\mathscr{E}(A)\times\mathscr{E}(B)), \end{aligned} \tag{2.5.2}$$

κ is the homomorphism induced by κ in (2.2.1) *and (φ, ψ) is the one given by φ and ψ in* (2.4.6).

This theorem can be seen by using the commutative diagram (2.4.4) with (2.4.5) as follows. Restricting κ in (2.2.1), we have the homomorphism κ: Im p^*

$\to \mathscr{E}(E_f)$ by (2.2.2), and $\kappa^{-1}(1) = \mathrm{Ker}\, i_* = \mathrm{Im}\,(\Omega f)_* \subset \mathrm{Im}\, p^*$ by (2.3.1) and the horizontal exact sequence; thus it induces the monomorphism κ in (2.5.1). (2.3.1), the two exact sequences and the definition (2.4.6) imply that $\mathrm{Im}\,\kappa = 1 + i_*\,\mathrm{Im}\, p^* = 1 + (\mathrm{Ker}\, p_*) \cap (\mathrm{Ker}\, i^*) = (\varphi, \psi)^{-1}(1)$, since p is an H-map by (2.1.4). $\mathrm{Im}\,(\varphi, \psi) = G(f)$ is seen by (2.4.6) and the following (2.5.3) and (2.5.5):

(2.5.3) *For $(h_1, h_2) \in G(f)$, there is $h \in \mathscr{E}(E_f)$ such that $ph = h_1 p\colon E_f \to A$ and $hi = i(\Omega h_2)$ in $[\Omega B, E_f]$.*

In fact, a homotopy $H\colon A \times I \to B$ of fh_1 to $h_2 f$ gives us such a map

(2.5.4) $h\colon E_f \to E_f$ defined by $h(a, \ell) = (h_1(a), \ell_a)$; $\ell_a(t/2) = H(a, t)$ $(0 \leqq t \leqq 1)$, $= h_2 \ell(t-1)$ $(1 \leqq t \leqq 2)$.

(2.5.5) *For $h \in \mathscr{E}(E_f)$, $h_1 = \varphi(h) \in \mathscr{E}(A)$ and $h_2 = \psi(h) \in \mathscr{E}(B)$ satisfy $fh_1 = h_2 f$ in $[A, B]$.*

In fact, by the cofiber sequence in (2.4.2) and as a dual to (2.5.3), a homotopy $\bar{H}\colon \Omega B \times I \to E_f$ of hi to $i(\Omega h_2)$ defines

$$\bar{h}_1\colon C_i\,(= E_f \cup_i C\Omega B) \to C_i$$
$$\text{by}\quad \bar{h}_1 \mid E_f = h,\ \bar{h}_1(\ell, t/2) = \bar{H}(\ell, t)\ (0 \leqq t \leqq 1),\ = (h_2 \ell, t-1)\ (1 \leqq t \leqq 2),$$

so that $\bar{h}_1 j = jh\colon E_f \to C_i$ and $k\bar{h}_1 = (S\Omega h_2)k$ in $[C_i, S\Omega B]$. Thus, because (2.4.2) is homotopy commutative and $j^*\colon [C_i, A] \to [E_f, A]$ and $q^*\colon [A, B] \to [C_i, B]$ are injective by (2.4.3), (2.4.1) and (1.1.1), we have

$$q\bar{h}_1 j = qjh = ph = h_1 p \ (\text{since } h_1 = \varphi(h)) = h_1 qj \ \text{ in } [E_f, A] \text{ and so } q\bar{h}_1 = h_1 q \ \text{ in } [C_i, A];$$

$$fh_1 q = fq\bar{h}_1 = ek\bar{h}_1 = e(S\Omega h_2)k = h_2 ek = h_2 fq \ \text{ in } [C_i, B], \text{ and so } fh_1 = h_2 f \ \text{ in } [A, B].$$

We now study the subgroup $\mathscr{E}_H(E_f)$ of $\mathscr{E}(E_f)$ for the H-space $E_f = (E_f, m)$ in (2.1.4).

LEMMA 2.6. *Assume that $Q = \mathrm{Im}\,(p^*\colon [A, \Omega B] \to [E_f, \Omega B])$ satisfies (2.3.2). Then*

$$\kappa^{-1}(\mathscr{E}_H(E_f)) = p^*(P)/(\mathrm{Im}\,(\Omega f)_*) \cap p^*(P),\ P = [A, \Omega B]_H,\ \textit{for } \kappa \text{ in } (2.5.1).$$

PROOF. If $\alpha \in p^*(P)$, then $\alpha \in [E_f, \Omega B]_H$ since p is an H-map by (2.1.4), and so $\kappa(\alpha) \in [E_f, E_f]_H$ by Lemma 2.3 (ii). Conversely, assume that $\alpha \in Q$ satisfies $\kappa(\alpha) \in [E_f, E_f]_H$. Then

(2.6.1) $\kappa(\alpha)=\kappa(\alpha')$ *for some* $\alpha' \in [E_f, \Omega B]_H$ *by Lemma* 2.3 (iii).

This implies that $\alpha'-\alpha \in \operatorname{Ker} i_*$ by Lemma 2.3 (i), and $\operatorname{Ker} i_* = \operatorname{Im}(\Omega f)_* \subset Q$. Thus

(2.6.2) $\alpha' \in [E_f, \Omega B]_H \cap Q$ *and* $\alpha' = p^*\beta$ *for some* $\beta \in [A, \Omega B]$.

On the other hand, by (2.4.3), (1.1.7), (2.4.1) and (1.1.1), we see that

(2.6.3) $p \wedge p\colon E_f \wedge E_f \to A \wedge A$ *is* $(m+n)$*-connected, and* $(p \wedge p)^*\colon [A \wedge A, \Omega B] \cong [E_f \wedge E_f, \Omega B]$.

Consider the homomorphism $\phi\colon [X, \Omega B] \to [X \wedge X, \Omega B]$ in (2.2.3) for $X=A$ and E_f. Then, $(p \wedge p)^*\phi = \phi p^*$ by the definition of ϕ, since p is an H-map by (2.1.4). Thus (2.6.1–3) and $[X, \Omega B]_H = \operatorname{Ker}\phi$ in (2.2.3) show that $(p \wedge p)^*\phi(\beta) = \phi(\alpha')=0$, $\phi(\beta)=0$, $\beta \in [A, \Omega B]_H = P$ and $\kappa(\alpha)=\kappa(p^*\beta) \in \kappa(p^*P)$. q.e.d.

Lemma 2.7. (i) *By restricting* (φ, ψ) *in* (2.5.1), *we have the homomorphisms*

(2.7.1)
$$\tilde\varphi\colon \mathscr{E}_H(E_f) \to \mathscr{E}_H(A), \quad \tilde\psi\colon \mathscr{E}_H(E_f) \to \mathscr{E}(B) = \mathscr{E}_H(B),$$
$$(\tilde\varphi, \tilde\psi)\colon \mathscr{E}_H(E_f) \to \bar G(f) = G(f) \cap (\mathscr{E}_H(A) \times \mathscr{E}_H(B)).$$

(ii) $\operatorname{Im}(\tilde\varphi, \tilde\psi)$ *is the subgroup of* $\bar G(f)$ *consisting of all* $(h_1, h_2) \in \mathscr{E}(A) \times \mathscr{E}(B)$ *satisfying the following property*:

(2.7.2) *There are homotopies* $H\colon A \times I \to B$ *of* fh_1 *to* $h_2 f$ (*i.e.*, $(h_1, h_2) \in G(f)$) *and*

$$H_1\colon A \times A \times I \to A \text{ rel } A \vee A \text{ of } h_1 m_1 \text{ to } m_1(h_1 \times h_1) \ (\textit{i.e., } h_1 \in \mathscr{E}_H(A)),$$
$$H_2\colon B \times B \times I \to B \text{ rel } B \vee B \text{ of } h_2 m_2 \text{ to } m_2(h_2 \times h_2) \ (\textit{i.e., } h_2 \in \mathscr{E}_H(B)),$$

and in addition, there is a secondary homotopy $D\colon A \times A \times I^2 \to B$ $(I^2 = I \times I)$ *such that* $D(a, a', s, t/2)$ $((s, t/2) \in \dot I^2)$ *is*

$$(*)\quad \begin{aligned} &fH_1(a, a', s)\,(t=0),\\ &H(m_1(a, a'), t)\,(s=0,\ 0 \leqq t \leqq 1),\ h_2F(a, a', t-1)\,(s=0,\ 1 \leqq t \leqq 2),\\ &H_2(f(a), f(a'), s)\,(t=2),\\ &F(h_1(a), h_1(a'), t)\,(s=1,\ 0 \leqq t \leqq 1),\ m_2(H(a, t-1), H(a', t-1))\,(s=1,\ 1 \leqq t \leqq 2), \end{aligned}$$

where $F\colon A \times A \times I \to B$ rel $A \vee A$ *is a homotopy of* fm_1 *to* $m_2(f \times f)$ *given in* (2.1.1).

Proof. (i) By (2.4.3), (2.4.1), (1.1.1) and (1.1.7), $(p \times p)^*\colon [A \times A, A] \to [E_f \times E_f, A]$ is bijective. Thus the H-map p induces $\tilde\varphi = p_!\colon \mathscr{E}_H(E_f) \to \mathscr{E}_H(A)$ in Lemma 1.6, which is $\varphi \mid \mathscr{E}_H(E_f)$. $\mathscr{E}_H(B) = \mathscr{E}(B)$ is seen by (2.4.1) and (1.5.9).

(ii) We can prove (ii) by the same proof as that of C.-K. Cheng [6, Th. 2.2] (where B is assumed to be $K(\pi, n+1)$) as follows. Consider $h \in \mathscr{E}(E_f)$ given by (2.5.4) for $(h_1, h_2) \in G$ and a homotopy H. Then, by the definition of m in (2.1.4),

(2.7.3) *we have* $\bar{D}_0(\sim hm)$, $\bar{D}_1(\sim m(h \times h))\colon E_f \times E_f \to E_f$ *such that* $p\bar{D}_0 = h_1 m_1(p \times p)$, $p\bar{D}_1 = m_1(h_1 p \times h_1 p)$ *and*

$$\begin{aligned} p'\bar{D}_s((a, \ell), (a', \ell'))(t/3) & (p'\colon E_f \to PB \text{ is the projection}) \\ &= D(a, a', s, t/2) \quad \text{in } (*)\ (s \in \dot{I},\ 0 \leqq t \leqq 2), \\ &= h_2 m_2(\ell(t-2), \ell'(t-2))\ (s=0,\ 2 \leqq t \leqq 3), \\ &= m_2(h_2\ell(t-2), h_2\ell'(t-2))\ (s=1,\ 2 \leqq t \leqq 3). \end{aligned}$$

Thus, if (h_1, h_2) satisfies (2.7.2), then D and H_i give us a homotopy of $\bar{D}_0$ to $\bar{D}_1$ immediately, and $h \in \mathscr{E}_H(E_f)$.

Conversely, assume that $h \in \mathscr{E}_H(E_f)$. To show the existence of H_i and D, we deform $\bar{D}_s$ in (2.7.3) to

(2.7.4) $\bar{D}'_s(\sim \bar{D}_s)\colon E_f \times E_f \to E_f$ $(s \in \dot{I})$ so that $\bar{D}'_0 = \bar{D}'_1$ on $E_f \vee E_f$, by setting $p\bar{D}'_s = p\bar{D}_s$ and

$$p'\bar{D}'_s(\ ,\ )(t'/4) = p'\bar{D}_s(\ ,\ )(t/3) \qquad \text{for} \quad 0 \leqq t' \leqq 4,$$

where $t = \min\{t', 2\}$ $(s=1,\ 0 \leqq t' \leqq 3)$, $= \max\{0, t'-1\}$ (otherwise).

On the other hand, since p is n-connected by (2.4.3), we see that

(2.7.5) $p\colon E_f \to A$ *has a cross section* $\tau\colon A^n \to E_f$ $(p\tau = j\colon A^n \subset A)$ *on the n-skeleton A^n of A.*

Then, since $\bar{D}'_0$ is homotopic to $\bar{D}'_1$ by the assumption, we see the following by [9, Cor. 4.4 and §3]:

(2.7.6) *There is a homotopy* $\bar{D}'\colon A^n \times A^n \times I \to E_f$ rel $A^n \vee A^n$ *of* $\bar{D}'_0(\tau \times \tau)$ *to* $\bar{D}'_1(\tau \times \tau)$.

Now, for any homotopy $H_2\colon B \times B \times I \to B$ rel $B \vee B$ of $h_2 m_2$ to $m_2(h_2 \times h_2)$, $p'\bar{D}' \cdot (a, a', s)(t'/4)$ for $3 \leqq t' \leqq 4$ is equal to $H_2(p'\tau(a)(t'-3), p'\tau(a')(t'-3), s)$ if $t'=4$ or $s \in \dot{I}$ or $(a, a') \in A^n \vee A^n$ by (2.7.3–4). Therefore, by the homotopy extension property, we can deform the map $A^n \times A^n \times I^2 \to B$ given by $p'\bar{D}'\colon A^n \times A^n \times I \to PB$ to

(2.7.7) $D'\colon A^n \times A^n \times I^2 \to B$ such that $D'(a, a', s, t'/3)$ is stationary on s if $(a, a') \in A^n \vee A^n$ and is equal to

$$fp\bar{D}'(a, a', s)(t'=0),\ H_2(f(a), f(a'), s)(t'=3),\ D(a, a', s, t/2)$$
$$\text{in } (*)\ (s \in \dot{I},\ 0 \leqq t' \leqq 3 \text{ and } t \text{ is the one in (2.7.4))}.$$

Furthermore, by the obstruction theory and (2.4.1), we can extend

(2.7.8) $p\bar{D}'\colon A^n \times A^n \times I \to A$ to a homotopy $H_1\colon A \times A \times I \to A$ rel $A \vee A$ of $h_1 m_1$ to $m_1(h_1 \times h_1)$, and then D' in (2.7.7) to $D'\colon A \times A \times I^2 \to B$ so that $D'(a, a', s, t'/3)$ is stationary on s if $(a, a') \in A \vee A$ and is equal to

$$D(a, a', s, t/2) \text{ in } (*) \text{ if } (s, t'/3) \in \dot{I}^2, \text{ where } t = \min\{t', 2\}\ (s=1 \text{ or } t'=3),$$
$$= \max\{0, t'-1\}\ (s=0 \text{ or } t'=0).$$

Thus D' can be deformed to D in (2.7.2), and (h_1, h_2) satisfies (2.7.2). q.e.d.

By Theorem 2.5 together with Lemmas 2.6–7, we see immediately the following theorem, which is Theorem I–1 in the introduction.

THEOREM 2.8. *Assume that H-complexes A and B satisfy* (2.4.1) *and consider the mapping track E_f in* (2.1.3) *of an H-map $f\colon A \to B$, which is an H-space by* (2.1.4).

(i) *Then the group $\mathscr{E}_H(E_f)$ of all self H-equivalences of E_f is in the exact sequence*

$$0 \longrightarrow \tilde{H}(f) \xrightarrow{\tilde{\kappa}} \mathscr{E}_H(E_f) \xrightarrow{(\tilde{\varphi}, \tilde{\psi})} \tilde{G}(f) \longrightarrow 1 \tag{2.8.1}$$

obtained by restricting the one in (2.5.1), *where $\tilde{H}(f) = \kappa^{-1}(\mathscr{E}_H(E_f))$ for κ in* (2.5.1) *and*

$$\tilde{G}(f) = \{(h_1, h_2) \in \mathscr{E}_H(A) \times \mathscr{E}_H(B) \mid (h_1, h_2) \text{ satisfies (2.7.2)}\} \subset G(f) \cap (\mathscr{E}_H(A) \times \mathscr{E}_H(B)). \tag{2.8.2}$$

(ii) *Furthermore, consider the diagram*

$$\begin{array}{ccccccc} & & [A, \Omega B] & & & & \\ & & \big\downarrow{\scriptstyle p^*} & & & & \\ [E_f, \Omega A] & \xrightarrow{(\Omega f)_*} & [E_f, \Omega B] & \xrightarrow{\phi} & [E_f \wedge E_f, \Omega B] & \xrightarrow{i_*} & [E_f \wedge E_f, E_f], \end{array} \tag{2.8.3}$$

where ϕ is the homomorphism defined by (2.2.3), *and assume that*

$$\operatorname{Im}(\phi p^*) \cap \operatorname{Ker} i_* \subset \operatorname{Im}(\phi(\Omega f)_*). \tag{2.8.4}$$

Then the group $\tilde{H}(f)$ in (2.8.1) *is given by*

$$\tilde{H}(f) = p^*([A, \Omega B]_H)/(\operatorname{Im}(\Omega f)_*) \cap p^*([A, \Omega B]_H). \tag{2.8.5}$$

Throughout this section, we have been concerned with the H-space (E_f, m) given in (2.1.4). We conclude this section with the following remark on any multiplication on E_f.

REMARK 2.9 (cf. [26, Th. 4], [5, Cor. 1.9]). *Let A and B be CW-complexes with* (2.4.1) *and $f: A \to B$ be a map, and assume that the mapping track E_f of f is an H-space with a multiplication m'. Then A is an H-space with a multiplication m_1 so that $p: E_f \to A$ and $f: A \to B$ are H-maps, where B is an H-space with unique multiplication m_2 by* (2.4.1) *and* (1.5.1). *Furthermore, there is a homotopy F rel $A \vee A$ of fm_1 to $m_2(f \times f)$ so that m' is homotopic to m given in* (2.1.4) *by using F.*

PROOF. Since $(p \times p)^*: [A \times A, A] \cong [E_f \times E_f, A]$ and $(p \vee p)^*: [A \vee A, A] \cong [E_f \vee E_f, A]$ by (2.4.3), (2.4.1) and (1.1.1), we have $m_1: A \times A \to A$ with $m_1(p \times p) = pm'$ in $[E_f \times E_f, A]$ and $m_1 | A \vee A = \nabla$. Consider

(2.9.1) $[A, B] \xrightarrow{\phi} [A \wedge A, B] \xrightarrow{(p \wedge p)^*} [E_f \wedge E_f, B]$, where $(p \wedge p)^*$ is injective by (2.6.3), (2.4.1) and (1.1.1),

and ϕ is the map in (1.5.7) for (A, m_1) and (B, m_2). Then we see $\phi(f) = 0$ and $f \in [A, B]_H$, because

$$(p \wedge p)^* \phi(f) = d(m_2(f \times f), fm_1)(p \wedge p) = d(m_2(fp \times fp), fpm') = 0$$

by (1.5.2–7), $m_1(p \times p) \sim pm'$ and $fp \sim *$.

To show the second half, consider the H-space (E_f, m) given in (2.1.4) by using a homotopy $F: A \times A \times I \to B$ rel $A \vee A$ of fm_1 to $m_2(f \times f)$. Furthermore, consider the sequence

$$[A \wedge A, \Omega B] \xrightarrow{(p \wedge p)^*} [E_f \wedge E_f, \Omega B] \xrightarrow{i_*} [E_f \wedge E_f, E_f] \xrightarrow{p_*} [E_f \wedge E_f, A],$$

where $(p \wedge p)^*$ is bijective by (2.6.3) and $\xrightarrow{i_*} \xrightarrow{p_*}$ is exact. Then, since $pm = m_1(p \times p) \sim pm'$,

(2.9.2) *the separation element $d(m, m') \in [E_f \wedge E_f, E_f]$ in* (1.5.6) *is $(p \wedge p)^* i_* \omega$ for some $\omega \in [A \wedge A, \Omega B]$.*

By using this ω, define the second homotopy $\bar{F}: A \times A \times I \to B$ rel $A \vee A$ of fm_1 to $m_2(f \times f)$ by

$$\bar{F}(a, a', t) = m_2(F(a, a', t), (\omega\pi(a, a'))(t))$$

$$(\pi: A \times A \to A \wedge A \quad \text{is the collapsing map}).$$

Then, by the definition of the multiplication in (2.1.4) and $\mu \sim m(1 \times i)$ in (2.1.5), we see that

(2.9.3) *the multiplication* $\bar{m}$ *on* E_f *given in* (2.1.4) *by* $\bar{F}$ *is equal to* $m+_m i\omega(p\wedge p)\pi$ *in* $[E_f\times E_f, E_f]$.

Thus, $m'=m+_m\pi^*d(m, m')=m+_m\pi^*(p\wedge p)^*i_*\omega=\bar{m}$ in $[E_f\times E_f, E_f]$ by (1.5.4) and (2.9.2–3). q.e.d.

§3. Some corollaries to Theorems 2.5 and 2.8

In this section, we give some corollaries to Theorems 2.5 and 2.8 under the situations given in §2 with suitable additional assumptions.

In the first place, we study the groups $G(f)$ in (2.5.2) and $\tilde{G}(f)$ in (2.8.2). Corresponding to these groups, the projection $\rho\colon \mathscr{E}(A)\times\mathscr{E}(B)\to\mathscr{E}(A)$ defines the epimorphisms

(3.1) $$\rho\colon G(f)\to\rho(G(f))(\subset\mathscr{E}(A)),\quad \tilde{\rho}\colon \tilde{G}(f)\to\rho(\tilde{G}(f))(\subset\mathscr{E}_H(A)).$$

COROLLARY 3.2. *In Theorem* 2.5 (*resp.* 2.8), *assume in addition that*

(3.2.1) *the induced map* $f^*\colon [B, B]\to[A, B]$ *is injective on* $\mathscr{E}(B)$ (*resp.* $\mathscr{E}_H(B)$).

Then $\rho\colon G(f)\to\rho(G(f))$ (*resp.* $\tilde{\rho}\colon \tilde{G}(f)\to\rho(\tilde{G}(f))$) *in* (3.1) *is an isomorphism.*

PROOF. If f^* is injective on $\mathscr{E}(B)$, then the second factor $h_2\in\mathscr{E}(B)$ of $(h_1, h_2)\in G(f)$ is determined by $h_1\in\mathscr{E}(A)$ and the condition $fh_1=h_2f$ in $[A, B]$. Thus ρ in (3.1) is isomorphic. The rest can be proved samely. q.e.d.

Let A'_i $(i=1, 2)$ and $f'\colon A'_1\to A'_2$ be given, and consider the case when

(3.3.1) $A=A_1$, $B=A_2$, $A_i=\Omega A'_i$ with the loop multiplication m_i, $f=\Omega f'\colon A=\Omega A'_1\to B=\Omega A'_2$, and
(3.3.2) the multiplication m on E_f given in (2.1.4) is defined by using the stationary homotopy $F\colon A\times A\times I\to B$ of $fm_1=m_2(f\times f)$ (where the equality holds by definition).

COROLLARY 3.4. *In case* (3.3.1–2), *assume in addition to Theorem* 2.8 *that*

(3.4.1) $\mathscr{E}_H(A_i)\subset\mathrm{Im}\ (\Omega\colon [A'_i, A'_i]\to[A_i, A_i])$, *e.g.*, $3m\geqq n-1$ *in* (2.4.1), *and*
(3.4.2) $\Omega\colon [A'_1, A'_2]\to[A_1, A_2]=[A, B]$ *is injective.*

Then $\tilde{G}(f)=\bar{G}(f)=\{(h_1, h_2)\in\mathscr{E}_H(A)\times\mathscr{E}_H(B)\,|\,fh_1=h_2f$ *in* $[A, B]\}$ *in* (2.8.2).

PROOF. If $h_i\in\mathscr{E}_H(A_i)$, then $h_i=\Omega h'_i$ for some $h'_i\in\mathscr{E}(A'_i)$ by (3.4.1) and we have the stationary homotopy $H_i\colon A_i\times A_i\times I\to A_i$ of $h_im_i=m_i(h_i\times h_i)$ $(i=1, 2)$. Assume that $fh_1=h_2f$ in $[A, B]$. Then $f'h'_1=h'_2f'$ in $[A'_1, A'_2]$ by (3.4.2); and a homotopy $H'\colon A'_1\times I\to A'_2$ of $f'h'_1$ to h'_2f' defines a homotopy $H\colon A\times I\to B$ of

fh_1 to $h_2 f$ by $H(a, t)(u) = H'(a(u), t)$ for $a \in A = \Omega A'_1$, which satisfies $H(m_1(a, a'), t) = m_2(H(a, t), H(a', t))$ by definition. Thus, a secondary homotopy $D: A \times A \times I^2 \to B$ in (2.7.2) can be defined immediately, and $(h_1, h_2) \in \tilde{G}(f)$. We see that (3.4.1) holds if $3m \geqq n-1$, because

(3.4.4) ([28, Lemma 7.4]) *$\mathrm{Im}(\Omega\colon [X, Y] \to [\Omega X, \Omega Y]) = [\Omega X, \Omega Y]_H$ if X is n-connected and $\pi_i(Y) = 0$ for $i > 3n+1$.* q.e.d.

In the rest of this section, we consider the Postnikov system of an H-space. On the Eilenberg-MacLane space, the following are well known:

(3.5.1) *An Eilenberg-MacLane space $K(\pi, i)$ $(i \geqq 2)$ is an H-space with unique multiplication which is the loop multiplication on $\Omega K(\pi, i+1) = K(\pi, i)$, and*

$$\mathscr{E}(K(\pi, i)) = \mathscr{E}_H(K(\pi, i)) = \mathrm{aut}\, \pi \text{ (cf. [10], (1.5.1) and (1.5.9))}.$$

(3.5.2) $[X, K(\pi, i)] = H^i(X; \pi)$, *and*

$[X, K(\pi, i)]_H = PH^i(X; \pi)$ *when X is an H-space* (cf. [27]),

where $PH^i(X; \pi)$ is the subgroup of $H^i(X; \pi)$ consisting of all primitive elements.

Now let $X = (X, m)$ be a given 1-connected H-space, and

(3.6.1) $\{X_n, f_n\colon X \to X_n,\ P_n\colon X_n \to X_{n-1},\ k^{n+1} \in H^{n+1}(X_{n-1}; \pi_n)\}$ $\quad (\pi_n = \pi_n(X))$

be the Postnikov system of X, that is (cf. [26, Th. 5] and Remark 2.9),

(3.6.2) $X_n = (X_n, m_n)$ is an H-space with $\pi_i(X_n) = 0$ for $i > n$ $(X_1 = *,\ X_2 = K(\pi_2, 2))$ and f_n is an $(n+1)$-connected H-map in (1.3.1) or (1.3.2) with (1.7.1),

(3.6.3) $k^{n+1} \in PH^{n+1}(X_{n-1}; \pi_n) = [X_{n-1}, K(\pi_n, n+1)]_H$ is the Postnikov invariant of X, p_n is an H-map with $p_n f_n = f_{n-1}$ in $[X, X_{n-1}]$, and we have a fiber sequence

(3.6.4) $$\Omega X_{n-1} \xrightarrow{\Omega k^{n+1}} K(\pi_n, n) \xrightarrow{i_n} X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{k^{n+1}} K(\pi_n, n+1)$$

which is homotopy equivalent to the one in (2.1.3) for $f = k^{n+1}$, and so is the H-space X_n to the H-space E_f in (2.1.4) for the H-map $f = k^{n+1}$.

Then, we have the homomorphisms

(3.6.5) $\Phi_n = f_{n!}\colon \mathscr{E}(X) \to \mathscr{E}(X_n)$ and $\tilde{\Phi}_n = \Phi_n \mid \mathscr{E}_H(X)\colon \mathscr{E}_H(X) \to \mathscr{E}_H(X_n)$

of (1.3.3) and (1.7.2), respectively. Furthermore, for $n \geqq 3$, $A = X_{n-1}$ and $B = K(\pi_n, n+1)$ satisfy the assumption (2.4.1) with $m = 2$, and we have the homomorphisms

(3.6.6) $\varphi_n = p_{n!}\colon \mathscr{E}(X_n) \to \mathscr{E}(X_{n-1})$ and $\tilde{\varphi}_n = \varphi_n \mid \mathscr{E}_H(X_n)\colon \mathscr{E}_H(X_n) \to \mathscr{E}_H(X_{n-1})$

of (2.4.6) and (2.7.1), respectively; and by definition, there hold the equalities

(3.6.7) $\varphi_n \Phi_n = \Phi_{n-1}$ *and* $\tilde{\varphi}_n \tilde{\Phi}_n = \tilde{\Phi}_{n-1}$ (since $p_n f_n \sim f_{n-1}$).

By applying Theorems 2.5 and 2.8 to the fiber sequence (3.6.4), we have the following corollary, which is Theorem I–2 in the introduction.

COROLLARY 3.7. *Let X be a 1-connected H-complex. Then the groups $\mathscr{E}(X_n)$ and $\mathscr{E}_H(X_n)$ of the n-stage X_n in the Postnikov system* (3.6.1) *of X have the following properties*:

(i) $\mathscr{E}(X_2) = \mathscr{E}_H(X_2) = \operatorname{aut} \pi_2 \quad (\pi_n = \pi_n(X))$.

(ii) *Let $n \geqq 3$, and consider the induced homomorphisms*

(3.7.1) $H^n(X_{n-1}; \pi_n) \xrightarrow{p_n^*} H^n(X_n; \pi_n) = [X_n, K(\pi_n, n)] \xrightarrow{(\Omega k^{n+1})_*} [X_n, \Omega X_{n-1}]$

for p_n and k^{n+1} in (3.6.4). Then we have the exact sequences

$$(3.7.2)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H_n & \xrightarrow{\kappa} & \mathscr{E}(X_n) & \xrightarrow{(\phi_n,\, \psi_n)} & G_n & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \tilde{H}_n & \xrightarrow{\tilde{\kappa}} & \mathscr{E}_H(X_n) & \xrightarrow{(\tilde{\varphi}_n,\, \tilde{\psi}_n)} & \tilde{G}_n & \longrightarrow & 1 \end{array}$$

of (2.5.1) and (2.8.1) for the fiber sequence (3.6.4), where

$$(3.7.3)\qquad \begin{array}{ll} H_n = H(k^{n+1}) = \operatorname{Im} p_n^* / \operatorname{Im} (\Omega k^{n+1})_*, & G_n = G(k^{n+1}) \subset \mathscr{E}(X_{n-1}) \times \operatorname{aut} \pi_n, \\ \tilde{H}_n = \tilde{H}(k^{n+1}) = \kappa^{-1}(\mathscr{E}_H(X_n)), & \tilde{G}_n = \tilde{G}(k^{n+1}) \subset G_n \cap (\mathscr{E}_H(X_{n-1}) \times \operatorname{aut} \pi_n). \end{array}$$

(iii) *Furthermore, in addition to* (3.7.1), *consider the sequence*

(3.7.4) $H^n(X_n; \pi_n) \xrightarrow{\phi} H^n(X_n \wedge X_n; \pi_n) = [X_n \wedge X_n, K(\pi_n, n)] \xrightarrow{i_{n*}} [X_n \wedge X_n, X_n]$,

where ϕ is defined by (2.2.3) *with $X = X_n$ and i_n is in* (3.6.4), *and assume that*

$$(3.7.5)\qquad \operatorname{Im} (\phi p_n^*) \cap \operatorname{Ker} i_{n*} \subset \operatorname{Im} (\phi (\Omega k^{n+1})_*).$$

Then the group $\tilde{H}_n$ in (3.7.2) *is given by*

$$(3.7.6)\qquad \tilde{H}_n = p_n^* P_n / (\operatorname{Im} (\Omega k^{n+1})_*) \cap p_n^* P_n \quad (P_n = PH^n(X_{n-1}; \pi_n)).$$

(iv) *If p_n^* in* (3.7.1) *is epimorphic, then the epimorphisms*

$$(3.7.7)\qquad \rho\colon G_n \to \rho(G_n) (\subset \mathscr{E}(X_{n-1})), \quad \tilde{\rho}\colon \tilde{G}_n \to \rho(\tilde{G}_n) (\subset \mathscr{E}_H(X_{n-1})),$$

defined by the projection $\rho\colon \mathscr{E}(X_{n-1}) \times \operatorname{aut} \pi_n \to \mathscr{E}(X_{n-1})$, are isomorphic.

PROOF. (i) is in (3.5.1), and (ii) and (iii) are the consequences of Theorems 2.5 and 2.8.

(iv) There holds the exact sequence $H^n(X_{n-1}; \pi_n) \xrightarrow{p_n^*} H^n(X_n; \pi_n) \xrightarrow{\tau} H^{n+1}(\pi_n, n+1; \pi_n) \xrightarrow{(k^{n+1})^*} H^{n+1}(X_{n-1}; \pi_n)$ of the fiber sequence (3.6.4). Therefore $(k^{n+1})^*$ is monomorphic since p_n^* is epimorphic. Thus we have (iv) by Corollary 3.2. q.e.d.

In the above corollary, the upper exact sequence of (3.7.2) has been obtained by J. W. Rutter [22, Cor. 3.2]. By D. W. Kahn [10], the homomorphisms Φ_n in (3.6.5) and φ_n in (3.6.6) have been considered and the group $\rho(G_n)$ has been investigated in [10, Lemma 2.1].

EXAMPLE 3.8. *Consider the case that the homotopy groups of an H-complex X are trivial except for $\pi_m = \pi_m(X)$ and $\pi_n = \pi_n(X)$ $(n > m \geqq 2)$. If the Postnikov invariant k is in the image of the cohomology suspension Ω: $H^{n+2}(\pi_m, m+1; \pi_n) \to H^{n+1}(\pi_m, m; \pi_n)$ and this Ω is monomorphic, then we have the exact sequence*

$$0 \to \tilde{H} \to \mathscr{E}_H(X) \to G \to 1,$$

where $\tilde{H} = f_n^* PH^n(\pi_m, m; \pi_n)$ $(f_n = p_n$: $X = X_n \to X_{n-1} = K(\pi_m, m))$ and G is the subgroup $G(k)$ of $\mathrm{aut}\,\pi_m \times \mathrm{aut}\,\pi_n$ *given in* (2.5.2) *for k: $K(\pi_m, m) \to K(\pi_n, n+1)$.*

PROOF. In the exact sequence $[X \wedge X, \Omega X_{n-1}] \xrightarrow{(\Omega k)_*} [X \wedge X, K(\pi_n, n)] \xrightarrow{i_*} [X \wedge X, X]$ $(X = X_n)$, the first term is $H^{m-1}(X \wedge X; \pi_m) = 0$. Thus $\mathrm{Ker}\, i_* = 0$ and (3.7.5) is satisfied. Further, $[X, \Omega X_{n-1}] = H^{m-1}(X; \pi_m) = 0$. Therefore we have the desired exact sequence by Corollaries 3.7 and 3.4. q.e.d.

The following lemma on $(\Omega k^{n+1})_*$ in (3.7.1) will be used in the later sections.

LEMMA 3.9. *Let X^ℓ be the ℓ-skeleton of a CW-complex X, and assume that $X^n = X^{n-1} \cup_g e^n$ for some g: $S^{n-1} \to X^{n-1}$. If $(Sg)^*$: $[SX^{n-1}, X] \to \pi_n(X)$ is trivial, then so is $(\Omega k^{n+1})_*$: $[X_n, \Omega X_{n-1}] \to [X_n, K(\pi_n, n)]$ in (3.7.1). Furthermore, the converse is also true when $X^{n-1} = X^{n-2}$.*

PROOF. We consider the commutative diagram

$$\begin{array}{ccccccccc}
[X_n, \Omega X_n] & \underset{\cong}{\xrightarrow{(f_n j_n)^*}} & [X^n, \Omega X_n] & \xrightarrow{j^*} & [X^{n-1}, \Omega X_n] & = & [SX^{n-1}, X_n] & \xrightarrow{(Sg)^*} & \pi_n(X_n) \\
\Big\downarrow{\scriptstyle (\Omega p_n)_*} & & & & \Big\downarrow{\scriptstyle (\Omega p_n)_*} & & \cong\Big\uparrow{\scriptstyle f_{n*}} & & \cong\Big\uparrow{\scriptstyle f_{n*}} \\
[X_n, \Omega X_{n-1}] & \underset{\cong}{\xrightarrow{(f_n j_n)^*}} & [X^n, \Omega X_{n-1}] & \underset{\cong}{\xrightarrow{j^*}} & [X^{n-1}, \Omega X_{n-1}] & & [SX^{n-1}, X] & \xrightarrow{(Sg)^*} & \pi_n(X),
\end{array} \tag{3.9.1}$$

where j_n: $X^n \subset X$ and j: $X^{n-1} \subset X^n$. Because j_n, j, f_n and p_n are n-, $(n-1)$-, $(n+1)$- and n-connected, respectively, by (1.1.6) and (3.6.2–3), we see the following by (1.1.1), (1.1.3) and (3.6.2):

(3.9.2) *In* (3.9.1), *the maps indicated by $\cong$ are all isomorphic; and*
(3.9.3) *the right $(\Omega p_n)_*$ is epimorphic, and is isomorphic if $X^{n-1} = X^{n-2}$.*

Furthermore, the upper $\xrightarrow{j^*}\xrightarrow{(Sg)^*}$ is exact by the Puppe sequence of the cofibering $S^{n-1} \xrightarrow{g} X^{n-1} \xrightarrow{j} X^n$, and

(3.9.4) *the lower $(Sg)^*$ is trivial if and only if the upper j^* is epimorphic.*

Since the left $(\Omega p_n)_*$ and $(\Omega k^{n+1})_*$ in the lemma form the exact sequence of the fiber sequence (3.6.4), these imply the lemma. q.e.d.

Part II. Application to H-complexes of rank 2 with 2-torsion

§4. The Postnikov system of the H-space $G_{2,b}$

We now recall the 1-connected H-complex $G_{2,b}$ of rank 2 with 2-torsion in homology.

Let G_2 be the compact exceptional Lie group of rank 2, and

(4.1.1) $V_{7,2}=SO(7)/SO(5)=M^6 \cup e^{11}$ ($M^6=S^5 \cup_2 e^6$ is the mapping cone of $2\iota_5$)

be the Stiefel manifold. Then we have the principal bundle

(4.1.2) $S^3 \xrightarrow{i} G_2 \xrightarrow{p} V_{7,2}$ with classifying map $f\colon V_{7,2} \longrightarrow BS^3$,

which has the following properties by [17, Lemmas 4.3, 4.2]:

(4.1.3) $G_2=(G_2)^9 \cup_\omega e^{11} \cup e^{14}$, $(G_2)^9$ *(the 9-skeleton of* $G_2)=p^{-1}(M^6)$, $\omega \in \pi_{10}((G_2)^9)=Z_{120}$ *is a generator, and the homomorphism* $\pi_{10}(S^3)(=Z_{15})\to \pi_{10}((G_2)^9)$ *induced by the inclusion* $S^3 \subset (G_2)^9$ *maps a generator* $\alpha \in \pi_{10}(S^3)$ *to* 8ω.

Now, for each integer b, consider $b\alpha \in \pi_{10}(S^3)=\pi_{11}(BS^3)$ and the composition

(4.1.4) $f_b=\nabla(f \vee b\alpha)\phi\colon V_{7,2} \xrightarrow{\phi} V_{7,2} \vee S^{11} \xrightarrow{f\vee b\alpha} BS^3 \vee BS^3 \xrightarrow{\nabla} BS^3$,

where ϕ is the map collapsing the equator $S^{10}\times\{1/2\}$ in $V_{7,2}=M^6 \cup CS^{10}$. Then we have

(4.1.5) the principal bundle $S^3 \xrightarrow{i} G_{2,b} \longrightarrow V_{7,2}$ with classifying map f_b in (4.1.4)

(e.g., $G_{2,0}\simeq G_2$), and Mimura-Nishida-Toda [17, §§5–6] proved the following

(4.1.6) $G_{2,b}$ *is a* 1*-connected H-complex of type* (3,11) *so that the inclusion* $S^3 \subset G_{2,b}$ *is an H-map with respect to the usual multiplication on* S^3.

In fact, consider the collection P_1 of all primes $\neq 3, 5$. Then, there are a P_1-equivalence $h_1\colon G_2 \to G_{2,b}$ and a $\{3, 5\}$-equivalence $h_2\colon E_b \to G_{2,b}$ such that

$h_j i \sim i$ (i: the inclusion), where E_b is the S^3-bundle over S^{11} induced by a $\{3, 5\}$-equivalence $S^{11} \to V_{7,2}$ from (4.1.5). There are also p-equivalences h_3: $E_b \to G_2$ or h_4: $S^3 \times S^{11} \to E_b$ for $p=3, 5$ such that $h_j i \sim i$. These h_j induce a multiplication on $G_{2,b}$ so that i is an H-map by [16], since i: $S^3 \to G_2$ and $i_{(p)}$: $S^3_{(p)} \to S^3_{(p)} \times S^{11}_{(p)}$, for odd prime p, are H-maps with respect to the usual multiplication on S^3.

Furthermore, they proved the following

(4.1.7) ([17, Th. 5.1]) *Let X be a* 1*-connected H-complex of rank* 2 *such that $H_*(X; Z)$ has a* 2*-torsion. Then X is homotopy equivalent to $G_{2,b}$ for some b; and there are just* 8 *homotopy types of such H-complexes*: *$G_{2,b}$ for* $-2 \leqq b \leqq 5$.

By the results obtained in [17], $G_{2,b}$ satisfies the following properties:

(4.2.1) $H^*(G_{2,b}; Z_2) = Z_2[x_3]/(x_3^4) \otimes \Lambda(x_5)$, $Sq^2 x_3 = x_5$, $Sq^4 x_5 = 0$ ($\deg x_i = i$),

$H^*(G_{2,b}; Z_p) = \Lambda(y_3, y_{11})$ *for each odd prime p* ($\deg y_i = i$).

(4.2.2) *$G_{2,b}$ has a cell structure given by*

$$G_{2,b} \simeq X = S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14} \quad (-2 \leqq b \leqq 5).$$

(4.2.3) *For the n-skeleton X^n of this H-complex X, $X^9 \simeq (G_2)^9$ in* (4.1.3) *and*

$$X^5 = S^3 \cup_{\eta_3} e^5 \quad (\eta_n \in \pi_{n+1}(S^n) = Z_2 \text{ is a generator, } n \geqq 3),$$

$$X^6/S^3 = M^6, \quad X^9/X^6 = M^9 \quad (M^{n+1} = S^n \cup_2 e^{n+1},\ 2 = 2\iota_n \in \pi_n(S^n)),$$

$$X^{11} = X^9 \cup_{\omega(b)} e^{11} \quad (\omega(b) = (1+8b)\omega \in \pi_{10}(X^9) = \pi_{10}((G_2)^9) = Z_{120}).$$

(4.2.4) ([18, Lemma 3.3]) $\pi_n = \pi_n(X) = \pi_n(G_{2,b})$ ($n \leqq 14$) *is* 0 *except for*

$$\pi_3 = Z, \quad \pi_6 = Z_3, \quad \pi_8 = Z_2, \quad \pi_9 = Z_6, \quad \pi_{11} = Z \oplus Z_2,$$

$$\pi_{10} = \begin{cases} Z_{15}\ (b=-2), \\ Z_3\ \ (b=1,4), \\ Z_5\ (b=3), \end{cases} \qquad \pi_{13} = Z_3\ (b=-2,1,4), \qquad \pi_{14} = Z_{168} \oplus \begin{cases} Z_6\ (b=-2,1,4), \\ Z_2\ (b=-1,0,2,3,5). \end{cases}$$

In the rest of this paper, we study the group $\mathscr{E}_H(X) = \mathscr{E}_H(G_{2,b})$ of self H-equivalences of the H-complex $X \simeq G_{2,b}$ in (4.2.2), by applying Corollary 3.7 and by using some results obtained in the previous paper [18], where the group $\mathscr{E}(X) = \mathscr{E}(G_{2,b})$ of self equivalences is determined up to extension (we notice that S. Oka [20, Th. 9.4] has determined it in case $b \neq -2$).

In this section, we prepare some results on the cohomology of the Postnikov system

(4.3.1) $\{X_n, f_n\colon X \to X_n,\ p_n\colon X_n \to X_{n-1},\ k^{n+1} \in PH^{n+1}(X_{n-1};\ \pi_n)\}$

$(\pi_n = \pi_n(X) = \pi_n(G_{2,b}))$

of the H-complex $X \simeq G_{2,b}$ in (4.2.2), (cf. (3.6.1)).

In the first place, we have the following lemma on the induced homomorphism

(4.3.2) $p_n^*\colon H^n(X_{n-1};\pi_n) \to H^n(X_n;\pi_n)$ of p_n in (4.3.1).

LEMMA 4.4. (i) $H^n(X_n;\pi_n)=0$ *if* $4 \leqq n \leqq 13$ *and* $n \neq 8$, 9 *and* 11.

(ii) *If* $n=8$, 9 *and* 14, *then* p_n^* *is isomorphic and*

$$H^n(X_n;\pi_n) \cong H^n(X_n;Z_2) = Z_2\ (n=8,9),\quad H^{14}(X_{14};\pi_{14}) = \pi_{14}.$$

(iii) *If* $n=11$, *then* $H^{11}(X_{11};\pi_{11})=\pi_{11}=Z\oplus Z_2$,

$$H^{11}(X_{10};\pi_{11}) \cong H^{11}(X_{10};Z_2) = Z_2 \text{ by } \iota_* \text{ where } \iota\colon Z_2 \subset Z \oplus Z_2 = \pi_{11},$$

and $p_{11}^*\colon Z_2 \to Z\oplus Z_2$ *is equal to the inclusion* ι.

PROOF. Since $p_n f_n = f_{n-1}$ in $[X, X_{n-1}]$, we have

(4.4.1) $$f_{n-1}^* = f_n^* p_n^*\colon H^n(X_{n-1};\pi_n) \xrightarrow{p_n^*} H^n(X_n;\pi_n) \underset{\cong}{\xrightarrow{f_n^*}} H^n(X;\pi_n),$$

where f_n^* is isomorphic because f_n is $(n+1)$-connected.

(i) follows immediately from the cell structure of X in (4.2.2), $\pi_5=0$ and $\pi_6=Z_3$ in (4.2.4) and $H^6(X;Z_3)=0$ in (4.2.1).

(ii) We notice that X^m is 2-connected by (4.2.2) and (X^n, X^m) is m-connected for $m<n$. Therefore by the Blakers-Massey theorem, $\pi_i(X^n, X^m) \cong \pi_i(X^n/X^m)$ if $i \leqq m+2$, and it holds the exact sequence

(4.4.2) $$\pi_i(X^m) \to \pi_i(X^n) \to \pi_i(X^n/X^m) \to \pi_{i-1}(X^m) \to \cdots \quad \text{for } i \leqq m+2.$$

Since $X^9/X^6 = M^9 = S^8 \cup_2 e^9$ by (4.2.3), we have the exact sequence

(4.4.3) $$\pi_8(X^6) \to \pi_8(X^9)(=\pi_8) \to \pi_8(M^9)(=Z_2) \to \pi_7(X^6) \to \pi_7(X^9)(=\pi_7),$$

where $\pi_7=0$, $\pi_8=Z_2$ by (4.2.4), and $\pi_7(X^6)=Z_2$ by [18, Lemma 3.7]. Therefore,

(4.4.4) $j_{6*}\colon \pi_8(X^6) \to \pi_8(X)$ $(=Z_2)$ *is epimorphic, where* $j_6\colon X^6 \subset X$.

This and the definition (1.3.2) of X_n imply that $(X_7)^9 = X^9 \cup e_1^9$ where e_1^9 is attached to X^6. Thus

(4.4.5) $f_{7*}\colon H_8(X) \cong H_8(X_7)$, *where* $H_*(\ ) = H_*(\ ;Z)$.

Furthermore $f_{n-1*}\colon H_{n-1}(X) \cong H_{n-1}(X_{n-1})$, and

(4.4.6) $H_7(X)=0$, $H_8(X)=Z_2$, $H_9(X)=0=H_9(X_8)$ (by (4.2.2–3) and (1.3.2)).

Therefore, for $n=8$ and 9, we see that f_{n-1}^* in (4.4.1) is isomorphic and (ii) holds since $\pi_8=Z_2$ and $\pi_9=Z_6$.

Since $X=X^{11}\cup e^{14}$ by (4.2.2), we have the exact sequence $\pi_{14}(X^{11})\to \pi_{14}(X)\to\pi_{14}(S^{14})(=Z)$ by (4.4.2), which implies that

(4.4.7) $j_{11*}\colon \pi_{14}(X^{11})\to\pi_{14}(X)(=\pi_{14}$ *in* (4.2.5)) *is epimorphic* (since π_{14} is finite).

Therefore, we have samely $f_{13*}\colon H_{14}(X)(=Z)\cong H_{14}(X_{13})$ and (ii) for $n=14$.

(iii) Consider the exact sequence

(4.5.1) $\pi_{11}(X^9)\xrightarrow{j_{9*}}\pi_{11}(X)(=\pi_{11})\xrightarrow{p_*}\pi_{11}(X/X^9)(\cong\pi_{11}(S^{11})=Z)\xrightarrow{\partial}\pi_{10}(X^9)$

of (4.4.2), where $j_9\colon X^9\subset X$. Then (4.1.3) and $X^{11}=X^9\cup_{\omega(b)}e^{11}$ in (4.2.3) show that

(4.5.2) $\pi_{10}(X^9)=Z_{120}$ *and* $\operatorname{Im}\partial$ *are generated by* ω *and* $(1+8b)\omega$, *respectively.*

Therefore,

(4.5.3) $\operatorname{Im}p_*=\operatorname{Ker}\partial=m_bZ$, *where* $m_b=120/(|1+8b|, 120)$, *and*

(4.5.4) $\operatorname{Im}j_{9*}=\operatorname{Ker}p_*=Z_2\subset Z\oplus Z_2=\pi_{11}$ (cf. (4.2.4)).

Thus, by (4.2.2) and the definition (1.3.2) of X_{10}, we have $X^{12}=X^9\cup e^{11}$ and

(4.5.5) $(X_{10})^{12}=X^9\cup e^{11}\cup e_1^{12}\cup e_2^{12}$ *with* $\partial e_1^{12}=m_be^{11}$, $\partial e_2^{12}=0$ *in the chain complex.*

Therefore $f_{10*}\colon H_{11}(X)=Z\to H_{11}(X_{10})=Z_{m_b}$ is epimorphic, and we see (iii) by (4.4.1) and by noticing that m_b in (4.5.3) is a non-zero even integer. q.e.d.

On the subgroup $PH^n(X_n;\pi)$ consisting of primitive elements, we have the following

LEMMA 4.6. *$PH^n(X_n;\pi_n)=0$ if $n=8$, 9, 14, and $PH^{11}(X_{11};Z_2)=0$.*

PROOF. By Lemma 4.4 (ii) and (4.2.1), $H^n(X_n;\pi_n)\cong H^n(X;Z_2)=Z_2$ $(n=8,9)$ and $H^{11}(X_{11};Z_2)\cong H^{11}(X;Z_2)=Z_2$ are generated by x_3x_5, x_3^3 and $x_3^2x_5$, respectively. We see easily that these elements are not primitive by definition, and the lemma holds for $n=8$, 9 and 11.

To show the lemma for $n=14$, it is sufficient to prove that

(4.6.1) $PH^{14}(X_{14};Z_q)\cong PH^{14}(X;Z_q)=0$ *for* $q=2$, 3, 7 *and* 8,

by (4.2.4) for π_{14}. When q is a prime, (4.2.1) shows that $H^{14}(X;Z_q)=Z_q$ is generated by $x_3^3x_5$ if $q=2$ and by y_3y_{11} if $q\neq 2$, which are not primitive. Thus (4.6.1) holds for $q=2$, 3 and 7.

$H^{14}(X; Z)=Z$ is generated by z_3z_{11} where $z_i\in H^i(X; Z)=Z$ $(i=3, 11)$ is a generator by (4.2.1). Therefore, by considering the reduction mod 8, we see that $H^{14}(X; Z_8)=Z_8$ is generated by u_3u_{11} where $u_i\in H^i(X; Z_8)=Z_8$ $(i=3, 11)$ is a generator. Suppose that $u=\ell u_3u_{11}$ is primitive. Then its reduction mod 2 is also primitive and hence is 0 by (4.6.1) for $q=2$. Thus $\ell=2\ell'$. Furthermore, we see that $2H^i(X; Z_8)=0$ if $4\leqq i\leqq 10$ by (4.2.2–3). Hence, for the i-th projections p_i: $X\times X\to X$ $(i=1, 2)$,

$$\begin{aligned} p_1^*u+p_2^*u &= m^*(u)=m^*(\ell'u_3)m^*(2u_{11})=\ell'(p_1^*u_3+p_2^*u_3)(2p_1^*u_{11}+2p_2^*u_{11}) \\ &= p_1^*u+p_2^*u+\ell(p_1^*u_3\cdot p_2^*u_{11}+p_2^*u_3\cdot p_1^*u_{11}) \quad \text{in } H^{14}(X\times X; Z_8), \end{aligned}$$

which shows $\ell\equiv 0$ mod 8. Thus (4.6.1) holds for $q=8$. q.e.d.

§5. The triviality of self H-equivalences of $G_{2,b}$

We now study the group $\mathscr{E}_H(X)=\mathscr{E}_H(G_{2,b})$ of self H-equivalences of the H-complex $X\simeq G_{2,b}$ in (4.2.2). The notations given in §4 are used continuously.

By the cell structure of X in (4.2.2), Proposition 1.4 and (1.7.3) show the following

LEMMA 5.1. (i) *f_nj_n: $X^n\subset X\to X_n$ induces the isomorphism*

$$(f_nj_n)^!: \mathscr{E}(X_n)\cong\mathscr{E}(X^n) \qquad \textit{for} \quad n=3, 6, 9, 11, 12 \textit{ and } 14.$$

(ii) *The induced homomorphism $\tilde{\Phi}_n=f_{n!}$: $\mathscr{E}_H(X)(=\mathscr{E}_H(G_{2,b}))\to\mathscr{E}_H(X_n)$ in (3.6.5) is monomorphic if $n\geqq 14$ and isomorphic if $n\geqq 28$.*

We investigate the group $\mathscr{E}_H(X_n)$ by using Corollary 3.7. Consider the exact sequence

(5.2.1) $\qquad 0\to\tilde{H}_n\to\mathscr{E}_H(X_n)\to\tilde{G}_n\to 1$ $(n\geqq 3)$ in (3.7.2) for $X\simeq G_{2,b}$,

and the diagram

(5.2.2)
$$\begin{array}{ccccccc} [X_n, \Omega X_{n-1}] & & H^n(X_{n-1}; \pi_n) & & H^{11}(X_n\wedge X_n; Z_2) & & [X_n\wedge X_n, X_n] \\ \big\downarrow{\scriptstyle (\Omega k^{n+1})_*} & & \big\downarrow{\scriptstyle p_n^*} & & \big\downarrow{\scriptstyle \iota_*(n=11)} & & \big\uparrow{\scriptstyle i_{n*}} \\ [X_n, K(\pi_n, n)] & = & H^n(X_n; \pi_n) & \xrightarrow{\phi} & H^n(X_n\wedge X_n; \pi_n) & = & [X_n\wedge X_n, K(\pi_n, n)] \end{array}$$

of (3.7.1), (3.7.4) and ι_* for $n=11$, where ι: $Z_2\subset Z\oplus Z_2=\pi_{11}$ (cf. (4.2.4)). Then we have the following assertion, which will be proved in §§6–7:

ASSERTION 5.3. *In (5.2.2), i_{n*} $(n=8, 9, 14)$ and $i_{11*}\iota_*$ are monomorphic.*

By this assertion, we see the following

LEMMA 5.4. *Let* $4 \leqq n \leqq 14$. *Then* $\mathrm{Im}(\phi p_n^*) \cap \mathrm{Ker}\, i_{n*} = 0$ *in* (5.2.2) *and* $\tilde{H}_n = 0$ *in* (5.2.1).

PROOF. Lemma 4.4 (i), (iii) and the above assertion imply the first equality which assures the assumption (3.7.5). Thus $\tilde{H}_n$ is the quotient group of $p_n^* \cdot (PH^n(X_{n-1}; \pi_n))$ by Corollary 3.7 (iii), and we see that $\tilde{H}_n = 0$ by Lemmas 4.4 (i), (iii) and 4.6. q. e. d.

Furthermore, by using some results obtained in [18], we can prove the following

LEMMA 5.5. $\tilde{\rho}\colon \tilde{G}_n \to \rho(\tilde{G}_n)(\subset \mathscr{E}_H(X_{n-1}))$ *in* (3.7.7) *is isomorphic for* $4 \leqq n \leqq 14$.

PROOF. When $4 \leqq n \leqq 14$ and $n \neq 11$, the lemma is seen immediately from Corollary 3.7 (iv) and Lemma 4.4 (i)–(ii). To show the lemma for $n = 11$, consider the commutative diagram

$$
\begin{array}{ccccccccc}
H_{11} & \xrightarrow{\kappa} & \mathscr{E}(X_{11}) & \xrightarrow{(\varphi_{11}, \psi_{11})} & G_{11} & \xrightarrow{\rho} & \rho(G_{11}) \subset \mathscr{E}(X_{10}) & \xrightarrow{\varphi_{10}} & \mathscr{E}(X_9) \\
 & & \downarrow \cong & & & & & & \downarrow \cong \\
 & & \mathscr{E}(X^{11}) & & & \xrightarrow{\qquad\qquad j^1 \qquad\qquad} & & & \mathscr{E}(X^9),
\end{array} \tag{5.5.1}
$$

where the two vertical isomorphisms are the ones in Lemma 5.1 (i), the homomorphism j^1 induced from $j\colon X^9 \subset X^{11}$ is defined by Proposition 1.4 (i) and (4.2.2), the upper homomorphisms are the ones in (3.7.2) and (3.7.7), and the commutativity is seen by the definition (1.2.1–2) and $p_n f_n = f_{n-1}$ in $[X, X_{n-1}]$ (cf. (3.6.3)). Then,

(5.5.2) $\quad \mathrm{Ker}\, j^1 = Z_2 \quad$ (by [18, Proof of Lemma 4.2]).

Furthermore, $H_{11} = \mathrm{Im}\, p_{11}^* / \mathrm{Im}\,(\Omega k^{12})_*$ (see (3.7.3)) is Z_2 because $\mathrm{Im}\, p_{11}^* = Z_2$ by Lemma 4.4 (iii) and $\mathrm{Im}\,(\Omega k^{12})_* = 0$ by $X^{11} = X^9 \cup_{\omega(b)} e^{11}$ in (4.2.3), Lemma 3.9 and [18, Lemma 3.11]. Thus

(5.5.3) $\quad G_{11} \cong \mathscr{E}(X_{11}) / \mathrm{Im}\, \kappa, \quad \mathrm{Im}\, \kappa \cong H_{11} = Z_2$ (by the exactness of (3.7.2)).

These imply that the epimorphism $\rho\colon G_{11} \to \rho(G_{11})$ in the commutative diagram (5.5.1) is isomorphic, and so is its restriction $\tilde{\rho}\colon \tilde{G}_{11} \to \rho(\tilde{G}_{11})$. q. e. d.

By the above two lemmas, we have the following

PROPOSITION 5.6. *For* $X \simeq G_{2,b}$ *(with any multiplication),* $\tilde{\Phi}_3 = f_{3!}\colon \mathscr{E}_H(X) \to \mathscr{E}_H(X_3)$ *in* (3.6.5) *is monomorphic, where* $X_3 = K(\pi_3, 3)$, $\pi_3 = Z$ *and* $\mathscr{E}_H(X_3) = \mathscr{E}_H(K(Z, 3)) = Z_2$. *Thus the group* $\mathscr{E}_H(G_{2,b})$ *is trivial or* Z_2.

Now, to prove Theorem II in the introduction, we notice the following

LEMMA 5.7. *The inclusion* $j_3: S^3 \subset X(\simeq G_{2,b})$ *induces the epimorphism*

$$j_{3*}: \pi_6(S^3)(=Z_{12}) \to \pi_6(X)(=Z_3) \quad \text{(cf. (4.2.4))}.$$

PROOF. Consider the exact sequence $\pi_6(S^3) \to \pi_6(X^6)$ $(=\pi_6(X)=Z_3) \to$ $\pi_6(M^6)$ of (4.4.2) for $(X^7, X^4)=(X^6, S^3)$ with $X^6/S^3=M^6=S^5 \cup_2 e^6$ (cf. (4.2.3)). Then $\pi_6(M^6)=Z_2$ and we see the lemma. q.e.d.

Consider the usual multiplication $\bar{m}: S^3 \times S^3 \to S^3$, $\bar{m}(x, y)=xy$ (the product of unit quaternions x and y). Then, we have the following

THEOREM 5.8. *The group* $\mathscr{E}_H(G_{2,b})$ *is trivial for the H-space* $G_{2,b}$ $(-2 \leqq b \leqq 5)$ *such that the inclusion* $j_3: S^3 \subset G_{2,b}$ *is an H-map with respect to the usual multiplication* $\bar{m}$ *on* S^3 (cf. (4.1.6)).

PROOF. Contrary to the theorem, suppose that $\mathscr{E}_H(X) \neq 1$ for $X \simeq G_{2,b}$, where

(5.8.1) *the inclusion* $j_3: (S^3, \bar{m}) \to (X, m)$ *is an H-map, i.e.,* $j_3\bar{m} \sim m(j_3 \times j_3)$: $S^3 \times S^3 \to X$.

Then, by Proposition 5.6 and the definition of $\Phi_3 = j_{3!}$, we see that

(5.8.2) *there is* $h \in \mathscr{E}_H(X)$ *with* $\Phi_3(h)=-1$ *in* $\mathscr{E}_H(X_3)=Z_2$, *i.e.,* $h_*=-1$: $\pi_3(X) \to \pi_3(X)$ $(=Z)$.

Consider the homeomorphism $\sigma: S^3 \to S^3$, $\sigma(x)=x^{-1}$ (the inverse of a unit quaternion x). Then $\sigma_*=-1: \pi_3(S^3) \to \pi_3(S^3)$, and by (5.8.1–2), we see the following

(5.8.3) $h: X \to X$ *satisfies* $hm \to m(h \times h): X \times X \to X$ *and* $hj_3\sigma \sim j_3: S^3 \to X$.

(5.8.4) *The maps* $\bar{m}$, $\bar{m}T: S^3 \times S^3 \to S^3$ $(T(x, y)=(y, x))$ *satisfies* $\bar{m}=\bar{m}T$ *on* $S^3 \vee S^3$ *and*

$$\begin{aligned} j_3\bar{m} &= m(hj_3\sigma \times hj_3\sigma) = hm(j_3\sigma \times j_3\sigma) = hj_3\bar{m}(\sigma \times \sigma) \\ &= j_3\sigma\bar{m}(\sigma \times \sigma) = j_3\bar{m}T \quad \text{in} \quad [S^3 \times S^3, X], \end{aligned}$$

i.e., the separation element $d=d(\bar{m}, \bar{m}T) \in \pi_6(S^3)$ *satisfies* $j_3d=0$ *in* $\pi_6(X)$ (cf. (1.5.4)).

On the other hand, since $\bar{m}$ is the usual multiplication on S^3,

(5.8.5) ([9, p. 176]) $\pi_6(S^3)=Z_{12}$ *is generated by* $d=d(\bar{m}, \bar{m}T)$ *in* (5.8.4).

Thus, $j_3d=0$ in (5.8.4) contradicts Lemma 5.7; and we see the theorem. q.e.d.

By this theorem, Theorem II in the introduction is proved except for the proof of Assertion 5.3.

§6. Proof of Assertion 5.3 for $n=8$, 9 and 11

To prove Assertion 5.3, consider the exact sequence

$$(6.1)\qquad [Y, \Omega X_n] \xrightarrow{(\Omega p_n)_*} [Y, \Omega X_{n-1}] \xrightarrow{(\Omega k^{n+1})_*} [Y, K(\pi_n, n)]$$
$$(=H^n(Y; \pi_n)) \xrightarrow{i_{n*}} [Y, X_n] \xrightarrow{p_{n*}} [Y, X_{n-1}]$$

of the fiber sequence (3.6.4) for $X \simeq G_{2,b}$. Then

LEMMA 6.2. *If $n=8$, 9, then $[X_n \wedge X_n, \Omega X_{n-1}]=0$ and Assertion 5.3 holds.*

PROOF. Since $f_n: X \to X_n$ is $(n+1)$-connected, (1.1.6–7) and (4.2.2) imply that

(6.2.1) *$h \wedge h: X^m \wedge X^m \to X_n \wedge X_n$ is $(m+3)$-connected, where $h=f_n j_m: X^m \subset X \to X_n$ $(m \leqq n+1)$.*

Therefore, by (1.1.1) and $\pi_i(\Omega X_{n-1})=\pi_{i+1}(X_{n-1})=0$ $(i \geqq n-1)$, we see that

$$(6.2.2)\qquad (h \wedge h)^*: [X_n \wedge X_n, \Omega X_{n-1}] \cong [X^{n-4} \wedge X^{n-4}, \Omega X_{n-1}]$$
$$(h = f_n j_{n-4}: X^{n-4} \subset X \to X_n).$$

When $n=8$, $X^4=S^3$ by (4.2.2) and $\pi_6(\Omega X_7)=\pi_7(X)=0$ by (4.2.4). Thus $[X_8 \wedge X_8, \Omega X_7]=0$.

When $n=9$, $X^5=S^3 \cup e^5$ by (4.2.2) and $X^5 \wedge X^5/S^3 \wedge S^3$ is 7-connected. Therefore, in the Puppe exact sequence

$$[X^5 \wedge X^5/S^3 \wedge S^3, \ \Omega X_8] \to [X^5 \wedge X^5, \ \Omega X_8] \to [S^3 \wedge S^3, \ \Omega X_8] \ (=\pi_7(X)=0),$$

the first term is 0 by (1.1.2). Thus $[X^5 \wedge X^5, \Omega X_8]=0$, and we see the lemma by (6.2.2) and (6.1). q.e.d.

We now study the case $n=11$. Consider the cofiber sequence

(6.3.1) $S^3 \xrightarrow{j} X^6 \xrightarrow{p} M^6 \xrightarrow{g} S^4 \xrightarrow{Sj} SX^6 \longrightarrow \cdots$ of $X^6/S^3=M^6=S^5 \cup_2 e^6$ in (4.2.3).

Then, because $X^5=S^3 \cup_{\eta_3} e^5$ by (4.2.3), we see that

(6.3.2) *$g: M^6 \to S^4$ in (6.3.1) is an extension* $\mathrm{ext}\,\eta_4$ *of $\eta_4=S\eta_3 \in \pi_5(S^4)$.*

The cofiber sequence obtained from (6.3.1) by smashing Y induces the Puppe exact sequence

$$(6.3.3)\qquad [Y \wedge S^3, W] \xleftarrow{(1 \wedge j)^*} [Y \wedge X^6, W] \xleftarrow{(1 \wedge p)^*}$$
$$[Y \wedge M^6, W] \xleftarrow{(1 \wedge g)^*} [Y \wedge S^4, W] \leftarrow \cdots.$$

The following (6.3.4) is proved in [18, Lemmas 3.2–3 and 3.5]:

(6.3.4) $\pi_8(X)=Z_2$ *is generated by* ρ_8 $(=\langle\eta_6^2\rangle)$, $\rho_8\eta_8\in\pi_9(X)=Z_6$ *is of order* 2, *and*

$[M^{10}, X]=Z_2$ *is generated by an extension* $\mathrm{ext}(\rho_8\eta_8)$ *of* $\rho_8\eta_8$.

LEMMA 6.4. (i) $(S^4g)^*\colon \pi_8(X)\to[M^{10}, X]$ *is isomorphic* $(M^{n+6}=S^nM^6)$.
(ii) $[M^{11}, W]$ *and* $[M^6\wedge M^6, W]$ *are trivial for* $W=X_9$, X_{10} *and* ΩX_{10}.
(iii) $[S^4\wedge X^6, W]$ *are trivial for* $W=X_n$ $(n\geqq 10)$ *and* ΩX_{10}.

PROOF. (i) (6.3.2) shows that $(S^4g)^*\rho_8=\mathrm{ext}(\rho_8\eta_8)$. Thus (6.3.4) implies (i).

(ii) For $W=X_9$ and ΩX_{10}, (ii) follows from (1.1.2) and (1.3.1), since M^{11} and $M^6\wedge M^6$ are 9-connected. (ii) for $W=X_{10}$ is seen by the exact sequence

$$H^{10}(Y;\pi_{10})\to[Y, X_{10}]\to[Y, X_9](=0)\ \text{in (6.1) for}\ Y=M^{11}, M^6\wedge M^6,$$

where the first term is 0 since $M^n=S^{n-1}\cup_2 e^n$ and $\pi_{10}=Z_3, Z_5, Z_{15}$ or 0 by (4.2.4).

(iii) The exact sequence (6.3.3) for $Y=S^4$ implies (iii) by (i) and (ii), because $\pi_7(X)=0$ by (4.2.4), $f_{n*}\colon[Y, X]\cong[Y, X_n]$ if $\dim Y\leqq n$ by (1.1.3) and (1.3.2), and $[Y, \Omega W]=[SY, W]$. q.e.d.

Denoting simply by $(Y)^{\wedge 2}=Y\wedge Y$, we consider the commutative diagrams

(6.5.1)

$$\begin{tikzcd}
{[(X_{11})^{\wedge 2}, \Omega X_{11}]} \arrow[r, "\bar{h}^*"] \arrow[d, "p'_*"] & {[(X^6)^{\wedge 2}, \Omega X_{11}]} \arrow[d, "p'_*"] & {[M^6\wedge X^6, \Omega X_{11}]} \arrow[l, "(p\wedge 1)^*"', "\mathrm{epi}"] \arrow[r, "(1\wedge j)^*"] \arrow[d, "p'_*"] & {[M^9, \Omega X_{11}]} \arrow[d, "p'_*"] \\
{[(X_{11})^{\wedge 2}, \Omega X_{10}]} \arrow[r, "\bar{h}^*", "\cong"'] \arrow[d, "k'_*"] & {[(X^6)^{\wedge 2}, \Omega X_{10}]} \arrow[d, "k'_*"] & {[M^6\wedge X^6, \Omega X_{10}]} \arrow[l, "(p\wedge 1)^*"', "\cong"] \arrow[r, "(1\wedge j)^*", "\cong"'] \arrow[d, "k'_*"] & {[M^9, \Omega X_{10}]} \\
H^{11}((X_{11})^{\wedge 2};\pi_{11}) \arrow[r, "\bar{h}^*"] & H^{11}((X^6)^{\wedge 2};\pi_{11}) & H^{11}(M^6\wedge X^6;\pi_{11}), \arrow[l, "(p\wedge 1)^*"', "\cong"] &
\end{tikzcd}$$

(6.5.2)

$$\begin{tikzcd}
H^{11}(M^6\wedge X^6;\pi_{11}) \arrow[d, "i_{11*}"] & H^{11}((M^6)^{\wedge 2};\pi_{11}) \arrow[l, "(1\wedge p)^*"', "\cong"] \arrow[r, "(i\wedge 1)^*"] \arrow[d, "\cong"', "i_{11*}"] & H^{11}(M^{11};\pi_{11}) \arrow[d, "\cong"', "i_{11*}"] & H^{11}(S^{11};\pi_{11}) \arrow[l, "q^*"', "\mathrm{epi}"] \arrow[d, "\cong"', "i_{11*}"] \\
{[M^6\wedge X^6, X_{11}]} & {[(M^6)^{\wedge 2}, X_{11}]} \arrow[l, "(1\wedge p)^*"'] \arrow[r, "(i\wedge 1)^*"] & {[M^{11}, X_{11}]} & \pi_{11}(X_{11}) \arrow[l, "q^*"'] \arrow[d, equal] \\
{[M^6\wedge SX^6, X_{11}]} \arrow[r, "(1\wedge Sj)^*"] & {[M^{10}, X_{11}](=Z_2)} \arrow[u, "(1\wedge g)^*"] \arrow[r, "i^*"] & \pi_9(X_{11}) \arrow[u, "(S^5g)^*"] & \pi_{11}(=Z\oplus Z_2, \text{see (4.2.4)}),
\end{tikzcd}$$

where $\bar{h}=h\wedge h$, $h=f_{11}j_6\colon X^6\subset X\to X_{11}$, $p'=\Omega p_{11}$, $k'=\Omega k^{12}$ and

(6.5.3) the vertical sequences in (6.5.1) continued to i_{11*} in (6.5.2) are the ones in (6.1), (6.5.4) j, p, g are the maps in (6.3.1) and $(1\wedge Sj)^*$, $(1\wedge g)^*$, $(1\wedge p)^*$ in (6.5.2) form the exact sequence (6.3.3),

(6.5.5) $S^n \xrightarrow{i} M^{n+1} (=S^n \cup_2 e^{n+1}) \xrightarrow{q} S^{n+1}$ is the cofibering, and

(6.5.6) $[M^{10}, X_{11}] \cong [M^{10}, X] = Z_2$ is generated by ext $(\rho_8\eta_8)$ and i^* ext $(\rho_8\eta_8) = \rho_8\eta_8$ (cf. (6.3.4)).

LEMMA 6.6. (i) *In (6.5.1–2), the homomorphisms indicated by* epi *or* $\cong$ *are epimorphic or isomorphic, respectively, and so are the ones on the cohomology for any coefficients instead of* π_{11}.

(ii) $[M^{11}, X_{11}] = Z_2 \oplus Z_2$ *and* $q^*: \pi_{11}(X_{11}) \to [M^{11}, X_{11}]$ *is epimorphic.*

(iii) $(i\wedge 1)^*(1\wedge g)^* \operatorname{ext}(\rho_8\eta_8) = (S^5 g)^*(\rho_8\eta_8)$ *is not contained in* $q^*(Z_2)$ $(\subset [M^{11}, X_{11}])$.

PROOF. (i) is proved for $\bar{h}^*$ by (6.2.2) and $X^7 = X^6$ in (4.2.2), for q^* by the Puppe exact sequence

(6.6.1) $$\pi_{n+1}(W) \xrightarrow{\times 2} \pi_{n+1}(W) \xrightarrow{q^*} [M^{n+1}, W] \xrightarrow{i^*} \pi_n(W) \xrightarrow{\times 2} \pi_n(W)$$

(of the cofibering in (6.5.5)) with $n=10$ and $W=K(\pi, 11)$, and for the others by the exact sequences (6.3.3), (6.1) and Lemma 6.4 (ii)–(iii).

(ii) is proved by the exact sequence (6.6.1) for $n=10$, $W=X_{11}$ and by (1.3.1) and (4.2.4).

(iii) Consider the commutative diagram ($j_9: X^9 \subset X$, $p: X \to X/X^9$ is the collapsing map)

(6.6.2)

$$\begin{array}{ccccc}
\pi_{11}(X^9) & \xrightarrow{j_{9*}} & \pi_{11}(X)\ (\cong \pi_{11}(X_{11}) = Z \oplus Z_2) & \xrightarrow{p_*} & \pi_{11}(X/X^9)\ (\cong \pi_{11}(S^{11}) = Z) \\
\downarrow q^* & & \downarrow q^* & & \downarrow q^* \\
[M^{11}, X^9] & \xrightarrow{j_{9*}} & [M^{11}, X](\cong [M^{11}, X_{11}] = Z_2 \oplus Z_2) & \xrightarrow{p_*} & [M^{11}, X/X^9](\cong [M^{11}, S^{11}] = Z_2) \\
\downarrow i^* & & & & \\
\pi_{10}(X^9)(= Z_{120} \text{ generated by } \omega, \text{ see (4.2.3) and (4.1.3)}), & & & &
\end{array}$$

where the left and upper sequences are the ones in (6.6.1) and (4.5.1), respectively, and the lower one is also exact by [7, Lemma 3.1] and (1.1.3). Then, by the exact sequence (6.6.1), we see that

(6.6.3) i^* *induces* $[M^{11}, X^9]/q^*\pi_{11}(X^9) \cong i^*[M^{11}, X^9] = Z_2$ $(\subset Z_{120})$, *and* $[M^{11}, S^{11}] = Z_2$.

The latter and (4.5.3) (where m_b is even) show that $p_* q^* = q^* p_* = 0$. Thus,

(6.6.4) *the lower* p_* *is trivial by* (ii) *and* $j_{9*}: [M^{11}, X^9] \to [M^{11}, X] = Z_2 \oplus Z_2$ *is epimorphic.*

Therefore, there is $\alpha_0 \in [M^{11}, X^9]$ with $j_{9*}\alpha_0 \notin q^*(Z_2)$, which satisfies $\alpha_0 \notin q^*\pi_{11}(X^9)$ since $j_{9*}\pi_{11}(X^9) = Z_2$ by (4.5.4). Thus,

(6.6.5) *if* $\alpha \in [M^{11}, X^9]$ *satisfies* $i^*\alpha = 60\omega \in \pi_{10}(X^9)$, *then* $\alpha = \alpha_0 + q^*\beta$ *for some* $\beta \in \pi_{11}(X^9)$ *by* (6.6.3), *and hence* $j_{9*}\alpha \notin q^*(Z_2)$.

Now, by (6.3.2), we have the commutative diagram

$$\text{(6.6.6)}\quad \begin{array}{ccccccc} \pi_9(X^9) & = & \pi_9(X^9) & \xrightarrow{j_{9*}} & \pi_9(X) & \xleftarrow{\eta_8^*} & \pi_8(X)(=Z_2) \xleftarrow[\cong]{j_{9*}} \pi_8(X^9) \\ \downarrow{\scriptstyle \eta_9^*} & & \downarrow{\scriptstyle (S^5g)^*} & & \downarrow{\scriptstyle (S^5g)^*} & & \\ \pi_{10}(X^9) & \xleftarrow{i^*} & [M^{11}, X^9] & \xrightarrow{j_{9*}} & [M^{11}, X] & \xleftarrow{q^*} & \pi_{11}(X). \end{array}$$

Consider the elements

(6.6.7) $\rho_8' \in \pi_8(X^9)$ with $j_{9*}\rho_8' = \rho_8 \in \pi_8(X)$ in (6.3.4), and $\rho_8'\eta_8 \in \pi_9(X^9)$.

Then, by the commutativity of (6.6.6), $\alpha = (S^5g)^*(\rho_8'\eta_8) \in [M^{11}, X^9]$ satisfies

(6.6.8) $j_{9*}\alpha = (S^5g)^*(\rho_8\eta_8) \in [M^{11}, X](\cong [M^{11}, X_{11}])$, $i^*\alpha = \rho_8'\eta_8\eta_9 \in \pi_{10}(X^9)$.

Therefore, (iii) can be proved by (6.6.5) and by showing the equality

(6.6.9) $\rho_8'\bar{\eta} = 60\omega$ *in* $\pi_{10}(X^9)$ *for the generator* $\bar{\eta} = \eta_8\eta_9 \in \pi_{10}(S^8) = Z_2$.

To show (6.6.9), we notice the following results due to [17, Lemmas 4.1–2 and their proofs]:

(6.6.10) *There are a CW-complex* $K = M^9 \cup CM^{10}$ *and a map* $f: K \to X^9$ $(\simeq (G_2)^9)$ *such that* $f_*: \pi_i(K) \to \pi_i(X^9)$ *is an isomorphism* mod 2 *for* $4 \leqq n \leqq 12$ *and, in the commutative diagram*

$$\text{(6.6.11)}\quad \begin{array}{ccccc} \pi_{10}(M^9)(=Z_4) & \xrightarrow{i_*} & \pi_{10}(K)(=Z_8) & \xrightarrow{f_*} & \pi_{10}(X^9)(=Z_{120}) \\ \uparrow{\scriptstyle \eta^*} & & \uparrow{\scriptstyle \eta^*} & & \uparrow{\scriptstyle \eta^*} \\ \pi_8(M^9)(=Z_2) & \xrightarrow{i_*} & \pi_8(K)(=Z_2) & \xrightarrow{f_*} & \pi_8(X^9)(\cong \pi_8(X) = Z_2) \end{array}$$

$(i: M^9 \subset K)$, *the upper homomorphisms are monomorphic and the lower ones are isomorphic.*

(6.6.10) implies immediately (6.6.9), because $\bar{\eta}^*$ for M^9 in (6.6.11) is known to be monomorphic (cf. Araki-Toda [1, (4.2)]). q.e.d.

By the above lemma, we can prove Assertion 5.3 for $n = 11$.

LEMMA 6.7. *Let* $\iota: Z_2 \subset Z \oplus Z_2 = \pi_{11}$. *Then* $\operatorname{Im} \iota_* \cap \operatorname{Ker} i_{11*} = 0$ *for*

(6.7.1) $H^{11}(X_{11} \wedge X_{11}; Z_2) \xrightarrow{\iota_*} H^{11}(X_{11} \wedge X_{11}; \pi_{11}) \xrightarrow{i_{11*}} [X_{11} \wedge X_{11}, X_{11}]$

in (5.2.2), *and Assertion* 5.3 *holds for* $n=11$.

PROOF. Consider the diagram (6.5.2). Then, Lemma 6.6 (iii) and (6.5.6) imply that

(6.7.2) $(1\wedge g)^*$ *is injective,* $(1\wedge Sj)^*=0$ *and* Ker (*the lower* $(1\wedge p)^*$) $=\mathrm{Im}(1\wedge g)^*=Z_2$ *by* (6.5.4),

(6.7.3) *the lower* $(i\wedge 1)^*$ *maps* $G=\mathrm{Im}(1\wedge g)^*$ *monomorphically and* $(i\wedge 1)^*G\cap q^*(Z_2)=0$, *and hence*

(6.7.4) *so does* $F=(i\wedge 1)^*(i_{11*})^{-1}=(i_{11*})^{-1}(i\wedge 1)^*$ *and* $F(G)\cap\mathrm{Im}(\iota_*: H^{11}(M^{11}; Z_2)\to H^{11}(M^{11}; \pi_{11}))=0$,

by Lemma 6.6 (i) and the naturality of ι_*. Consider also the diagram (6.5.1). Then, the upper $(1\wedge j)^*$ $(=(1\wedge Sj)^*)$ is trivial by (6.7.2), and so are the left three p'_*'s by Lemma 6.6 (i). Thus, (6.5.3) shows that k'_*'s are all monomorphic and

(6.7.5) *the composition* $F'=i_{11*}((p\wedge 1)^*(1\wedge p)^*)^{-1}\bar{h}^*: H^{11}((X_{11})^{\wedge 2}; \pi_{11})\to [(M^6)^{\wedge 2}, X_{11}]$ *in* (6.5.1–2) *maps* $\mathrm{Ker}\, i_{11*}$ *in* (6.7.1) *isomorphically onto* $G=\mathrm{Im}(1\wedge g)^*$ *in* (6.7.2–4); *and hence*

(6.7.6) *the composition* $F''=FF'=(i\wedge 1)^*((p\wedge 1)^*(1\wedge p)^*)^{-1}\bar{h}^*: H^{11}((X_{11})^{\wedge 2}; \pi_{11})\to H^{11}(M^{11}; \pi_{11})$ *in* (6.5.1–2) *maps* $\mathrm{Ker}\, i_{11*}$ *in* (6.7.1) *monomorphically and* $F''(\mathrm{Ker}\, i_{11*})\cap\mathrm{Im}(\iota_*$ *in* (6.7.4)$)=0$.

Therefore, considering F'' in (6.7.6) for the coefficient Z_2 instead of π_{11} by the latter half of Lemma 6.6 (i), we see the lemma by the last equality in (6.7.6) and the naturality of $\iota_*: H^*(\ ; Z_2)\to H^*(\ ; \pi_{11})$. q. e. d.

§7. Proof of Assertion 5.3 for $n=14$

In the first place, we notice the following

LEMMA 7.1. $S^4X^9\simeq S^4X^6\vee M^{13}$ *on the suspension of* $X^9=X^6\cup e^8\cup e^9$ *in* (4.2.3).

PROOF. Since $X^9\simeq(G_2)^9$ by (4.2.3), it is sufficient to prove the lemma for $X=G_2$.

Let $X=G_2$. Then, we have the fiberings (cf. [30, p. 714])

(7.1.1) $S^3 \longrightarrow SU(3)(=S^3\cup e^5\cup e^8)\xrightarrow{\pi} S^5,\quad SU(3)\longrightarrow X(=G_2)\xrightarrow{\bar{\pi}} S^6.$

Consider

(7.1.2) the 8-skeleton $X^8=SU(3)\cup e^6$, the cofibering $SU(3)\to X^8\xrightarrow{\bar{p}} X^8/SU(3)$ $(=S^6)$ and $j_8: X^8\subset X$.

Then, since $\bar{\pi}(SU(3))=*$, we have a map $\varepsilon: S^6(=X^8/SU(3))\to S^6$ such that

$\varepsilon\bar{p}=\bar{\pi}j_8$ in $[X^8, S^6]$. Thus, by noticing that $\bar{p}_*\colon \pi_6(X^8, SU(3))\cong\pi_6(S^6)$, we have the commutative diagram of the exact sequences of the homotopy groups induced by $\bar{p}$ and $\bar{\pi}$ including $\varepsilon_*\colon \pi_6(S^6)\to\pi_6(S^6)$, which shows that ε_* is isomorphic and so $\varepsilon=\pm\iota_6$. Therefore,

(7.1.3) *$\bar{p}f=0$ in $\pi_8(S^6)$, where $f\colon S^8\to X^8$ is the attaching map in $X^9=X^8\cup_f e^9$,*

because $(\pm\iota_6)\bar{p}f=\varepsilon\bar{p}f=\bar{\pi}j_8 f$ in $\pi_8(S^6)$ and $j_8 f=0$ in $\pi_8(X)$. On the other hand, we have

(7.1.4) *$S^4X^8\simeq S^4X^6\vee S^{12}$ where $X^6=X^5\cup e^6=S^3\cup e^5\cup e^6$, and $S^4X^9=S^4X^8\cup_{S^4f}e^{13}$,*

because $S^4SU(3)\simeq S^4X^5\vee S^{12}$ by [15, Lemma 2.1]. Thus, by the exact sequences induced by the cofiberings $S^7\to S^4X^6\xrightarrow{\tilde{p}}M^{10}$ ($\tilde{p}=S^4p$) in (6.3.1) and $S^9\xrightarrow{i}M^{10}\xrightarrow{q}S^{10}$ in (6.5.3), and by using $\pi_{11}(S^7)=0=\pi_{12}(S^7)$ in [29, Prop. 5.8–9], we see that

(7.1.5) *$j_*\colon \pi_{12}(S^4X^6)\to\pi_{12}(S^4X^8)$ (j is the inclusion) is monomorphic,*
$\tilde{p}_*\colon \pi_{12}(S^4X^6)\cong\pi_{12}(M^{10})$ $(=Z_2\oplus Z_2$ generated by $\beta_1=i_*\nu_9$, $\beta_2=$ (coext $\eta_{10})\eta_{11}$, cf. [1, (4.2)]), and
$\pi_{12}(S^4X^8)=Z_2\oplus Z_2\oplus Z$ generated by α_1, α_2 and α $(\alpha_i=j_*\tilde{p}_*^{-1}(\beta_i)$ $(i=1, 2)$, $Z\cong\pi_{12}(S^{12}))$,

where $\nu_9\in\pi_{12}(S^9)=Z_{24}$ and $q_*\beta_2=\eta_{10}\eta_{11}\in\pi_{12}(S^{10})=Z_2$ are the elements of order 8 and 2, respectively.

Therefore, the attaching map $S^4f\in\pi_{12}(S^4X^8)$ in (7.1.4) is represented by

(7.1.6) $S^4f=a_1\alpha_1+a_2\alpha_2+a\alpha$ for some $a_i=0, 1$ and some integer a;

and we see that $a=2$ because $S^4X^9/S^4X^6=M^{13}=S^{12}\cup_2 e^{13}$ by (4.2.3), $a_2=0$ by (7.1.3) because $(S^4\bar{p})j=q\tilde{p}$, and $a_1=0$ because $Sq^4x_5=0$ in $H^*(X; Z_2)$ by (4.2.1) and $\nu_9\in\pi_{12}(S^9)$ is detected by Sq^4. Thus, we have $S^4f=2\alpha$ and the lemma.

q.e.d.

In addition to the cofiber sequence (6.3.1), consider the ones

(7.2.1) $X^6\xrightarrow{j'}X^9\xrightarrow{p'}M^9(=X^9/X^6)\xrightarrow{g'}SX^6$, $X^9\xrightarrow{j''}X^{11}\xrightarrow{p''}S^{11}(=X^{11}/X^9)$,

due to (4.2.3). Then these induce the Puppe exact sequences

(7.2.2) $$[Y\wedge X^6, W]\xleftarrow{(1\wedge j')^*}[Y\wedge X^9, W]\xleftarrow{(1\wedge p')^*}[Y\wedge M^9, W]$$
$$\xleftarrow{(1\wedge g')^*}[Y\wedge SX^6, W]\leftarrow\cdots,$$

(7.2.3) $\quad [Y\wedge X^9, W] \xleftarrow{(1\wedge j'')^*} [Y\wedge X^{11}, W] \xleftarrow{(1\wedge p'')^*} [Y\wedge S^{11}, W]$
$\leftarrow [Y\wedge SX^9, W] \leftarrow \cdots.$

LEMMA 7.3. (i) $(1\wedge p')^*\colon [Y\wedge M^9, X_{14}]\to[Y\wedge X^9, X_{14}]$ is *monomorphic for* $Y=S^4Y'$, X^6 *and* X^9.

(ii) $(1\wedge p'')^*\colon [X^m\wedge S^{11}, X_{14}]\to[X^m\wedge X^{11}, X_{14}]$ *is monomorphic for any* $m\geqq 3$.

PROOF. (i) By Lemma 7.1, (i) holds for $Y=S^4Y'$. Consider the commutative diagram

$$\begin{array}{ccccccc}
[M^6\wedge SX^6, X_{14}] & \xrightarrow{(p\wedge 1)^*} & [X^6\wedge SX^6, X_{14}] & \xrightarrow{(j\wedge 1)^*} & [S^3\wedge SX^6, X_{14}](=0) & & \\
\downarrow{\scriptstyle (1\wedge g')^*} & & \downarrow{\scriptstyle (1\wedge g')^*} & & & & \\
[M^6\wedge M^9, X_{14}] & \xrightarrow{(p\wedge 1)^*} & [X^6\wedge M^9, X_{14}] & \xleftarrow{(j'\wedge 1)^*} & [X^9\wedge M^9, X_{14}] & \xleftarrow{(p'\wedge 1)^*} & [M^9\wedge M^9, X_{14}](=0) \\
\downarrow{\scriptstyle (1\wedge p')^*} & & \downarrow{\scriptstyle (1\wedge p')^*} & & \downarrow{\scriptstyle (1\wedge p')^*} & & \\
[M^6\wedge X^9, X_{14}] & & [X^6\wedge X^9, X_{14}] & \longleftarrow & [X^9\wedge X^9, X_{14}], & &
\end{array}$$

where the upper sequence is the one in (6.3.3), the others are in (7.2.2), and $(=0)$'s are seen by Lemma 6.4 (iii) and (1.1.2). Then the left $(1\wedge g')^*$ is trivial by (i) for $Y=M^6=S^4M^2$, and hence so is the middle $(1\wedge g')^*$. Thus the middle $(1\wedge p')^*$ is monomorphic, and hence so is the right $(1\wedge p')^*$.

(ii) To prove (ii), we notice that

(7.3.2) $\quad [M^{13}, X_n]=0$ *for any* n, *and* $[S^4X^9, X_{14}]=0$.

In fact, $[M^{13}, X_n]=0$ is seen by the exact sequence (6.6.1) for M^{13} and $\pi_{12}(X_n)=0$, $\pi_{13}(X_n)=0$ or Z_3 in (4.2.4). Hence $[S^4X^9, X_{14}]=0$ is seen by Lemma 6.4 (iii) and the exact sequence (7.2.2) for $Y=S^4$ and $W=X_{14}$.

By the latter half of (7.3.2) and the exact sequence (7.2.3) for $Y=S^3=X^3=X^4$, (ii) holds for $m=3$ and 4. Therefore we see (ii) for $m\geqq 4$, because the inclusion $X^4\wedge S^{11}\subset X^m\wedge S^{11}$ is 15-connected and induces the isomorphism $[X^m\wedge S^{11}, X_{14}]\cong[X^4\wedge S^{11}, X_{14}]$ by (1.1.1). q.e.d.

We now consider the exact sequence (6.1) for $n=14$.

LEMMA 7.4. (i) $i_{14*}\colon H^{14}(X^m\wedge X^n; \pi_{14})\to[X^m\wedge X^n, X_{14}]$ *is monomorphic for* $(m, n)=(6, 9)$, $(9, 9)$, $(9, 11)$ *and* $(11, 11)$.

(ii) $i_{14*}\colon H^{14}(X_{14}\wedge X_{14}; \pi_{14})\to[X_{14}\wedge X_{14}, X_{14}]$ *is monomorphic, and Assertion* 5.3 *holds for* $n=14$.

PROOF. (i) To prove (i), we notice that

(7.4.1) $\quad [M^m\wedge M^9, \Omega X_{13}]=0=[X^m\wedge M^9, \Omega X_{13}] \quad$ *for* $m=6$ *and* 9.

In fact, the first equality is seen by (1.1.2). Therefore the second one is shown by (7.3.2) and by the exact sequences (6.3.3) and (7.2.2) for $Y=M^9$ and $W=\Omega X_{13}$.

We now consider the commutative diagrams

$$\begin{array}{ccccc} H^{14}(X^m\wedge X^6;\pi_{14}) & \leftarrow & H^{14}(X^m\wedge X^9;\pi_{14}) & \xleftarrow{(1\wedge p')^*} & H^{14}(X^m\wedge M^9;\pi_{14}) \\ \downarrow{\scriptstyle i_{14*}} & & \downarrow{\scriptstyle i_{14*}} & & \text{mono}\downarrow{\scriptstyle i_{14*}} \\ [X^m\wedge X^6,\ X_{14}] & \leftarrow & [X^m\wedge X^9,\ X_{14}] & \xleftarrow[\text{mono}]{(1\wedge p')^*} & [X^m\wedge M^9,\ X_{14}] \end{array} \qquad (m=6 \text{ and } 9) \tag{7.4.2}$$

of the exact sequences in (7.2.2), and

$$\begin{array}{ccccc} H^{14}(X^m\wedge X^9;\pi_{14}) & \leftarrow & H^{14}(X^m\wedge X^{11};\pi_{14}) & \leftarrow & H^{14}(X^m\wedge S^{11};\pi_{14}) \\ \downarrow{\scriptstyle i_{14*}} & & \downarrow{\scriptstyle i_{14*}} & & \text{mono}\downarrow{\scriptstyle i_{14*}} \\ [X^m\wedge X^9,\ X_{14}] & \leftarrow & [X^m\wedge X^{11},\ X_{14}] & \xleftarrow[\text{mono}]{(1\wedge p'')^*} & [X^m\wedge S^{11},X_{14}] \end{array} \qquad (m=9 \text{ and } 11) \tag{7.4.3}$$

of the exact sequences in (7.2.3). In these diagrams, the homomorphisms indicated by mono are monomorphic by Lemma 7.3 and by the exact sequence (6.1), (7.4.1) and $[X^m\wedge S^{11}, \Omega X_{13}]=0$. Therefore, in each diagram, if the left i_{14*} is monomorphic, then so is the middle one. Thus, noticing that $H^{14}(X^6\wedge X^6;\pi)=0$, we see (i) successively for $(m, n)=(6, 9)$, $(9, 9)$, $(9, 11)$ and $(11, 11)$.

(ii) Consider $h=f_{14}j_{11}\colon X^{11}\subset X\to X_{14}$ and the commutative diagram

$$\begin{array}{ccc} H^{14}(X^{11}\wedge X^{11};\pi_{14}) & \xleftarrow{(h\wedge h)^*} & H^{14}(X_{14}\wedge X_{14};\pi_{14}) \\ \downarrow{\scriptstyle i_{14*}} & & \downarrow{\scriptstyle i_{14*}} \\ [X^{11}\wedge X^{11},\ X_{14}] & \xleftarrow{(h\wedge h)^*} & [X_{14}\wedge X_{14},\ X_{14}]. \end{array} \tag{7.4.4}$$

Then the upper $(h\wedge h)^*$ is isomorphic by (1.1.1), because $h\wedge h$ is 16-connected by (6.2.1) and $X^{11}=X^{13}$ in (4.2.2). Thus we see (ii) by (i) for $m=n=11$. q.e.d.

Thus, Assertion 5.3 is proved in Lemmas 6.2, 6.7 and 7.4 (ii); and the proof of Theorem II in the introduction is completed by the note given in the end of §5.

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