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Oscillation of functional differential equations with general deviating arguments

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Introduction

In this paper we consider linear and nonlinear functional differential equations with deviating arguments of the forms

(LE) $$L_n x(t) + \sigma \sum_{h=1}^{N} q_h(t) x(g_h(t)) = 0,$$

(NE) $$L_n x(t) + \sigma \sum_{h=1}^{N} q_h(t) f_h(x(g_h(t))) = 0,$$

where L_n is a disconjugate differential operator defined recursively by

(1) $$L_0 x = x, \quad L_i x = \frac{1}{p_i} \frac{d}{dt} L_{i-1} x \ (1 \le i \le n), \quad p_n \equiv 1.$$

The following conditions are assumed to hold throughout this paper:

(a) $n \geq 2$, $\sigma = \pm 1$;
(b) $p_i \in C(R_+, R_+ \backslash \{0\})$, $\int_0^\infty p_i(t)dt = \infty \ (1 \le i \le n-1)$, $R_+ = [0, \infty)$;
(c) $q_h \in C(R_+, R_+)$, $g_h \in C(R_+, R)$, $\lim_{t \to \infty} g_h(t) = \infty \ (1 \le h \le N)$;
(d) $f_h \in C(R, R)$ is nondecreasing and $x f_h(x) > 0$ for $x \neq 0 \ (1 \le h \le N)$.

The domain of L_n, $\mathscr{D}(L_n)$, is defined to be the set of all functions x which have the continuous "quasi-derivatives" $L_i x$, $0 \le i \le n$, on $[T_x, \infty)$. Our attention is restricted to those solutions $x \in \mathscr{D}(L_n)$ of (LE) or (NE) which satisfy

$$\sup \{|x(t)| : t \geq T\} > 0 \qquad \text{for any} \quad T \geq T_x.$$

Such a solution is said to be a proper solution. We make the standing hypothesis that (LE) or (NE) possesses proper solutions. A proper solution of (LE) or (NE) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

We denote the sets of all proper solutions, all oscillatory solutions and all nonoscillatory solutions of (LE) or (NE) by $\mathscr{S}$, $\mathscr{O}$ and $\mathscr{N}$, respectively. It is clear that $\mathscr{S} = \mathscr{O} \cup \mathscr{N}$. Because of the conditions (a)–(d) $\mathscr{N}$ has a decomposition such that (see [2], [13] or [45]):

$$\mathscr{N} = \mathscr{N}_1 \cup \mathscr{N}_3 \cup \cdots \cup \mathscr{N}_{n-1} \qquad \text{if} \quad \sigma = 1 \text{ and } n \text{ is even},$$

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{n-1} \quad \text{if} \quad \sigma = 1 \text{ and } n \text{ is odd},$$
$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_n \quad \text{if} \quad \sigma = -1 \text{ and } n \text{ is even},$$
$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_n \quad \text{if} \quad \sigma = -1 \text{ and } n \text{ is odd},$$

where $\mathcal{N}_l$ denotes the subset of $\mathcal{N}$ consisting of all x satisfying

(2) $\quad xL_ix > 0\ (0 \leq i \leq l)$ and $(-1)^{i-l}xL_ix > 0\ (l \leq i \leq n)$ on $[T_x, \infty)$.

Equation (LE) or (NE) is said to be oscillatory if all of its proper solutions are oscillatory. Equation (LE) or (NE) is said to be almost oscillatory if:

(i) for $\sigma=1$ and n even, every proper solution of (LE) or (NE) is oscillatory;

(ii) for $\sigma=1$ and n odd, every proper solution x of (LE) or (NE) is either oscillatory or strongly decreasing in the sense that

(3) $$|L_ix(t)| \downarrow 0 \quad \text{as} \quad t \uparrow \infty, \quad 0 \leq i \leq n-1;$$

(iii) for $\sigma=-1$ and n even, every proper solution x of (LE) or (NE) is oscillatory, strongly decreasing or else strongly increasing in the sense that

(4) $$|L_ix(t)| \uparrow \infty \quad \text{as} \quad t \uparrow \infty, \quad 0 \leq i \leq n-1;$$

(iv) for $\sigma=-1$ and n odd, every proper solution x of (LE) or (NE) is either oscillatory or strongly increasing.

The main purpose of this paper is to develop criteria for (LE) or (NE) to be oscillatory or almost oscillatory. The nonlinear equation (NE) and the linear equation (LE) are studied in Part I and Part II, respectively. In each part we obtain conditions which imply that $\mathcal{N}_l=\emptyset$ by analyzing the three cases: $1 \leq l \leq n-1$, $l=0$ and $l=n$, separately, and then combine these conditions to derive the desired oscillation criteria for the equation under study. It is shown that there exists a class of genuinely nonlinear equations (NE) for which the oscillation situation can be completely characterized. No such characterization results are known for the linear equation (LE).

The oscillatory behavior of functional differential equations with deviating arguments has been intensively studied in the last two decades. Most of the literature on this subject has been concerned with equations of the form

(5) $$x^{(n)}(t) + F(t, x(g_1(t)),\dots, (g_N(t))) = 0,$$

which is a special case of (LE) or (NE) with all $p_i \equiv 1$, $1 \leq i \leq n$. For typical results regarding (5) we refer to [3], [4], [8], [12], [16], [18], [23], [28] and [36]. There is, however, much current interest in the study of the oscillation properties of higher-order differential equations of the forms (LE) and (NE) involving general disconjugate operators L_n defined by (1); see, for example, the papers [1, 2, 5–7, 9–11, 13–15, 17, 19–27, 29–35, 37–47]. In the present paper we proceed further in this direction to establish new oscillation results for equations (LE) and (NE) with *general* deviating arguments $g_h(t)$ which extend and unify many of the previous results obtained in the above-mentioned papers.

Part I Nonlinear equations

0. Preliminaries. We begin by listing the notation and formulas which will be extensively used in the subsequent discussions.

For functions $\psi_i \in C(R_+, R)$, $i=1, 2,\dots$, we define

$$(0.1)\quad \begin{cases} I_0 \equiv 1, \\ I_i(t, s; \psi_i,\dots, \psi_1) = \int_s^t \psi_i(r) I_{i-1}(r, s; \psi_{i-1},\dots, \psi_1)dr, \quad i = 1, 2,\dots. \end{cases}$$

From the definition of I_i it follows that

$$(0.2)\qquad I_i(t, s; \psi_1,\dots, \psi_i) = (-1)^i I_i(s, t; \psi_i,\dots, \psi_1)$$

and

$$(0.3)\qquad I_i(t, s; \psi_1,\dots, \psi_i) = \int_s^t \psi_i(r) I_{i-1}(t, r; \psi_1,\dots, \psi_{i-1})dr.$$

Let x be a function which has the continuous "quasi-derivatives" $L_i x$ $(0 \le i \le n)$ (see (1)) on R_+. Then it is easy to verify that for $t, s \in R_+$

$$(0.4)\quad L_i x(t) = \textstyle\sum_{j=i}^{k-1} I_{j-i}(t, s; p_{i+1},\dots, p_j) L_j x(s) + \displaystyle\int_s^t I_{k-i-1}(t, r; p_{i+1},\dots, p_{k-1}) p_k(r) L_k x(r)dr, \quad 0 \le i < k \le n,$$

which is an extension of the Taylor's formula with remainder encountered in calculus. In view of (0.2), (0.4) can be rewritten as

$$(0.5)\quad L_i x(t) = \textstyle\sum_{j=i}^{k-1} (-1)^{j-i} I_{j-i}(s, t; p_j,\dots, p_{i+1}) L_j x(s) + (-1)^{k-i} \displaystyle\int_t^s I_{k-i-1}(r, t; p_{k-1},\dots, p_{i+1}) p_k(r) L_k x(r)dr, \quad 0 \le i < k \le n.$$

For a function $g \in C(R_+, R)$ we put

$$(0.6)\quad g_*(t) = \min\{g(t), t\},$$

$$(0.7)\quad \mathscr{A}[g] = \{t \in R_+ : g(t) > t\},$$

$$(0.8)\quad \mathscr{R}[g] = \{t \in R_+ : g(t) < t\},$$

(0.9) $H_l[g](t, T)$

$$= \int_T^{g_*(t)} I_{l-1}(g(t), s; p_1,\dots, p_{l-1})p_l(s)I_{n-l-1}(t, s; p_{n-1},\dots, p_{l+1})ds,$$
$$T \geq 0, \quad 1 \leq l \leq n-1.$$

We simply write

$$H_l[g](t) = H_l[g](t, 0).$$

As easily verified, for any fixed $T>0$, there exist positive constants c_1 and c_2 such that

(0.10) $$c_1 H_l[g](t) \leq H_l[g](t, T) \leq c_2 H_l[g](t)$$
for all sufficiently large t.

1. Basic results. We first consider the differential inequalities

(NI$^+$) $$\sigma L_n x(t) + q(t)f(x(g(t))) \leq 0$$

and

(NI$^-$) $$\sigma L_n x(t) + q(t)f(x(g(t))) \geq 0,$$

under the conditions:

(c′) $q \in C(R_+, R_+)$, $g \in C(R_+, R)$ and $\lim_{t\to\infty} g(t) = \infty$,

(d′) $f \in C(R, R)$ is nondecreasing and $xf(x) > 0$ for $x \neq 0$.

We introduce the notation:

$$\mathcal{N}^+ = \{x\colon x \text{ is a positive solution of (NI}^+) \text{ on } [T_x, \infty) \text{ for some } T_x\},$$

$$\mathcal{N}^- = \{x\colon x \text{ is a negative solution of (NI}^-) \text{ on } [T_x, \infty) \text{ for some } T_x\},$$

and

$$\mathcal{N}_l^\pm = \{x \in \mathcal{N}^\pm\colon x \text{ satisfies (2) for some } T_x\}.$$

$\mathcal{N}^\pm$ has a decomposition such that

$$\mathcal{N}^\pm = \mathcal{N}_1^\pm \cup \mathcal{N}_3^\pm \cup \cdots \cup \mathcal{N}_{n-1}^\pm \quad \text{if } \sigma = 1 \text{ and } n \text{ is even},$$
$$\mathcal{N}^\pm = \mathcal{N}_0^\pm \cup \mathcal{N}_2^\pm \cup \cdots \cup \mathcal{N}_{n-1}^\pm \quad \text{if } \sigma = 1 \text{ and } n \text{ is odd},$$
$$\mathcal{N}^\pm = \mathcal{N}_0^\pm \cup \mathcal{N}_2^\pm \cup \cdots \cup \mathcal{N}_n^\pm \quad \text{if } \sigma = -1 \text{ and } n \text{ is even},$$
$$\mathcal{N}^\pm = \mathcal{N}_1^\pm \cup \mathcal{N}_3^\pm \cup \cdots \cup \mathcal{N}_n^\pm \quad \text{if } \sigma = -1 \text{ and } n \text{ is odd}.$$

We first give conditions implying that $\mathcal{N}_l^\pm = \emptyset$ in the case $1 \leq l \leq n-1$.

THEOREM 1.1. *Let* $1 \le l \le n-1$ *and* $(-1)^{n-l}\sigma = -1$.

(i) *Assume that*

$$\int^{\infty} \frac{dx}{f(x)} < \infty. \tag{1.1}$$

Then $\mathcal{N}_l^+ = \emptyset$ *for* (NI$^+$) *if*

$$\int^{\infty} H_l[g](t)q(t)dt = \infty. \tag{1.2}$$

(ii) *Assume that*

$$\int^{-\infty} \frac{dx}{f(x)} < \infty. \tag{1.3}$$

Then $\mathcal{N}_l^- = \emptyset$ *for* (NI$^-$) *if* (1.2) *holds.*

PROOF. (i) Assume that $\mathcal{N}_l^+$ has an element x. Then there exists $T_0 > 0$ such that

$$L_i x > 0 \ (0 \le i \le l) \text{ and } (-1)^{i-l} L_i x > 0 \ (l \le i \le n) \text{ on } [T_0, \infty). \tag{1.4}$$

Choose T_1 and T_2 ($T_0 \le T_1 \le T_2$) so large that

$$\inf\{g(t)\colon t \ge T_1\} > T_0 \text{ and } \inf\{g_*(t)\colon t \ge T_2\} > T_1. \tag{1.5}$$

We fix T_3 ($\ge T_2$) arbitrarily and define

$$T_4 = \max\{T_3, \max\{g(t)\colon T_1 \le t \le T_3\}\}. \tag{1.6}$$

Assume first that $l \ge 2$. From (0.5) with i, k, t and s replaced by l, n, s and T_4, respectively, we obtain in view of (1.4)

$$\begin{aligned} L_l x(s) &= \textstyle\sum_{j=l}^{n-1} (-1)^{j-l} I_{j-l}(T_4, s; p_j, \dots, p_{l+1}) L_j x(T_4) \\ &\quad + (-1)^{n-l} \int_s^{T_4} I_{n-l-1}(t, s; p_{n-1}, \dots, p_{l+1}) L_n x(t) dt \\ &\ge \int_s^{T_4} I_{n-l-1}(t, s; p_{n-1}, \dots, p_{l+1}) \cdot (-1)^{n-l} L_n x(t) dt, \quad T_1 \le s \le T_4. \end{aligned}$$

Since

$$(-1)^{n-l} L_n x(t) = -\sigma L_n x(t) \ge q(t) f(x(g(t))), \quad t \ge T_1,$$

we have

$$L_l x(s) \ge \int_s^{T_4} I_{n-l-1}(t, s; p_{n-1}, \dots, p_{l+1}) q(t) f(x(g(t))) dt, \quad T_1 \le s \le T_4.$$

On the other hand, using (0.4) with i, k, t and s replaced by 1, l, r and T_1, respectively, and (1.4), we get

$$\begin{aligned} x'(r) &= p_1(r)L_1x(r) \\ &= p_1(r)\{\textstyle\sum_{j=1}^{l-1} I_{j-1}(r, T_1; p_2,\dots, p_j)L_jx(T_1) \\ &\quad + \int_{T_1}^{r} I_{l-2}(r, s; p_2,\dots, p_{l-1})p_l(s)L_lx(s)ds\} \\ &\geq p_1(r)\int_{T_1}^{r} I_{l-2}(r, s; p_2,\dots, p_{l-1})p_l(s)L_lx(s)ds, \quad r \geq T_1. \end{aligned}$$

Combining these inequalities, dividing the result by $f(x(r))$ and then integrating it on $[T_1, T_4]$, we obtain

$$\begin{aligned} &\int_{T_1}^{T_4} [x'(r)/f(x(r))]dr \\ &\geq \int_{T_1}^{T_4}\int_{T_1}^{r}\int_{s}^{T_4} p_1(r)I_{l-2}(r, s; p_2,\dots, p_{l-1})p_l(s) \\ &\qquad \cdot I_{n-l-1}(t, s; p_{n-1},\dots, p_{l+1})q(t)\,[f(x(g(t)))/f(x(r))]dtdsdr. \end{aligned} \tag{1.7}$$

Interchanging the order of integration shows that

$$\begin{aligned} \int_{X_1}^{X_4}\frac{dx}{f(x)} &\geq \int_{T_1}^{T_4}\int_{T_1}^{r}\int_{s}^{T_4} \psi\, dtdsdr = \int_{T_1}^{T_4}\int_{s}^{T_4}\int_{s}^{T_4} \psi\, dtdrds \\ &= \int_{T_1}^{T_4}\int_{s}^{T_4}\int_{s}^{T_4} \psi\, drdtds = \int_{T_1}^{T_4}\int_{T_1}^{t}\int_{s}^{T_4} \psi\, drdsdt \\ &\geq \int_{T_2}^{T_3}\int_{T_1}^{g_*(t)}\int_{s}^{g(t)} \psi\, drdsdt, \end{aligned}$$

where $X_i = x(T_i)$, $i=1, 4$, and $\psi = \psi(r, s, t)$ denotes the integrand in the last integral of (1.7). Taking account of the inequality

$$f(x(g(t)))/f(x(r)) \geq 1, \quad T_1 \leq r \leq g(t), \tag{1.8}$$

which is a consequence of the increasing nature of f and x, we obtain

$$\begin{aligned} \int_{X_1}^{X_4}\frac{dx}{f(x)} &\geq \int_{T_2}^{T_3} q(t)\int_{T_1}^{g_*(t)} p_l(s)I_{n-l-1}(t, s; p_{n-1},\dots, p_{l+1}) \\ &\qquad \cdot \int_{s}^{g(t)} p_1(r)I_{l-2}(r, s; p_2,\dots, p_{l-1})drdsdt \\ &= \int_{T_2}^{T_3} q(t)\int_{T_1}^{g_*(t)} I_{l-1}(g(t), s; p_1,\dots, p_{l-1})p_l(s) \\ &\qquad \cdot I_{n-l-1}(t, s; p_{n-1},\dots, p_{l+1})dsdt, \end{aligned}$$

where we have used (0.1). Since T_3 is fixed arbitrarily, letting $T_3 \to \infty$ in the above inequality and using the assumption (1.1), we conclude that

$$\int_{T_2}^{\infty} q(t) \int_{T_1}^{g_*(t)} I_{l-1}(g(t), s; p_1,\dots, p_{l-1})p_l(s)I_{n-l-1}(t, s; p_{n-1},\dots, p_{l+1})dsdt$$
$$\leq \int_{X_1}^{\infty} \frac{dx}{f(x)} < \infty.$$

This contradicts the assumption (1.2).

Assume next that $l=1$. Replacing i, k, t and s by 1, n, s and T_3, respectively, in (0.5), and using (1.4), we obtain

$$\begin{aligned} x'(s) &= p_1(s)L_1x(s) \\ &= p_1(s)\{\textstyle\sum_{j=1}^{n-1}(-1)^{j-1}I_{j-1}(T_3, s; p_j,\dots, p_2)L_jx(T_3) \\ &\qquad + (-1)^{n-1}\displaystyle\int_s^{T_3} I_{n-2}(t, s; p_{n-1},\dots, p_2)L_nx(t)dt\} \\ &\geq p_1(s)\int_s^{T_3} I_{n-2}(t, s; p_{n-1},\dots, p_2)q(t)f(x(g(t)))dt, \quad T_1 \leq s \leq T_3. \end{aligned}$$

Dividing this by $f(x(s))$ and integrating on $[T_1, T_3]$ yields

$$\int_{T_1}^{T_3} [x'(s)/f(x(s))]ds$$
$$\geq \int_{T_1}^{T_3}\int_s^{T_3} p_1(s)I_{n-2}(t, s; p_{n-1},\dots, p_2)q(t)\,[f(x(g(t)))/f(x(s))]dtds.$$

We denote by $\psi=\psi(s, t)$ the integrand in the last integral. It follows that

$$\int_{X_1}^{X_3}\frac{dx}{f(x)} \geq \int_{T_1}^{T_3}\int_s^{T_3} \psi\, dtds = \int_{T_1}^{T_3}\int_{T_1}^{t} \psi\, dsdt \geq \int_{T_2}^{T_3}\int_{T_1}^{g_*(t)} \psi\, dsdt,$$

where $X_i=x\,(T_i)$, $i=1$, 3. Using the inequality

$$f(x(g(t)))/f(x(s)) \geq 1, \quad T_1 \leq s \leq g(t), \tag{1.9}$$

we obtain

$$\int_{X_1}^{X_3}\frac{dx}{f(x)} \geq \int_{T_2}^{T_3} q(t) \int_{T_1}^{g_*(t)} p_1(s)I_{n-2}(t, s; p_{n-1},\dots, p_2)\; dsdt,$$

which gives in the limit as $T_3 \to \infty$

$$\int_{T_2}^{\infty} q(t) \int_{T_1}^{g_*(t)} p_1(s)I_{n-2}(t, s; p_{n-1},\dots,p_2)\; dsdt \leq \int_{X_1}^{\infty} \frac{dx}{f(x)} < \infty.$$

This contradicts the assumption (1.2).

(ii) can be proved similarly.

REMARK 1.1. In Theorem 1.1 the monotonicity of f can be slightly relaxed, that is, it suffices to assume the existence of a positive number M such that

$$f(x) \le f(y) \quad \text{for} \quad M \le x \le y \tag{1.10}$$

in (i), and

$$f(x) \le f(y) \quad \text{for} \quad x \le y \le -M \tag{1.11}$$

in (ii).

We are next concerned with the case where $\mathcal{N}_0^{\pm} = \emptyset$.

THEOREM 1.2. *Let* $(-1)^n\sigma = -1$.

(i) *Assume that*

$$\int_{+0} \frac{dx}{f(x)} < \infty. \tag{1.12}$$

Then $\mathcal{N}_0^{+} = \emptyset$ *for* (NI$^+$) *if*

$$\int_{\mathscr{R}[g]} I_{n-1}(t, g(t); p_{n-1},\dots, p_1)q(t)dt = \infty. \tag{1.13}$$

(ii) *Assume that*

$$\int_{-0} \frac{dx}{f(x)} < \infty. \tag{1.14}$$

Then (1.13) *implies that* $\mathcal{N}_0^{-} = \emptyset$ *for* (NI$^-$).

PROOF. We only give a proof of (i), since (ii) can be proved similarly.

Assume that $x \in \mathcal{N}_0^{+}$. Then there exists T_0 such that

$$(-1)^i L_i x > 0 \ (0 \le i \le n) \quad \text{on} \quad [T_0, \infty). \tag{1.15}$$

We fix T_1, T_2 which satisfy (1.5), and choose T_3 ($\ge T_2$) arbitrarily. Replacing i, k, t and s by 1, n, s and T_3, respectively, in (0.5), we have with the use of (1.15)

$$\begin{aligned} -x'(s) &= -p_1(s)L_1x(s) \\ &= p_1(s)\{\textstyle\sum_{j=1}^{n-1}(-1)^j I_{j-1}(T_3, s; p_j,\dots, p_2)L_jx(T_3) \\ &\qquad + (-1)^n \displaystyle\int_s^{T_3} I_{n-2}(t, s; p_{n-1},\dots, p_2)L_nx(t)dt\} \\ &\ge p_1(s)\int_s^{T_3} I_{n-2}(t, s; p_{n-1},\dots, p_2)q(t)f(x(g(t)))dt, \quad T_1 \le s \le T_3. \end{aligned}$$

Dividng the above by $f(x(s))$ and integrating on $[T_1, T_3]$, we obtain

(1.16) $$\int_{T_3}^{T_1} [x'(s)/f(x(s))]ds$$
$$\geq \int_{T_1}^{T_3}\int_{s}^{T_3} p_1(s)I_{n-2}(t, s; p_{n-1},\dots, p_2)q(t)\,[f(x(g(t)))/f(x(s))]dtds,$$

from which it follows that

(1.17)
$$\int_{X_3}^{X_1}\frac{dx}{f(x)} \geq \int_{T_1}^{T_3}\int_{s}^{T_3} \psi\, dtds = \int_{T_1}^{T_3}\int_{T_1}^{t} \psi\, dsdt \geq \int_{\mathscr{R}[g]\cap[T_2,T_3]}\int_{g(t)}^{t} \psi\, dsdt,$$

where $X_i=x(T_i)$, $i=1$, 3, and $\psi=\psi(s, t)$ denotes the integrand in the repeated integral of (1.16). Noting that

(1.18) $$f(x(g(t)))/f(x(s)) \geq 1 \qquad \text{for} \quad s \geq g(t) \quad \text{and} \quad t \geq T_2,$$

and taking (0.3) into account, we see from (1.17) that

$$\int_{X_3}^{X_1}\frac{dx}{f(x)} \geq \int_{\mathscr{R}[g]\cap[T_2,T_3]} q(t)\int_{g(t)}^{t} p_1(s)I_{n-2}(t, s; p_{n-1},\dots,p_2)dsdt$$
$$= \int_{\mathscr{R}[g]\cap[T_2,T_3]} I_{n-1}(t, g(t); p_{n-1},\dots, p_1)q(t)dt.$$

Letting $T_3\to\infty$ and using (1.12), we conclude that

$$\int_{\mathscr{R}[g]\cap[T_2,\infty)} I_{n-1}(t, g(t); p_{n-1},\dots, p_1)q(t)dt \leq \int_{0}^{X_1}\frac{dx}{f(x)} < \infty,$$

which contradicts the condition (1.13). This completes the proof of (i).

REMARK 1.2. In Theorem 1.2 the monotonicity of f can be slightly relaxed, that is, it suffices to assume the existence of a positive number m such that

(1.19) $$f(x) \leq f(y) \qquad \text{for} \quad 0 < x \leq y \leq m$$

in (i), and

(1.20) $$f(x) \leq f(y) \qquad \text{for} \quad -m \leq x \leq y < 0$$

in (ii).

Finally, we discuss when it happens that $\mathscr{N}_n^{\pm}=\emptyset$.

THEOREM 1.3. *Let* $\sigma=-1$.
(i) *Assume that* (1.1) *holds. Then* $\mathscr{N}_n^{+}=\emptyset$ *for* (NI$^+$) *if*

(1.21) $$\int_{\mathscr{A}[g]} I_{n-1}(g(t), t; p_1,\dots, p_{n-1})q(t)dt = \infty.$$

(ii) *Assume that* (1.3) *holds. Then* (1.21) *implies that* $\mathscr{N}_n^- = \emptyset$ *for* (NI$^-$).

PROOF. We need only to prove (i). Assume that $x \in \mathscr{N}_n^+$. Then there exists $T_0 > 0$ such that

$$(1.22) \qquad L_i x > 0 \ (0 \le i \le n) \qquad \text{on} \quad [T_0, \infty).$$

We fix T_1, T_2 which satisfy (1.5), choose T_3 ($\ge T_2$) arbitrarily and define T_4 by (1.6). Using (1.22) in (0.4) with i, k, t and s replaced by 1, n, s and T_1, respectively, we have

$$\begin{aligned} x'(s) &= p_1(s) L_1 x(s) \\ &= p_1(s) \{ \textstyle\sum_{j=1}^{n-1} I_{j-1}(s, T_1; p_2, \dots, p_j) L_j x(T_1) \\ &\qquad + \int_{T_1}^{s} I_{n-2}(s, t; p_2, \dots, p_{n-1}) L_n x(t) dt \} \\ &\ge p_1(s) \int_{T_1}^{s} I_{n-2}(s, t; p_2, \dots, p_{n-1}) q(t) f(x(g(t))) dt, \quad s \ge T_1, \end{aligned}$$

which, after integration, yields

$$\begin{aligned} (1.23) \quad & \int_{T_1}^{T_4} [x'(s)/f(x(s))] ds \\ & \ge \int_{T_1}^{T_4} \int_{T_1}^{s} p_1(s) I_{n-2}(s, t; p_2, \dots, p_{n-1}) q(t) [f(x(g(t)))/f(x(s))] dt ds. \end{aligned}$$

It follows that

$$\int_{X_1}^{X_4} \frac{dx}{f(x)} \ge \int_{T_1}^{T_4} \int_{T_1}^{s} \psi \, dt ds = \int_{T_1}^{T_4} \int_{t}^{T_4} \psi \, ds dt \ge \int_{\mathscr{A}[g] \cap [T_1, T_3]} \int_{t}^{g(t)} \psi \, ds dt,$$

where $X_i = x(T_i)$, $i = 1, 4$, and $\psi = \psi(s, t)$ stands for the integrand in the last integral of (1.23). Since

$$f(x(g(t)))/f(x(s)) \ge 1 \qquad \text{for} \quad T_1 \le s \le g(t) \quad \text{and} \quad t \ge T_1,$$

we obtain

$$\begin{aligned} \int_{X_1}^{X_4} \frac{dx}{f(x)} &\ge \int_{\mathscr{A}[g] \cap [T_1, T_3]} q(t) \int_{t}^{g(t)} p_1(s) I_{n-2}(s, t; p_2, \dots, p_{n-1}) ds dt \\ &= \int_{\mathscr{A}[g] \cap [T_1, T_3]} I_{n-1}(g(t), t; p_1, \dots, p_{n-1}) q(t) dt, \end{aligned}$$

which implies that

$$\int_{\mathscr{A}[g] \cap [T_1, \infty)} I_{n-1}(g(t), t; p_1, \dots, p_{n-1}) q(t) dt \le \int_{X_1}^{\infty} \frac{dx}{f(x)} < \infty.$$

This contradicts (1.21), and the proof is complete.

REMARK 1.3. If $x \in \mathcal{N}_n^{\pm}$, then $x(\infty) = \pm\infty$. Therefore, we can relax the monotonicity condition on f by assuming the existence of a positive number M such that (1.10) holds in (i) of Theorem 1.3, and (1.11) holds in (ii) of Theorem 1.3.

We now apply the above results to the differential equation (NE). Let $x \in \mathcal{N}_l$ be a solution of (NE). If $x(t) > 0$ for sufficiently large t then

$$f_h(x(g_h(t))) > 0 \qquad (1 \le h \le N)$$

for all large t. Therefore,

$$\begin{aligned} 0 &= \sigma L_n x(t) + \textstyle\sum_{h=1}^N q_h(t) f_h(x(g_h(t))) \\ &\ge \sigma L_n x(t) + q_h(t) f_h(x(g_h(t))), \quad 1 \le h \le N. \end{aligned}$$

This means that if x is an eventually possitive solution of (NE) belonging to $\mathcal{N}_l$, then it belongs to $\mathcal{N}_l^+$ with respect to the differential inequality

$$\sigma L_n x(t) + q_h(t) f_h(x(g_h(t))) \le 0$$

for any $h = 1, \dots, N$. Likewise, an eventually negative solution $x \in \mathcal{N}_l$ of (NE) belongs to $\mathcal{N}_l^-$ with respect to the differential inequality

$$\sigma L_n x(t) + q_h(t) f_n(x(g_h(t))) \ge 0$$

for any $h = 1, \dots, N$. Hence we obtain the following results for equation (NE).

THEOREM 1.4. *Let $1 \le l \le n-1$ and $(-1)^{n-l}\sigma = -1$. Suppose that there exist integers $\alpha_+, \alpha_- \in \{1, \dots, N\}$ such that*

$$\int^{\pm\infty} \frac{dx}{f_{\alpha_\pm}(x)} < \infty \tag{1.24}$$

and

$$\int^{\infty} H_l[g_{\alpha_\pm}](t) q_{\alpha_\pm}(t) dt = \infty. \tag{1.25; l}$$

Then $\mathcal{N}_l = \emptyset$ for (NE).

THEOREM 1.5. *Let $(-1)^n \sigma = -1$. Suppose that there exist integers β_+, $\beta_- \in \{1, \dots, N\}$ such that*

$$\int_{\pm 0} \frac{dx}{f_{\beta_\pm}(x)} < \infty \tag{1.26}$$

and

$$\int_{\mathscr{B}[g_{\beta\pm}]} I_{n-1}(t, g_{\beta\pm}(t); p_{n-1},\dots, p_1)q^{\beta\pm}(t)dt = \infty. \tag{1.27}$$

Then $\mathcal{N}_0=\emptyset$ *for* (NE).

THEOREM 1.6. *Let* $\sigma=-1$. *Suppose that there exist integers* γ_+, $\gamma_- \in \{1,\dots, N\}$ *such that*

$$\int^{\pm\infty} \frac{dx}{f_{\gamma\pm}(x)} < \infty \tag{1.28}$$

and

$$\int_{\mathscr{A}[g_{\gamma\pm}]} I_{n-1}(t, g_{\gamma\pm}(t); p_1,\dots, p_{n-1})q_{\gamma\pm}(t)dt = \infty. \tag{1.29}$$

Then $\mathcal{N}_n=\emptyset$ *for* (NE).

These theorems seem to be new even when specialized to the equation

$$x^{(n)}(t) + \sigma q(t)f(x(g(t))) = 0, \tag{1.30}$$

for which conditions (c′) and (d′) are satisfied. So, we state them below as corollaries, by noting that in this case

$$I_{n-1}(t, s; p_1,\dots, p_{n-1}) = I_{n-1}(t, s; p_{n-1},\dots, p_1) = (t-s)^{n-1}/(n-1)!$$

and

$$H_l[g](t) \geq t^{n-l-1}g^{l-1}(t)g_*(t)/[(n-1)(l-1)!(n-l-1)!], \quad 1 \leq l \leq n-1.$$

COROLLARY 1.1. *Let* $1\leq l\leq n-1$ *and* $(-1)^{n-l}\sigma=-1$. *Then* $\mathcal{N}_l=\emptyset$ *for* (1.30) *if*

$$\int^{\pm\infty} \frac{dx}{f(x)} < \infty \quad \text{and} \quad \int^{\infty} t^{n-l-1}g^{l-1}(t)g_*(t)q(t)dt = \infty.$$

COROLLARY 1.2. *Let* $(-1)^n\sigma=-1$. *Then* $\mathcal{N}_0=\emptyset$ *for* (1.30) *if*

$$\int_{\pm 0} \frac{dx}{f(x)} < \infty \quad \text{and} \quad \int_{\mathscr{B}[g]} (t-g(t))^{n-1}q(t)dt = \infty.$$

COROLLARY 1.3. *Let* $\sigma=-1$. *Then* $\mathcal{N}_n=\emptyset$ *for* (1.30) *if*

$$\int^{\pm\infty} \frac{dx}{f(x)} < \infty \quad \text{and} \quad \int_{\mathscr{A}[g]} (g(t)-t)^{n-1}q(t)dt = \infty.$$

2. Variants of the basic results. The purpose of this section is to obtain variants of the results of the preceding section under stronger nonlinearity con-

dition on (NE) and ($NI^{\pm}$). The conditions we assume for (NE) and ($NI^{\pm}$) are as follows:

(e) $\inf\{f_h(\xi x)/f_h(\xi): \xi \neq 0\} > 0$ for any $x > 0$ and $h = 1,\dots, N$;

(e′) $\inf\{f(\xi x)/f(\xi): \xi \neq 0\} > 0$ for any $x > 0$.

Let us first examine ($NI^{\pm}$). For this purpose we need the function defined by

$$(2.1) \qquad \omega[f](x) = \begin{cases} \operatorname{sgn} x \cdot \inf\{f(\xi|x|)/f(\xi): \xi x > 0\} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Because of (d′) and (e′), $\omega[f]$ has the following properties:

(2.2) $\omega[f]$ is nondecreasing on R and $x\omega[f](x) > 0$ for $x \neq 0$;

(2.3) $|f(\xi|x|)| \geq |f(\xi)|\,|\omega[f](x)|$ for $\xi x > 0$.

THEOREM 2.1. *Let* $1 \leq l \leq n-1$ *and* $(-1)^{n-l}\sigma = -1$.

(i) *Assume that*

$$(2.4) \qquad \int_{+0} \frac{dx}{\omega[f](x)} < \infty.$$

Then $\mathcal{N}_l^+ = \emptyset$ *for* (NI^+) *if*

$$(2.5) \qquad \int^{\infty} q(t) f(H_l[g](t))dt = \infty.$$

(ii) *Assume that*

$$(2.6) \qquad \int_{-0} \frac{dx}{\omega[f](x)} < \infty.$$

Then $\mathcal{N}_l^- = \emptyset$ *for* (NI^-) *if*

$$(2.7) \qquad \int^{\infty} q(t)\,|f(-H_l[g](t))|dt = \infty.$$

PROOF. Suppose that $x \in \mathcal{N}_l^+$. Then (1.4) holds for some $T_0 > 0$. We fix T_1 and T_2 satisfying (1.5) and choose T_3 ($\geq T_2$) arbitrarily. We first claim that

$$(2.8) \qquad L_l x(s) \geq I_{n-l-1}(t, s; p_{n-1},\dots, p_{l+1})\,|L_{n-1}x(t)|, \quad T_1 \leq s \leq t.$$

When $l = n-1$, this is obvious because $L_l x$ is decreasing. When $l \leq n-2$, replacing i, k, t and s in (0.5) by l, $n-1$, s and t, respectively, and using (1.4) and the decreasing nature of $|L_{n-1}x|$, we obtain

$$\begin{aligned}
L_l x(s) &= \textstyle\sum_{j=l}^{n-2}(-1)^{j-l} I_{j-l}(t, s; p_j, \dots, p_{l+1}) L_j x(t) \\
&\quad + (-1)^{n-l-1} \int_s^t I_{n-l-2}(r, s; p_{n-2}, \dots, p_{l+1}) p_{n-1}(r) L_{n-1} x(r) dr \\
&\geq \int_s^t I_{n-l-2}(r, s; p_{n-2}, \dots, p_{l+1}) p_{n-1}(r) |L_{n-1} x(r)| dr \\
&\geq \int_s^t p_{n-1}(r) I_{n-l-2}(r, s; p_{n-2}, \dots, p_{l+1}) dr \cdot |L_{n-1} x(t)| \\
&= I_{n-l-1}(t, s; p_{n-1}, \dots, p_{l+1}) |L_{n-1} x(t)|, \quad T_1 \leq s \leq t.
\end{aligned}$$

On the other hand, from (0.4) with i, k, t and s replaced by 0, l, $g(t)$ and T_1, respectively, we get

$$\begin{aligned}
x(g(t)) &= \textstyle\sum_{j=0}^{l-1} I_j(g(t), T_1; p_1, \dots, p_j) L_j x(T_1) \\
&\quad + \int_{T_1}^{g(t)} I_{l-1}(g(t), s; p_1, \dots, p_{l-1}) p_l(s) L_l x(s) ds \\
&\geq \int_{T_1}^{g_*(t)} I_{l-1}(g(t), s; p_1, \dots, p_{l-1}) p_l(s) L_l x(s) ds, \quad t \geq T_2.
\end{aligned}$$

Combining this with (2.8), we obtain

$$\begin{aligned}
&x(g(t)) \\
&\geq \int_{T_1}^{g_*(t)} I_{l-1}(g(t), s; p_1, \dots, p_{l-1}) p_l(s) I_{n-l-1}(t, s; p_{n-1}, \dots, p_{l+1}) ds \cdot |L_{n-1} x(t)| \\
&\geq c H_l[g](t) |L_{n-1} x(t)|, \quad t \geq T_2
\end{aligned}$$

for some constant $c>0$. Substitute $u(t)$ for $c|L_{n-1}x(t)|$. Using (2.3), we see that

$$f(x(g(t))) \geq f(H_l[g](t) u(t)) \geq f(H_l[g](t)) \omega[f](u(t)), \quad t \geq T_2.$$

Therefore

$$\begin{aligned}
-u'(t) = (-1)^{n-l} c L_n x(t) &= -\sigma c L_n x(t) \geq c q(t) f(x(g(t))) \\
&\geq c q(t) f(H_l[g](t)) \omega[f](u(t)), \quad t \geq T_2.
\end{aligned}$$

Dividing the both sides by $c\omega[f](u(t))$ and integrating the result on $[T_2, T_3]$, we find

$$\int_{T_2}^{T_3} q(t) f(H_l[g](t)) dt \leq c^{-1} \int_{T_3}^{T_2} \frac{u'(t)}{\omega[f](u(t))} dt = c^{-1} \int_{U_3}^{U_2} \frac{du}{\omega[f](u)},$$

where $U_i = u(T_i)$, $i=2, 3$. Letting $T_3 \to \infty$ and using the assumption (2.4), we conclude that

$$\int_{T_2}^{\infty} q(t)f(H_i[g](t))dt \leq c^{-1}\int_0^{U_2} \frac{du}{\omega[f](u)} < \infty,$$

which contradicts (2.5). This completes the proof of (i).

The proof of (ii) is similar.

THEOREM 2.2. *Let* $(-1)^n\sigma = -1$.

(i) *Assume that* (2.4) *holds. Then the condition* (1.13) *or*

$$\int_{\mathscr{R}[g]} q(t)f(I_{n-1}(t, g(t); p_{n-1},\dots, p_1))dt = \infty \tag{2.9}$$

implies that $\mathscr{N}_0^+ = \emptyset$ *for* (NI$^+$).

(ii) *Assume that* (2.6) *holds. Then the condition* (1.13) *or*

$$\int_{\mathscr{R}[g]} q(t)|f(-I_{n-1}(t, g(t); p_{n-1},\dots, p_1))|dt = \infty \tag{2.10}$$

implies that $\mathscr{N}_0^- = \emptyset$ *for* (NI$^-$).

PROOF. (i) We first assume the condition (1.13). Sine, by (2.3) with $\xi = 1$,

$$f(x) \geq f(1)\omega[f](x) \qquad \text{for} \quad x > 0,$$

it follows that

$$\int_0^m \frac{dx}{f(x)} \leq \frac{1}{f(1)}\int_0^m \frac{dx}{\omega[f](x)} \qquad \text{for} \quad m > 0.$$

Thus (2.4) yields (1.12) and hence $\mathscr{N}_0^+ = \emptyset$ by (i) of Theorem 1.2.

We next assume the condition (2.9). Suppose that $\mathscr{N}_0^+ \neq \emptyset$. Then (1.15) holds for some $T_0 > 0$. Fix T_1 so large that

$$T_1 \geq T_0 \quad \text{and} \quad \inf\{g(t): t \geq T_1\} > T_0.$$

Consider (0.5) with i, k, t and s replaced by 0, $n-1$, $g(t)$ and t, respectively. In view of (1.15) and the fact that $|L_{n-1}x|$ is decreasing, we have

$$\begin{aligned} x(g(t)) &= \textstyle\sum_{j=0}^{n-2}(-1)^j I_j(t, g(t); p_j,\dots, p_1)L_jx(t) \\ &\quad + (-1)^{n-1}\int_{g(t)}^{t} I_{n-2}(r, g(t); p_{n-2},\dots, p_1)p_{n-1}(r)L_{n-1}x(r)dr \\ &\geq \int_{g(t)}^{t} I_{n-2}(r, g(t); p_{n-2},\dots, p_1)p_{n-1}(r)|L_{n-1}x(r)|dr \\ &\geq \int_{g(t)}^{t} I_{n-2}(r, g(t); p_{n-2},\dots, p_1)p_{n-1}(r)dr \cdot |L_{n-1}x(t)| \\ &= I_{n-1}(t, g(t); p_{n-1},\dots, p_1)|L_{n-1}x(t)|, \quad t \in \mathscr{R}[g] \cap [T_1, \infty). \end{aligned}$$

Set $u(t)=|L_{n-1}x(t)|$. Since by (2.3)

$$f(x(g(t))) \geq f(I_{n-1}(t, g(t); p_{n-1},..., p_1))\omega[f](u(t))$$

for $t\in \mathscr{R}[g] \cap [T_1, \infty)$, we obtain

$$\begin{aligned} -u'(t) &= (-1)^n L_n x(t) = -\sigma L_n x(t) \\ &\geq q(t)f(I_{n-1}(t, g(t); p_{n-1},..., p_1))\omega[f](u(t)), \quad t\in \mathscr{R}[g] \cap [T_1, \infty). \end{aligned}$$

Choose T_2 ($\geq T_1$) arbitrarily. Dividing both sides by $\omega[f](u(t))$ and integrating the result on $\mathscr{R}[g] \cap [T_1, T_2]$, we see that

$$\begin{aligned} &\int_{\mathscr{R}[g]\cap[T_1,T_2]} q(t)\, f(I_{n-1}(t, g(t); p_{n-1},..., p_1))\,dt \\ &\qquad \leq \int_{T_2}^{T_1} \frac{u'(t)}{\omega[f](u(t))}\,dt = \int_{U_2}^{U_1} \frac{du}{\omega[f](u)} \end{aligned}$$

where $U_i=u(T_i)$, $i=1, 2$. Letting $T_2\to\infty$, we conclude that

$$\int_{\mathscr{R}[g]\cap[T_1,\infty)} q(t)f(I_{n-1}(t, g(t); p_{n-1},..., p_1))dt \leq \int_0^{U_1} \frac{du}{\omega[f](u)} < \infty,$$

which contradicts (2.9)

(ii) can be shown similarly.

THEOREM 2.3. *Let* $\sigma=-1$.

(i) *Assume that*

$$\int^{\infty} \frac{du}{\omega[f](u)} < \infty. \tag{2.11}$$

Then the condition (1.21) *or*

$$\int_{\mathscr{A}[g]} q(t)f(I_{n-1}(g(t), t; p_1,..., p_{n-1}))dt = \infty \tag{2.12}$$

implies that $\mathscr{N}_n^+=\emptyset$ *for* (NI$_+$).

(ii) *Assume that*

$$\int^{-\infty} \frac{du}{\omega[f](u)} < \infty. \tag{2.13}$$

Then the condition (1.21) *or*

$$\int_{\mathscr{A}[g]} q(t)\,|f(-I_{n-1}(g(t), t; p_1,..., p_{n-1}))|dt = \infty \tag{2.14}$$

implies that $\mathscr{N}_n^-=\emptyset$ *for* (NI$^-$).

PROOF. (i) Since (2.11) implies (1.1), the condition (1.21) ensures that $\mathcal{N}_n^+=\emptyset$ by (i) of Theorem 1.3. So, it remains to show that $\mathcal{N}_n^+=\emptyset$ if (2.12) is satisfied. Suppose to the contrary that $\mathcal{N}_n^+\neq\emptyset$. Let $x\in\mathcal{N}_n^+$. Then, (1.22) holds for some T_0. Replacing i, k, t and s in (0.4) by 0, $n-1$, $g(t)$ and t, respectively, we get

$$\begin{aligned} x(g(t)) &= \textstyle\sum_{j=0}^{n-2} I_j(g(t), t; p_1,\dots, p_j)L_jx(t) \\ &\quad + \int_t^{g(t)} I_{n-2}(g(t), r; p_1,\dots, p_{n-2})p_{n-1}(r)L_{n-1}x(r)dr \\ &\geq \int_t^{g(t)} I_{n-2}(g(t), r; p_1,\dots, p_{n-2})p_{n-1}(r)L_{n-1}x(r)dr \end{aligned}$$

for $t\in\mathscr{A}[g]\cap[T_0,\infty)$, which, in view of the increasing nature of $L_{n-1}x$ and (0.3), implies

$$x(g(t)) \geq I_{n-1}(g(t), t; p_1,\dots, p_{n-1})L_{n-1}x(t), \quad t\in\mathscr{A}[g]\cap[T_0,\infty).$$

Set $u(t)=L_{n-1}x(t)$. Then, $u(t)$ satisfies

$$\begin{aligned} u'(t) &= L_nx(t) = -\sigma L_nx(t) \geq q(t)f(x(g(t))) \\ &\geq q(t)f(I_{n-1}(g(t), t; p_1,\dots, p_{n-1}))\omega[f](u(t)) \end{aligned}$$

for $t\in\mathscr{A}[g]\cap[T_0,\infty)$. Dividing the above by $\omega[f](u(t))$ and integrating on $\mathscr{A}[g]\cap[T_0, T_1]$, we obtain

$$\begin{aligned} &\int_{\mathscr{A}[g]\cap[T_0,T_1]} q(t)\, f(I_{n-1}(g(t), t; p_1,\dots, p_{n-1}))dt \\ &\qquad \leq \int_{T_0}^{T_1}\frac{u'(t)}{\omega[f](u(t))}dt = \int_{U_0}^{U_1}\frac{du}{\omega[f](u)}, \end{aligned}$$

where $U_i=u(T_i)$, $i=0, 1$. Letting $T_1\to\infty$, we find

$$\int_{\mathscr{A}[g]\cap[T_0,\infty)} q(t)f(I_{n-1}(g(t), t; p_1,\dots, p_{n-1}))dt \leq \int_{U_0}^{\infty}\frac{du}{\omega[f](u)} < \infty.$$

This contradicts (2.12). This finishes the proof of (i).

A parallel argument holds in order to prove statement (ii).

Applying Theorem 2.1–2.3 to the equation (NE), we obtain the following results.

THEOREM 2.4. *Let* $1\leq l\leq n-1$ *and* $(-1)^{n-l}\sigma=-1$. *Suppose that there exist* $\alpha_+, \alpha_-\in\{1,\dots, N\}$ *such that*

$$\int_{\pm 0}\frac{dx}{\omega[f_{\alpha_\pm}](x)} < \infty. \tag{2.15}$$

Then $\mathcal{N}_l=\emptyset$ *for* (NE) *if*

$$\int^{\infty} q_{\alpha_\pm}(t)\,|f_{\alpha_\pm}(\pm H_l[g_{\alpha_\pm}](t))|dt = \infty. \tag{2.16; l}$$

THEOREM 2.5. *Let* $(-1)^n\sigma=-1$. *Suppose that there exists* $\beta_+,\ \beta_-\in\{1,\dots,N\}$ *such that*

$$\int_{\pm 0} \frac{dx}{\omega[f_{\beta_\pm}](x)} < \infty. \tag{2.17}$$

Then $\mathcal{N}_0=\emptyset$ *for* (NE) *if* (1.27) *holds or*

$$\int_{\mathscr{R}[g_{\beta_\pm}]} q_{\beta_\pm}(t)\,|f_{\beta_\pm}(\pm I_{n-1}(t,\ g_{\beta_\pm}(t);\ p_{n-1},\dots,\ p_1))|dt = \infty. \tag{2.18}$$

THEOREM 2.6. *Let* $\sigma=-1$. *Suppose that there exist* $\gamma_+,\ \gamma_-\in\{1,\dots,N\}$ *such that*

$$\int^{\pm\infty} \frac{dx}{\omega[f_{\gamma_\pm}](x)} < \infty. \tag{2.19}$$

Then $\mathcal{N}_n=\emptyset$ *for* (NE) *if* (1.29) *holds or*

$$\int_{\mathscr{A}[g_{\gamma_\pm}]} q_{\gamma_\pm}(t)\,|f_{\gamma_\pm}(\pm I_{n-1}(g_{\gamma_\pm}(t),\ t;\ p_1,\dots,\ p_{n-1}))|dt = \infty. \tag{2.20}$$

These theorems applied to the special equation (1.30) yields the following corollaries.

COROLLARY 2.1. *Let* $1\le l\le n-1$ *and* $(-1)^{n-l}\sigma=-1$. *Then* $\mathcal{N}_l=\emptyset$ *for* (1.30) *if*

$$\int_{\pm 0} \frac{dx}{\omega[f](x)} < \infty$$

and

$$\int^{\infty} q(t)\,|f(\pm t^{n-l-1}g^{l-1}(t)g_*(t))|dt = \infty.$$

COROLLARY 2.2. *Let* $(-1)^n\sigma=-1$. *Then* $\mathcal{N}_0=\emptyset$ *for* (1.30) *if*

$$\int_{\pm 0} \frac{dx}{\omega[f](x)} < \infty$$

and either

$$\int_{\mathscr{R}[g]} (t-g(t))^{n-1}q(t)dt = \infty \quad or \quad \int_{\mathscr{R}[g]} q(t)\,|f(\pm(t-g(t))^{n-1})|dt = \infty.$$

COROLLARY 2.3. *Let* $\sigma=-1$. *Then* $\mathcal{N}_n=\emptyset$ *for* (1.30) *if*

$$\int^{\pm\infty}\frac{dx}{\omega[f](x)}<\infty$$

and either

$$\int_{\mathscr{A}[g]}(g(t)-t)^{n-1}q(t)dt=\infty \quad or \quad \int_{\mathscr{A}[g]}q(t)\,|f(\pm(g(t)-t)^{n-1})|dt=\infty.$$

REMARK 2.1. Corollary 2.2 is an extension of Theorem 4 of Kusano and Onose [23].

EXAMPLE 2.1. Define the function f by

$$f(x)=\begin{cases}(\beta/\alpha)^{\beta}|x|^{\alpha}\operatorname{sgn} x & (|x|\geq e^{-\beta/\alpha}),\\ |x|^{\alpha}|\log|x|\,|^{\beta}\operatorname{sgn} x & (0<|x|\leq e^{-\beta/\alpha}),\\ 0 & (x=0),\end{cases}$$

where α and β are positive constants. This function satisfies (d′) and (e′), and we obtain

$$\omega[f](x)=\begin{cases}|x|^{\alpha}\{\beta/(\beta+\alpha\log|x|)\}^{\beta}\operatorname{sgn} x & (|x|\geq 1),\\ |x|^{\alpha}\operatorname{sgn} x & (|x|\leq 1).\end{cases}$$

Consider the equation (1.30) with this f under the assumptions

$$(-1)^n\sigma=-1,\quad \alpha<1,\quad t-e^{-\beta/\alpha}\leq g(t)\leq t.$$

By Corollary 2.2, $\mathcal{N}_0=\emptyset$ for this equation if either

$$\int^{\infty}(t-g(t))^{n-1}q(t)dt=\infty$$

or

$$\int^{\infty}(t-g(t))^{\alpha(n-1)}|\log(t-g(t))|^{\beta}q(t)dt=\infty.$$

On the other hand, the condition

$$|f(xy)|\geq|f(x)f(y)| \qquad \text{for all}\quad x,\ y$$

is not satisfied, since

$$f(xy)/[f(x)f(y)]=|(\log x)^{-1}+(\log y)^{-1}|^{\beta}\longrightarrow 0 \text{ as } x,\ y\longrightarrow 0+,$$

and hence Theorem 4 in [23] can not be applied to this equation.

3. Oscillation. We want to develop criteria for equation (NE) to be oscillatory or almost oscillatory. To do this we need conditions which ensure the existence of a nonoscillatory solution $x(t)$ of (NE) such that

$$x(t) = c + o(1) \text{ as } t \to \infty \text{ for some } c \neq 0 \tag{3.1}$$

or

$$x(t) = I_{n-1}(t, 0; p_1, \dots, p_{n-1})[c + o(1)] \text{ as } t \to \infty \text{ for some } c \neq 0. \tag{3.2}$$

THEOREM 3.1. *Suppose that the conditions* (a)–(d) *hold.*

(i) *A necessary and sufficient condition for* (NE) *to have a solution* $x(t)$ *satisfying* (3.1) *is that*

$$\int^{\infty} \textstyle\sum_{h=1}^{N} I_{n-1}(t, 0; p_{n-1}, \dots, p_1)(t) q_h(t) dt < \infty. \tag{3.3}$$

(ii) *A necessary and sufficient condition for* (NE) *to have a solution* $x(t)$ *satisfying* (3.2) *is that*

$$\int^{\infty} \textstyle\sum_{h=1}^{N} q_h(t) |f_h(c' I_{n-1}(g_h(t), 0; p_1, \dots, p_{n-1}))| dt < \infty \tag{3.4}$$

for some c' *such that* $cc' > 0$.

For the proof of this theorem see Kitamura and Kusano [11] and Fink and Kusano [5].

We first discuss the almost oscillatory behavior of (NE). For convenience, we employ the notation:

$$J(t, s) = \operatorname{sgn}(t-s) \cdot |I_{n-1}(t, s; p_1, \dots, p_{n-1})|, \quad J(t) = J(t, 0),$$
$$K(t, s) = \operatorname{sgn}(t-s) \cdot |I_{n-1}(t, s; p_{n-1}, \dots, p_1)|, \quad K(t) = K(t, 0).$$

THEOREM 3.2. *Suppose that* (a)–(d) *hold. A sufficient condition for* (NE) *to be almost oscillatory is that*:

(i) *when* $\sigma = 1$ *and* n *is even, there exist* $\alpha_+, \alpha_- \in \{1, \dots, N\}$ *such that* (1.24) *and* (1.25; l) ($l = 1, 3, \dots, n-1$) *hold*;

(ii) *when* $\sigma = 1$ *and* n *is odd, there exist* $\alpha_+, \alpha_- \in \{1, \dots, N\}$ *such that* (1.24) *and* (1.25; l) ($l = 2, 4, \dots, n-1$) *hold, and*

$$\int^{\infty} \textstyle\sum_{h=1}^{N} K(t) q_h(t) dt = \infty; \tag{3.5}$$

(iii) *when* $\sigma = -1$ *and* n *is even, there exist* $\alpha_+, \alpha_- \in \{1, \dots, N\}$ *such that* (1.24), (1.25; l) ($l = 2, 4, \dots, n-2$) *and* (3.5) *hold, and*

$$\int^{\infty} \textstyle\sum_{h=1}^{N} q_h(t) |f_h(c J(g_h(t)))| dt = \infty \qquad \textit{for all} \quad c \neq 0; \tag{3.6}$$

(iv) *when* $\sigma=-1$ *and* n *is odd, there exist* α_+, $\alpha_-\in\{1,\dots,N\}$ *such that* (1.24), (1.25; l) ($l=1, 3,\dots, n-2$) *and* (3.6) *hold.*

PROOF. (i) By Theorem 1.4 we have $\mathcal{N}=\mathcal{N}_1\cup\mathcal{N}_3\cup\cdots\cup\mathcal{N}_{n-1}=\emptyset$, that is, $\mathcal{S}=\mathcal{O}$.

(ii) Theorem 1.4 implies that $\mathcal{S}=\mathcal{O}\cup\mathcal{N}_0$. Let x be a positive solution of (NE) belonging to $\mathcal{N}_0$. Then (1.15) holds for some $T_0>0$. Thus x is decreasing on $[T_0, \infty)$ and $x(t)$ has a nonnegative limit as $t\to\infty$. This limit must be 0 by (i) of Theorem 3.1. It can now be shown easily that $x(t)$ satisfies (3). A similar argument applies to a negative solution of (NE) belonging to $\mathcal{N}_0$.

(iii) From Theorem 1.4 it follows that $\mathcal{S}=\mathcal{O}\cup\mathcal{N}_0\cup\mathcal{N}_n$. Let $x\in\mathcal{N}_n$ be a positive solution of (NE). Then (1.22) holds for some $T_0>0$, so that x is increasing on $[T_0, \infty)$ and tends to a finite or infinite limit as $t\to\infty$. By (ii) of Theorem 3.1 this limit must be infinite under the condition (3.6), and clearly, $x(t)$ satisfies (4). Similarly, a negative solution $x\in\mathcal{N}_n$ is also strongly increasing.

(iv) We have $\mathcal{S}=\mathcal{O}\cup\mathcal{N}_n$ by Theorem 1.4. Exactly as above, we can show that a solution belonging to $\mathcal{N}_n$ is strongly increasing. This completes the proof.

Using Theorem 2.4 instead of Theorem 1.4, we can easily obtain the following result.

THEOREM 3.3. *Suppose that* (a)–(e) *hold. A sufficient condition for* (NE) *to be almost oscillatory is that*:

(i) *when* $\sigma=1$ *and* n *is even, there exist* α_+, $\alpha_-\in\{1,\dots,N\}$ *such that* (2.15) and (2.16; l) ($l=1, 3,\dots, n-1$) *hold*;

(ii) *when* $\sigma=1$ *and* n *is odd, there exist* α_+, $\alpha_-\in\{1,\dots,N\}$ *such that* (2.15) (2.16; l) ($l=2, 4,\dots, n-1$) *and* (3.5) *hold*;

(iii) *when* $\sigma=-1$ *and* n *is even, there exist* α_+, $\alpha_-\in\{1,\dots,N\}$ *such that* (2.15), (2.16; l) ($l=2, 4,\dots, n-2$), (3.5) *and* (3.6) *hold*;

(iv) *when* $\sigma=-1$ *and* n *is odd, there exist* α_+, $\alpha_-\in\{1,\dots,N\}$ *such that* (2.15), (2.16; l) ($l=1, 3,\dots, n-2$) *and* (3.6) *hold.*

We are able to obtain necessary and sufficient conditions for a certain class of equations of the form (NE) to be almost oscillatory, as the following theorem shows.

THEOREM 3.4. *Suppose that* (a)–(d) *hold. Suppose moreover that*

$$\int^{\pm\infty}\frac{dx}{f_h(x)}<\infty,\quad 1\le h\le N \tag{3.7}$$

and

$$\liminf_{t\to\infty} H_l[g_h](t)/K(t)>0,\quad 1\le h\le N \tag{3.8}$$

for any l such that

$$1 \leq l \leq n-1, \quad (-1)^{n-l}\sigma = -1.$$

Then a necessary and sufficient condition for (NE) *to be almost oscillatory is that*:

(i) *when $\sigma=1$, the condition* (3.5) *holds*;

(ii) *when $\sigma=-1$, the conditions* (3.5) *and* (3.6) *hold.*

PROOF. (i) Let $\sigma=1$ and n be even. From (3.5), there exists a $k \in \{1,\dots, N\}$ such that

$$\int^{\infty} K(t)q_k(t)dt = \infty.$$

Put $\alpha_+ = \alpha_- = k$. Then (3.7) implies (1.24). Since, from (3.8),

$$H_l[g_{\alpha_\pm}](t) \geq cK(t), \quad l = 1, 3,\dots, n-1$$

for some positive constant c and all large t, we have

$$\int_T^{\infty} H_l[g_{\alpha_\pm}](t)q_{\alpha_\pm}(t)dt \geq c\int_T^{\infty} K(t)q_k(t)dt = \infty$$

for sufficiently large T. Thus (1.25; l) ($l=1, 3,\dots, n-1$) hold. Hence, by (i) of Theorem 3.2, (NE) is almost oscillatory.

Conversely, if (3.5) is not satisfied, then, by Theorem 3.1, equation (NE) has a solution $x(t)$ which converges to some non-zero constant as $t \to \infty$, so that (NE) is not almost oscillatory. Therefore the condition (3.5) is necessary for (NE) to be almost oscillatory.

The case when $\sigma=1$ and n is odd can be treated similarly.

(ii) Let $\sigma=-1$ and n be even. Since (3.8) implies (1.25; l) ($l=2, 4,\dots, n-2$) as in (i), from (iii) of Theorem 3.2 it follows that (3.5) and (3.6) are sufficient for (NE) to be almost oscillatory.

If (3.5) is not satisfied, then, by (i) of Theorem 3.1, (NE) has a nonoscillatory solution which is not strongly decreasing. Similarly, if (3.6) is not satisfied, then (NE) has a nonoscillatory solution which is not strongly increasing. This shows that (3.5) and (3.6) are necessary conditions for (NE) to be almost oscillatory.

The proof for the case when $\sigma=-1$ and n is odd is analogous.

A variant of Theorem 3.4 is obtained if the condition (e) is added.

THEOREM 3.5. *Suppose that* (a)–(e) *hold. Suppose moreover that*

$$\int_{\pm 0} \frac{dx}{\omega[f_h](x)} < \infty, \quad 1 \leq h \leq N \tag{3.9}$$

and

(3.10) $$\liminf_{t\to\infty} H_l[g_h](t)/J(g_h(t)) > 0, \quad 1 \le h \le N$$

for any l *such that*

$$1 \le l \le n-1, \quad (-1)^{n-l}\sigma = -1.$$

Then a necessary and sufficient condition for (NE) *to be almost oscillatory is that*:

(i) *when* $(-1)^n\sigma = 1$, *the condition* (3.6) *holds*;

(ii) *when* $(-1)^n\sigma = -1$, *the conditions* (3.5) *and* (3.6) *holds.*

PROOF. (i) Let $\sigma=1$ and n be even. From (3.10),

$$H_l[g_h](t) \ge cJ(g_h(t)), \quad 1 \le h \le n, \quad l = 1, 3,\dots, n-1$$

for some positive constant c and all large t, and from (3.6) there exist α_+, $\alpha_- \in \{1,\dots, N\}$ such that

$$\int^\infty q_{\alpha_\pm}(t)\,|f_{\alpha_\pm}(\pm cJ(g_{\alpha_\pm}(t)))|dt = \infty.$$

Thus

$$\int_T^\infty q_{\alpha_\pm}(t)\,|f_{\alpha_\pm}(\pm H_l[g_{\alpha_\pm}](t))|dt \ge \int_T^\infty q_{\alpha_\pm}(t)\,|f_{\alpha_\pm}(\pm cJ(g_{\alpha_\pm}(t)))|dt = \infty$$

for $l=1, 3,\dots, n-1$ and any $T>0$. From (i) of Theorem 3.3 it follows that (NE) is almost oscillatory.

The necessity of the condition (3.6) is shown as in the proof of (i) of Theorem 3.4. The case when $\sigma = -1$ and n is odd can be discussed similarly.

(ii) The proof is quite similar to the above, and so the details will be omitted.

We are next concerned with the situation in which (NE) is oscillatory. Since the oscillatory nature is equivalent to the almost oscillatory nature when $\sigma=1$ and n is even, we omit this case and discuss the remaining cases.

THEOREM 3.6. *Suppose that* (a)–(d) *hold. A sufficient condition for* (NE) *to be oscillatory is that*:

(i) *when* $\sigma=1$ *and* n *is odd, there exist* α_+, $\alpha_- \in \{1,\dots, N\}$ *satisfying* (1.24) *and* (1.25; l) ($l=2, 4,\dots, n-1$), *and* β_+, $\beta_- \in \{1,\dots, N\}$ *satisfying* (1.26) *and* (1.27);

(ii) *when* $\sigma = -1$ *and* n *is even, there exist* α_+, $\alpha_- \in \{1,\dots, N\}$ *satisfying* (1.24) *and* (1.25; l) ($l=2, 4,\dots, n-2$), β_+, $\beta_- \in \{1,\dots, N\}$ *satisfying* (1.26) *and* (1.27), *and* γ_+, $\gamma_- \in \{1,\dots, N\}$ *satisfying* (1.28) *and* (1.29);

(iii) *when* $\sigma=-1$ *and* n *is odd, there exist* α_+, $\alpha_- \in \{1,\dots, N\}$ *satisfying* (1.24) *and* (1.25; l) ($l=1, 3,\dots, n-2$), *and* γ_+, $\gamma_- \in \{1,\dots, N\}$ *satisfying* (1.28) *and* (1.29)

This theorem is an immediate consequence of Theorems 1.4–1.6. The theorem below follows readily from Theorems 2.4–2.6.

THEOREM 3.7. *Suppose that* (a)–(e) *hold. A sufficient condition for* (NE) *to be oscillatory is that*:

(i) *when* $\sigma=1$ *and* n *is odd, there exist* $\alpha_+, \alpha_- \in \{1,\dots, N\}$ *satisfying* (2.15) *and* (2.16; l) ($l=2, 4,\dots, n-1$), *and* β_+, $\beta_- \in \{1,\dots, N\}$ *satisfying* (2.17) *and either* (1.27) *or* (2.18);

(ii) *when* $\sigma=-1$ *and* n *is even, there exist* α_+, $\alpha_- \in \{1,\dots, N\}$ *satisfying* (2.15) *and* (2.16; l) ($l=2, 4,\dots, n-2$), β_+, $\beta_- \in \{1,\dots, N\}$ *satisfying* (2.17) *and either* (1.27) *or* (2.18), *and* γ_+, $\gamma_- \in \{1,\dots, N\}$ *satisfying* (2.19) *and either* (1.29) *or* (2.20);

(iii) *when* $\sigma=-1$ *and* n *is odd, there exist* α_+, $\alpha_- \in \{1,\dots, N\}$ *satisfying* (2.15) *and* (2.16; l) ($l=1, 3,\dots, n-2$), *and* γ_+, $\gamma_- \in \{1,\dots, N\}$ *satisfying* (2.19) *and either* (1.29) *or* (2.20)

We show that there is a class of equations (NE) for which necessary and sufficient conditions for oscillation can be obtaind.

THEOREM 3.8. *Suppose that* (a)–(d) *hold. Let* $\sigma=1$ *and* n *be odd. Suppose moreover that* (3.8) *holds and*

$$\int_{\pm 0}^{\pm\infty} \frac{dx}{f_h(x)} < \infty, \quad 1 \le h \le N \tag{3.11}$$

and

$$\liminf_{t\to\infty} K(t, g_h(t))/K(t) > 0, \quad 1 \le h \le N. \tag{3.12}$$

Then (3.5) *is a necessary and sufficient condition for* (NE) *to be oscillatory.*

PROOF. The necessity of (3.5) follows from (i) of Theorem 3.1, so we prove the sufficiency. Note that (3.11) implies (1.24) and (1.26) for any $\alpha_\pm$ and $\beta_\pm$. Using (3.8) and (3.5), we see that

$$H_l[g_h](t) \ge c_1 K(t), \quad l = 2, 4,\dots, n-1, \quad 1 \le h \le N$$

for some positive constant c_1 and all large t, and

$$\int^{\infty} K(t)q_k(t)dt = \infty$$

for some $k \in \{1,\dots, N\}$. Put $\alpha_+ = \alpha_- = k$. Then (1.25; l) ($l=2, 4,\dots, n-1$) are satisfied. On the other hand, using (3.12), we obtain

$$K(t, g_k(t)) \geq c_2 K(t)$$

for some positive constant c_2 and all large t. This means that $\mathscr{R}[g_k] \supset [T, \infty)$ for sufficiently large $T>0$, and hence

$$\int_{\mathscr{R}[g_k]} K(t, g_k(t)) q_k(t) dt \geq c_2 \int_T^\infty K(t) q_k(t) dt = \infty.$$

Thus (1.27) is satisfied if we take $\beta_+ = \beta_- = k$. Therefore, by (i) of Theorem 3.6, we conclude that equation (NE) is oscillatory.

THEOREM 3.9. *Suppose that* (a)–(d) *hold. Let* $\sigma = -1$ *and* n *be odd. Suppose moreover that* (3.7) *and* (3.8) *hold, and*

(3.13) $$\liminf_{t\to\infty} J(g_h(t), t)/K(t) > 0, \quad 1 \leq h \leq N.$$

Then (3.5) *is a necessary and sufficient condition for* (NE) *to be oscillatory.*

PROOF. We only prove the sufficiency of (3.5). Since (3.9) implies (1.24) and (1.28) for any $\alpha_\pm$ and $\gamma_\pm$, it suffices to show that there exist $\alpha_\pm$ satisfying (1.25; l) ($l = 1, 3, \dots, n-2$), and $\gamma_\pm$ satisfying (1.29). The former is an easy consequence of the same argument as in the previous proof. To prove the latter notice that (3.5) and (3.13) imply that

$$J(g_k(t), t) \geq cK(t) \qquad \text{for all large } t,$$

and

$$\int^\infty K(t) q_k(t) dt = \infty$$

for some $c>0$ and $k \in \{1, \dots, N\}$. It follows that $\mathscr{A}[g_k] \supset [T, \infty)$ for sufficiently large $T>0$, and

$$\int_{\mathscr{A}[g_k]} J(g_k(t), t) q_k(t) dt \geq c \int_T^\infty K(t) q_k(t) dt = \infty.$$

Thus (1.29) is satisfied with $\gamma_+ = \gamma_- = k$, and the conclusion follows from (iii) of Theorem 3.6.

THEOREM 3.10. *Assume that* (a)–(e) *hold. Let* $\sigma = 1$ *and* n *be odd. Assume moreover that* (3.9) *and* (3.10) *hold, and*

(3.14) $$\liminf_{t\to\infty} K(t, g_h(t))/J(g_h(t)) > 0, \quad 1 \leq h \leq N$$

Then (3.6) *is a necessary and sufficient condition for* (NE) *to be oscillatory.*

PROOF. The necessity of (3.6) is obvious from (ii) of Theorem 3.1. To prove its sufficiency we note that (3.9) implies (2.15) and (2.17) for any $\alpha_{\pm}$ and $\beta_{\pm}$. From (3.10) and (3.6) we see that

$$H_l[g_h](t) \geq c_1 J(g_h(t)), \quad l = 2, 4, \dots, n-1, \quad 1 \leq h \leq N$$

for some positive constant c_1 and all large t, and

$$\int^{\infty} q_{\alpha_\pm}(t)\,|f_{\alpha_\pm}(\pm c_1 J(g_{\alpha_\pm}(t)))|dt = \infty$$

for some α_+, $\alpha_- \in \{1, \dots, N\}$. Thus the conditions (2.16; l) ($l=2, 4, \dots, n-1$) are satisfied. On the other hand, using (3.14) and (3.6), we obtain

$$K(t, g_h(t)) \geq c_2 J(g_h(t)), \quad 1 \leq h \leq N$$

for some positive constant c_2 and all large t, and

$$\int^{\infty} q_{\beta_\pm}(t)\,|f_{\beta_\pm}(\pm c_2 J(g_{\beta_\pm}(t)))|dt = \infty$$

for some β_+, $\beta_- \in \{1, \dots, N\}$. Hence $\mathscr{R}[g_{\beta_\pm}] \supset [T, \infty)$ for sufficiently large $T>0$, and we find that

$$\begin{aligned}&\int_{\mathscr{R}[g_{\beta\pm}]} q_{\beta_\pm}(t)|f_{\beta_\pm}(\pm K(t, g_{\beta_\pm}(t)))|dt \\ &\geq \int_T^{\infty} q_{\beta_\pm}(t)\,|f_{\beta_\pm}(\pm c_2 J(g_{\beta_\pm}(t)))|dt = \infty.\end{aligned}$$

Thus (2.18) holds. Therefore, (NE) is oscillatory by (i) of Theorem 3.7.

THEOREM 3.11. *Assume that* (a)–(e) *hold. Let* $\sigma = -1$ *and* n *be odd. Assume moreover that* (3.10) *holds, and*

$$\int_{\pm 0}^{\pm\infty} \frac{dx}{\omega[f_h](x)} < \infty, \quad 1 \leq h \leq N \tag{3.15}$$

and

$$\liminf_{t\to\infty} J(g_h(t), t)/J(g_h(t)) > 0, \quad 1 \leq h \leq N. \tag{3.16}$$

Then (3.6) *is a necessary and sufficient condition for* (NE) *to be oscillatory.*

PROOF. We only prove the sufficiency of (3.6). Since (3.15) implies (2.15) and (2.19) for any $\alpha_{\pm}$ and $\gamma_{\pm}$, we show that there exist $\alpha_{\pm}$ satisfying (2.16; l) ($l=1, 3, \dots, n-2$) and $\gamma_{\pm}$ satisfying (2.20). The former can be shown exactly as above. The latter is derived from (3.16) and (3.6). Indeed, there exist a positive constant c and γ_+, $\gamma_- \in \{1, \dots, N\}$ such that

$$J(g_h(t), t) \geq c\,J(g_h(t)), \quad 1 \leq h \leq N$$

for all large t, and

$$\int_{\mathscr{A}[g_{\gamma\pm}]} q_{\gamma\pm}(t)\,|f_{\gamma\pm}(\pm cJ(g_{\gamma\pm}(t)))|dt = \infty.$$

Then $\mathscr{A}[g_{\gamma\pm}] \supset [T, \infty)$ for sufficiently large $T > 0$, and hence we find that

$$\int_{\mathscr{A}[g_{\gamma\pm}]} q_{\gamma\pm}(t)\,|f_{\gamma\pm}(\pm J(g_{\gamma\pm}(t), t))|dt$$
$$\geq \int_T^\infty q_{\gamma\pm}(t)\,|f_{\gamma\pm}(\pm cJ(g_{\gamma\pm}(t)))|dt = \infty.$$

Thus (2.20) holds. Applying (iii) of Theorem 3.7, we see that (NE) is oscillatory.

We now list, as corollaries, the results in this section applied to the particular equation (1.30).

COROLLARY 3.1. *Let $\sigma = 1$ and n be even.*

(i) *Assume that* (a), (b), (c′) *and* (d′) *hold. Suppose that*

$$\int^{\pm\infty} \frac{dx}{f(x)} < \infty.$$

Then the condition

$$\int^\infty g_*^{n-1}(t)q(t)dt = \infty$$

is sufficient for (1.30) *to be oscillatory.*

If in addition

$$\liminf_{t\to\infty} g(t)/t > 0,$$

then the condition

$$\int^\infty t^{n-1}q(t)dt = \infty$$

is necessary and sufficient for (1.30) *to be oscillatory.*

(ii) *Assume that* (a), (b), (c′), (d′) *and* (e′) *hold. Suppose that*

$$\int_{\pm 0}^{+\infty} \frac{dx}{\omega[f](x)} < \infty.$$

Then the condition

$$\int^\infty q(t)\,|f(\pm g_*^{n-1}(t))|dt = \infty$$

is sufficient for (1.30) *to be oscillatory.*

If in addition

$$\limsup_{t\to\infty} g(t)/t < \infty,$$

then the condition

$$\int^{\infty} q(t)\,|f(\pm g^{n-1}(t))|dt = \infty$$

is necessary and sufficient for (1.30) *to be oscillatory.*

COROLLARY 3.2. *Let $\sigma=1$ and n be odd.*

(i) *Assume that* (a), (b), (c′) *and* (d′) *hold. Suppose that*

$$\int^{\pm\infty} \frac{dx}{f(x)} < \infty.$$

Then the condition

$$\int^{\infty} g_*^{n-1}(t)q(t)dt = \infty$$

is sufficient for (1.30) *to be almost oscillatory.*

If in addition

$$\liminf_{t\to\infty} g(t)/t > 0,$$

then the condition

$$\int^{\infty} t^{n-1}q(t)dt = \infty$$

is necessary and sufficient for (1.30) *to be almost oscillatory.*

(ii) *Assume that* (a), (b), (c′), (d′) *and* (e′) *hold. Suppose that*

$$\int_{\pm 0} \frac{dx}{\omega[f](x)} < \infty.$$

Then the conditions

$$\int^{\infty} t^{n-1}q(t)dt = \infty$$

and

$$\int^{\infty} q(t)\,|f(\pm g(t)g_*^{n-2}(t))|dt = \infty$$

are sufficient for (1.30) *to be almost oscillatory.*

If in addition

$$\limsup_{t\to\infty} g(t)/t < \infty,$$

then the conditions

$$\int^{\infty} t^{n-1} q(t)dt = \infty$$

and

$$\int^{\infty} q(t)\,|f(\pm g^{n-1}(t))|dt = \infty$$

are necessary and sufficient for (1.30) *to be almost oscillatory.*

(iii) *Assume that* (a), (b), (c′) *and* (d′) *hold. Suppose that*

$$\int_{\pm 0}^{\pm\infty} \frac{dx}{f(x)} < \infty.$$

Then the conditions

$$\int^{\infty} g(t) g_*^{n-2}(t) q(t)dt = \infty$$

and

$$\int_{\mathscr{R}[g]} (t-g(t))^{n-1} q(t)dt = \infty$$

are sufficient for (1.30) *to be oscillatory.*

If in addition

$$0 < \liminf_{t\to\infty} g(t)/t \leq \limsup_{t\to\infty} g(t)/t < 1,$$

then the condition

$$\int^{\infty} t^{n-1} q(t)dt = \infty$$

is necessary and sufficient for (1.30) *to be oscillatory.*

(iv) *Assume that* (a), (b), (c′), (d′) *and* (e′) *hold. Suppose that*

$$\int_{\pm 0}^{\pm\infty} \frac{dx}{\omega[f](x)} < \infty.$$

Then the conditions

$$\int^{\infty} q(t)\,|f(\pm g(t) g_*^{n-2}(t))|dt = \infty$$

and

$$\int_{\mathscr{P}[g]} (t-g(t))^{n-1}q(t)dt = \infty \quad or \quad \int_{\mathscr{P}[g]} q(t)|f(\pm(t-g(t))^{n-1})|dt = \infty$$

are sufficient for (1.30) *to be oscillatory.*

If in addition

$$\limsup_{t\to\infty} g(t)/t > 1,$$

then the condition

$$\int^{\infty} q(t)|f(\pm g^{n-1}(t))|dt = \infty$$

is necessary and sufficient for (1.30) *to be oscillatory.*

COROLLARY 3.3. *Let* $\sigma=-1$ *and* n *be even.*

(i) *In addition to* (a), (b), (c′) *and* (d′) *suppose that*

$$\int^{\pm\infty} \frac{dx}{f(x)} < \infty.$$

Then (1.30) *is almost oscillatory if*

$$\int^{\infty} tg_*^{n-2}(t)q(t)dt = \infty$$

and

$$\int^{\infty} q(t)|f(cg^{n-1}(t))|dt = \infty \qquad for\ any \quad c \neq 0.$$

Suppose moreover that

$$\liminf_{t\to\infty} g(t)/t > 0.$$

Then (1.30) *is almost oscillatory if and only if*

$$\int^{\infty} t^{n-1}q(t)dt = \infty$$

and

$$\int^{\infty} q(t)|f(cg^{n-1}(t))|dt = \infty \qquad for\ any \quad c \neq 0.$$

(ii) *In addition to* (a), (b), (c′), (d′) *and* (e′) *suppose that*

$$\int_{\pm 0} \frac{dx}{\omega[f](x)} < \infty.$$

Then (1.30) *is almost oscillatory if*

$$\int^{\infty} t^{n-1}q(t)dt = \infty$$

and

$$\int^{\infty} q(t)\,|f(\pm g(t)g_*^{n-2}(t))|dt = \infty.$$

Suppose moreover that

$$\limsup_{t\to\infty} g(t)/t < \infty.$$

Then (1.30) *is almost oscillatory if and only if*

$$\int^{\infty} t^{n-1}q(t)dt = \infty$$

and

$$\int^{\infty} q(t)\,|f(\pm g^{n-1}(t))|dt = \infty,$$

(iii) *In addition to* (a), (b), (c′), *and* (d′) *suppose that*

$$\int_{\pm 0}^{\pm\infty} \frac{dx}{f(x)} < \infty.$$

Then (1.30) *is oscillatory if*

$$\int^{\infty} tg(t)g_*^{n-3}(t)q(t)dt = \infty \quad \textit{for} \quad n > 2,$$

$$\int_{\mathscr{R}[g]} (t-g(t))^{n-1}q(t)dt = \infty$$

and

$$\int_{\mathscr{A}[g]} (g(t)-t)^{n-1}q(t)dt = \infty.$$

(iv) *In addition to* (a), (b), (c′), (d′) *and* (e′) *suppose that*

$$\int_{\pm 0}^{\pm\infty} \frac{dx}{\omega[f](x)} < \infty.$$

Then (1.30) *is oscillatory if*

$$\int^{\infty} q(t)\,|f(\pm tg(t)g_*^{n-3}(t))|dt = \infty \quad \textit{for} \quad n > 2,$$

$$\int_{\mathscr{R}[g]} (t-g(t))^{n-1}q(t)dt = \infty \quad \textit{or} \quad \int_{\mathscr{R}[g]} q(t)\,|f(\pm(t-g(t))^{n-1})|dt = \infty$$

and

$$\int_{\mathscr{A}[g]} (g(t)-t)^{n-1}q(t)dt = \infty \quad or \quad \int_{\mathscr{A}[g]} q(t)\,|f(\pm(g(t)-t)^{n-1})|dt = \infty.$$

COROLLARY 3.4. *Let $\sigma=-1$ and n be odd.*

(i) *In addition to* (a), (b), (c′) *and* (d′) *suppose that*

$$\int^{\pm\infty} \frac{dx}{f(x)} < \infty.$$

Then (1.30) *is almost oscillatory if*

$$\int^{\infty} t g_*^{n-2}(t)q(t)dt = \infty$$

and

$$\int^{\infty} q(t)\,|f(cg^{n-1}(t))|dt = \infty \qquad for\ any \quad c \neq 0.$$

Suppose moreover that

$$\liminf_{t\to\infty} g(t)/t > 0.$$

Then (1.30) *is almost oscillatory if and only if*

$$\int^{\infty} t^{n-1}q(t)dt = \infty$$

and

$$\int^{\infty} q(t)\,|f(cg^{n-1}(t))|dt = \infty \qquad for\ any \quad c \neq 0.$$

(ii) *In addition to* (a), (b), (c′), (d′) *and* (e′) *suppose that*

$$\int_{\pm 0} \frac{dx}{\omega[f](x)} < \infty.$$

Then (1.30) *is almost oscillatory if*

$$\int^{\infty} q(t)\,|f(\pm g_*^{n-1}(t))|dt = \infty.$$

Suppose moreover that

$$\limsup_{t\to\infty} g(t)/t < \infty.$$

Then (1.30) *is almost oscillatory if and only if*

$$\int^{\infty} q(t)\,|f(\pm g^{n-1}(t))|dt = \infty.$$

(iii) *In addition to* (a), (b), (c′) *and* (d′) *suppose that*

$$\int^{\pm\infty} \frac{dx}{f(x)} < \infty.$$

Then (1.30) *is oscillatory if*

$$\int^{\infty} t g_*^{n-2}(t) q(t) dt = \infty$$

and

$$\int_{\mathscr{A}[g]} (g(t)-t)^{n-1} q(t) dt = \infty.$$

Suppose morevoer that

$$\liminf_{t\to\infty} g(t)/t > 1.$$

Then (1.30) *is oscillatory if and only if*

$$\int^{\infty} t^{n-1} q(t) dt = \infty.$$

(iv) *In addition to* (a), (b), (c′), (d′) *and* (e′) *suppose that*

$$\int_{\pm 0}^{\pm\infty} \frac{dx}{\omega[f](x)} < \infty.$$

Then (1.30) *is oscillatory if*

$$\int^{\infty} q(t) |f(\pm t g_*^{n-2}(t))| dt = \infty$$

and

$$\int_{\mathscr{A}[g]} (g(t)-t)^{n-1} q(t) dt = \infty \quad \textit{or} \quad \int_{\mathscr{A}[g]} q(t) |f(\pm (g(t)-t)^{n-1})| dt = \infty.$$

Suppose moreover that

$$1 < \liminf_{t\to\infty} g(t)/t \le \limsup_{t\to\infty} g(t)/t < \infty.$$

Then (1.30) *is oscillatory if and only if*

$$\int^{\infty} q(t) |f(\pm g^{n-1}(t))| dt = \infty.$$

Corollaries 3.1–3.4 are improvements over Corollaries 1–4 of Ivanov, Kitamura, Kusano and Shevelo [8].

EXAMPLE 3.1. Consider the third order equation

(3.17) $$(t^{-b}(t^{-a}x')')' + \sigma q(t)|x(t^{\tau})|^{\gamma}\cdot \operatorname{sgn} x(t^{\tau}) = 0,$$

where

$$\sigma = \pm 1, \quad \gamma, \tau > 0, \quad a, b > -1.$$

(i) Let $\sigma=1$ and $\gamma, \tau<1$. In this case we obtain

$$H_2[g](t) = J(g(t)) = t^{(a+b+2)\tau}/[(b+1)(a+b+2)]$$

and

$$K(t, g(t)) = t^{a+b+2}\{1/[(a+1)(a+b+2)] + o(1)\} \quad \text{as} \quad t \longrightarrow \infty.$$

Therefore, from Theorem 3.10 it follows that (3.17) is oscillatory if and only if

$$\int^{\infty} t^{(a+b+2)\tau\gamma} q(t)dt = \infty.$$

(ii) Let $\sigma=-1$ and $\gamma, \tau>1$. We then have

$$H_1[g](t) = K(t) = t^{a+b+2}/[(a+1)(a+b+2)]$$

and

$$J(g(t), t) = t^{(a+b+2)\tau}\{1/[(b+1)(a+b+2)] + o(1)\} \quad \text{as} \quad t \longrightarrow \infty.$$

Therefore, from Theorem 3.9 it follows that (3.17) is oscillatory if and only if

$$\int^{\infty} t^{a+b+2} q(t)dt = \infty.$$

EXAMPLE 3.2. Consider the fourth order equation

(3.18) $$(t^{-1}x''(t))'' = t^{-2}f(x(t+\sin t)),$$

where the function $f \in C(R, R)$ satisfies

$$xf(x) > 0 \quad (x \neq 0) \quad \text{and} \quad \int^{\infty} \frac{dx}{f(x)} < \infty.$$

Since we can take

$$\sigma = -1,\ p_1(t) = p_3(t) \equiv 1,\ p_2(t) = t,\ g(t) = t + \sin t \quad \text{and} \quad q(t) = t^{-2},$$

we obtain

$$H_2[g](t) = \begin{cases} t^3(t+2\sin t)/12, & 2m\pi \le t \le (2m+1)\pi, \\ (t+\sin t)^3(t-\sin t)/12, & (2m+1)\pi \le t \le 2(m+1)\pi \end{cases}$$

for $m=1, 2,\dots$, which shows that $H_2[g](t) \sim t^4/12$ $(t \to \infty)$ and

$$\int^{\infty} H_2[g](t)q(t)dt = \infty.$$

The conditions (1.27) and (1.29) also hold for this equation, since

$$\int_{\mathscr{A}[g]} I_3(g(t), t; p_1, p_2, p_3)q(t)dt$$
$$= (1/12) \textstyle\sum_{m=1}^{\infty} \displaystyle\int_{2m\pi}^{(2m+1)\pi} (2t+\sin t) \sin^3 t \cdot t^{-2} dt$$
$$\geq (1/6) \textstyle\sum_{m=1}^{\infty} \displaystyle\int_{2m\pi}^{(2m+1)\pi} t^{-1} \sin^3 t dt = \infty$$

and

$$\int_{\mathscr{R}[g]} I_3(t, g(t); p_3, p_2, p_1)q(t)dt$$
$$= (1/12) \textstyle\sum_{m=1}^{\infty} \displaystyle\int_{(2m+1)\pi}^{2(m+1)\pi} (2t+\sin t)|\sin^3 t| t^{-2} dt$$
$$\geq (1/6) \textstyle\sum_{m=1}^{\infty} \displaystyle\int_{(2m+1)\pi}^{2(m+1)\pi} t^{-1} |\sin^3 t| dt = \infty.$$

Thus all the assumptions of (ii) of Theorem 3.6 are satisfied and hence equation (3.18) is oscillatory.

Part II Linear equations

4. Basic results. We now turn our attention to the linear equation (LE). First, we wish to establish conditions guaranteeing the nonexistence of solutions of (LE) belonging to $\mathscr{N}_l$ ($l=0, 1,\dots, n$). To do this, we start with the linear inequalities of the form

(LI$^+$) $$\sigma L_n x(t) + q(t)x(g(t)) \leq 0$$

and

(LI$^-$) $$\sigma L_n x(t) + q(t)x(g(t)) \geq 0,$$

for which the condition (c′) is satisfied.

We introduce the notation:

$$\tau[g](t) = \max\{\min\{s, g(s)\} : 0 \leq s \leq t\},$$
$$\rho[g](t) = \min\{\max\{s, g(s)\} : s \geq t\}.$$

Note that the following inequalities hold:

(4.1) $$g(s) \leq \tau[g](t) \quad \text{for} \quad \tau[g](t) < s < t$$

and

$$g(s) \geq \rho[g](t) \qquad \text{for} \quad t < s < \rho[g](t). \tag{4.2}$$

As in Part I we discuss the nonexistence of $\mathcal{N}_l^{\pm}$ for (LI$^{\pm}$) by distinguishing the three cases: $1 \leq l \leq n-1$, $l=0$ and $l=n$.

THEOREM 4.1. *Let* $1 \leq l \leq n-1$ *and* $(-1)^{n-l}\sigma = -1$. *Assume that there exists a nondecreasing function* $\psi \in C(R_+, R_+)$ *which satisfies*

$$\int^{\infty} \frac{dx}{x\psi(x)} < \infty. \tag{4.3}$$

Then the condition

$$\int^{\infty} [\psi(I_l(g(t), 0; p_1, \dots, p_l))]^{-1} H_l[g](t) q(t) dt = \infty \tag{4.4}$$

implies that $\mathcal{N}_l^{\pm} = \emptyset$ *for* (LI$^{\pm}$).

PROOF. We need only to consider (LI$^+$). Suppose that (LI$^+$) has a solution $x \in \mathcal{N}_l^+$. Then there exists $T_0 > 0$ such that (1.4) holds. From (0.4) with i, k and s replaced by 0, l and T_0, respectively, and in view of the increasing nature of $L_l x$, we see that

$$\begin{aligned} x(t) &= \textstyle\sum_{j=0}^{l-1} I_j(t, T_0; p_1, \dots, p_j) L_j x(T_0) \\ &\quad + \int_{T_0}^{t} I_{l-1}(t, s; p_1, \dots, p_{l-1}) p_l(s) L_l x(s) ds \\ &\leq \textstyle\sum_{j=0}^{l} I_j(t, T_0; p_1, \dots, p_j) L_j x(T_0), \quad t \geq T_0. \end{aligned}$$

Hence there is a positive constant c such that

$$cx(t) \leq I_l(t, 0; p_1, \dots, p_l), \quad t \geq T_0.$$

Put $Q(t) = [\psi(cx(g(t)))]^{-1} q(t)$ and $f(x) = x\psi(cx)$. Then $x(t)$ is a solution of the differential inequality

$$\sigma L_n x(t) + Q(t) f(x(g(t))) \leq 0$$

which belongs to $\mathcal{N}_l^+$. On the other hand, since

$$\int_M^{\infty} \frac{dx}{f(x)} = \int_M^{\infty} \frac{dx}{x\psi(cx)} < \infty$$

and

$$\int_T^{\infty} H_l[g](t) Q(t) dt$$

$$= \int_T^\infty [\psi(cx(g(t)))]^{-1} H_l[g](t) Q(t) dt$$

$$\geq \int_T^\infty [\psi(I_l(g(t), 0; p_1, \dots, p_l))]^{-1} H_l[g](t) Q(t) dt = \infty,$$

it follows from (i) of Theorem 1.1 that $\mathcal{N}_l^+ = \emptyset$. This is a contradiction.

THEOREM 4.2. *Let* $(-1)^n \sigma = -1$. *Assume that*

$$\limsup_{t\to\infty} \int_{\tau[g](t)}^t I_{n-i-1}(s, \tau[g](t); p_{n-1}, \dots, p_{i+1}) \cdot I_i(\tau[g](t), g(s); p_i, \dots, p_1) q(s) ds > 1 \tag{4.5}$$

for some $i = 0, 1, \dots, n-1$. *Then* $\mathcal{N}_0^\pm = \emptyset$ *for* (LI$^\pm$).

PROOF. We assume that (LI$^+$) has a solution $x \in \mathcal{N}_0^+$. Then there exists $T_0 > 0$ such that (1.15) holds. Choose T_1 $(> T_0)$ so that $\inf\{g(t) : t \geq T_1\} > T_0$. We now claim that the inequality

$$\int_{\tau[g](t)}^t I_{n-i-1}(s, \tau[g](t); p_{n-1}, \dots, p_{i+1}) \cdot I_i(\tau[g](t), g(s); p_i, \dots, p_1) q(s) ds < 1 \tag{4.6}$$

holds for $t \geq T_1$ and any $i = 0, 1, \dots, n-1$.

If $\tau[g](t) = t$, (4.6) is trivial, so we assume that t $(\geq T_1)$ satisfies $\tau[g](t) < t$. We first show that

$$x(g(s)) \geq I_i(\tau[g](t), g(s); p_i, \dots, p_1) |L_i x(\tau[g](t))| \tag{4.7}$$

for $\tau[g](t) < s < t$ and any $i = 0, 1, \dots, n-1$. If $i = 0$, this follows from (4.1) and the fact that x is decreasing. Let $i \geq 1$. Then, replacing i, k, t and s in (0.5) by 0, i, $g(s)$ and $\tau[g](t)$, respectively, we obtain

$$x(g(s)) = \sum_{j=0}^{i-1} (-1)^j I_j(\tau[g](t), g(s); p_j, \dots, p_1) L_j x(\tau[g](t)) + (-1)^i \int_{g(s)}^{\tau[g](t)} I_{i-1}(r, g(s); p_{i-1}, \dots, p_1) p_i(r) L_i x(r) dr,$$

which, in view of the decreasing nature of $|L_i x|$, readily implies (4.7). Next, replacing k, t and s in (0.5) by n, $\tau[g](t)$ and t, respectively, we have

$$|L_i x(\tau[g](t))| = (-1)^i L_i x(\tau[g](t))$$
$$= \sum_{j=i}^{n-1} (-1)^j I_{j-i}(t, \tau[g](t); p_j, \dots, p_{i+1}) L_j x(t) + (-1)^n \int_{\tau[g](t)}^t I_{n-i-1}(s, \tau[g](t); p_{n-1}, \dots, p_{i+1}) L_n x(s) ds,$$

whence it follows that

$$|L_i x(\tau[g](t))| \geq |L_i x(t)|$$
$$+ \int_{\tau[g](t)}^{t} I_{n-i-1}(s, \tau[g](t); p_{n-1}, \dots, p_{i+1}) q(s) x(g(s)) ds.$$

Combining this with (4.7) yields

$$|L_i x(\tau[g](t))| \geq |L_i x(t)|$$
$$+ |L_i x(\tau[g](t))| \int_{\tau[g](t)}^{t} I_{n-i-1}(s, \tau[g](t); p_{n-1}, \dots, p_{i+1})$$
$$\cdot I_i(\tau[g](t), g(s); p_i, \dots, p_1) q(s) ds,$$

which implies

$$\int_{\tau[g](t)}^{t} I_{n-i-1}(s, \tau[g](t); p_{n-1}, \dots, p_{i+1}) I_i(\tau[g](t), g(s); p_i, \dots, p_1) q(s) ds$$
$$\leq 1 - |L_i x(t)|/|L_i x(\tau[g](t))| < 1.$$

This prove (4.6), a contradiction to the hypothesis (4.5), so that $\mathcal{N}_0^+ = \emptyset$.

A similar argument holds if we assume that (LI^-) has a solution $x \in \mathcal{N}_0^-$.

REMARK 4.1. It is possible to relax (4.5) in Theorem 4.2 as follows: There exists a sequence $\{t_m\}$ such that $t_m \to \infty$ as $m \to \infty$ and

$$\int_{\tau[g](t_m)}^{t_m} I_{n-i-1}(s, \tau[g](t_m); p_{n-1}, \dots, p_{i+1}) I_i(\tau[g](t_m), g(s); p_i, \dots, p_1) q(s) ds \geq 1$$

for some $i = 0, 1, \dots, n-1$.

THEOREM 4.3. *Let* $\sigma = -1$. *Assume that*

$$\limsup_{t\to\infty} \int_t^{\rho[g](t)} I_i(g(s), \rho[g](t); p_1, \dots, p_i)$$
$$\cdot I_{n-i-1}(\rho[g](t), s; p_{i+1}, \dots, p_{n-1}) q(s) ds > 1 \tag{4.8}$$

for some $i = 0, 1, \dots, n-1$. *Then* $\mathcal{N}_n^{\pm} = \emptyset$ *for* ($\mathrm{LI}^{\pm}$).

PROOF. Suppose that (LI^+) has a solution x belonging to $\mathcal{N}_n^+$. Then there exists $T_0 > 0$ such that (1.22) holds. We will show that the inequality

$$\int_t^{\rho[g](t)} I_i(g(s), \rho[g](t); p_1, \dots, p_i)$$
$$\cdot I_{n-i-1}(\rho[g](t), s; p_{n-1}, \dots, p_{i+1}) q(s) ds < 1 \tag{4.9}$$

holds for $t \geq T_0$ and any $i = 0, 1, \dots, n-1$. We then have $\mathcal{N}_n^+ = \emptyset$, since (4.9) contradicts (4.8).

If $\rho[g](t)=t$, (4.9) is trivial, and so we suppose that $t\ (\geq T_0)$ satisfies $\rho[g](t)>t$. We first show the fact that

$$x(g(s)) \geq I_i(g(s), \rho[g](t); p_1,\dots, p_i)L_ix(\rho[g](t)) \tag{4.10}$$

for $t<s<\rho[g](t)$ and any $i=0, 1,\dots, n-1$. If $i=0$, (4.10) follows from (4.2) and the increasing nature of x. Let $i\geq 1$. From (0.4) with i, k, t and s replaced by 0, i, $g(s)$ and $\rho[g](t)$, respectively,

$$\begin{aligned} x(g(s)) &= \textstyle\sum_{j=0}^{i-1} I_j(g(s), \rho[g](t); p_1,\dots, p_j)L_jx(\rho[g](t)) \\ &\quad + \int_{\rho[g](t)}^{g(s)} I_{i-1}(g(s), r; p_1,\dots, p_{i-1})p_i(r)L_ix(r)dr. \end{aligned}$$

Using (4.2) and noting that L_ix is increasing, we easily get (4.10) from the above equation. Next, from (0.4) with k, t and s replaced by n, $\rho[g](t)$ and t, respectively,

$$\begin{aligned} L_ix(\rho[g](t)) &= \textstyle\sum_{j=i}^{n-1} I_{j-i}(\rho[g](t), t; p_{i+1},\dots, p_j)L_jx(t) \\ &\quad + \int_t^{\rho[g](t)} I_{n-i-1}(\rho[g](t), s; p_{i+1},\dots, p_{n-1})L_nx(g(s))ds, \end{aligned}$$

in particular

$$L_ix(\rho[g](t)) \geq L_ix(t) + \int_t^{\rho[g](t)} I_{n-i-1}(\rho[g](t), s; p_{i+1},\dots, p_{n-1})q(s)x(g(s))ds.$$

Combining this with (4.10), we obtain

$$\begin{aligned} L_ix(\rho[g](t)) &\geq L_ix(t) \\ &\quad + L_ix(\rho[g](t))\int_t^{\rho[g](t)} I_i(g(s), \rho[g](t); p_1,\dots, p_i) \\ &\qquad\qquad \cdot I_{n-i-1}(\rho[g](t), s; p_{i+1},\dots, p_{n-1})q(s)ds, \end{aligned}$$

which implies

$$\begin{aligned} &\int_t^{\rho[g](t)} I_i(g(s), \rho[g](t); p_1,\dots, p_i)I_{n-i-1}(\rho[g](t), s; p_{i+1},\dots, p_{n-1})q(s)ds \\ &\leq 1 - L_ix(t)/L_ix(\rho[g](t)) < 1. \end{aligned}$$

This is the desired contradiction (4.9).

That (4.8) ensures $\mathcal{N}_n^-=\emptyset$ for (LI$^-$) can be proved analogously.

REMARK 4.2. It is possible to relax (4.8) in Theorem 4.3 as follows: There exists a sequence $\{t_m\}$ such that $t_m\to\infty$ as $m\to\infty$ and

$$\int_{t_m}^{\rho[g](t_m)} I_i(g(s), \rho[g](t_m); p_1,\dots, p_i)I_{n-i-1}(\rho[g](t_m), s; p_{i+1},\dots, p_{n-1})q(s)ds\geq 1$$

for some $i=0, 1,\ldots, n-1$.

We now consider the equation (LE). Let x be a solution of (LE) belonging to $\mathcal{N}_l$. Then it is a solution of

$$\sigma L_n x(t) + q_h(t)x(g_h(t)) \leq 0, \quad 1 \leq h \leq N,$$

belonging to $\mathcal{N}_l^+$ if x is positive, and it is a solution of

$$\sigma L_n x(t) + q_h(t)x(g_h(t)) \geq 0, \quad 1 \leq h \leq N$$

belonging to $\mathcal{N}_l^-$ if x is negative. Therefore, from Theorems 4.1–4.3, we obtain the following theorems on the absence of $\mathcal{N}_l$ $(l=0, 1,\ldots, n)$ for (LE).

THEOREM 4.4. *Let* $1 \leq l \leq n-1$ *and* $(-1)^{n-l}\sigma = -1$. *If there exist* α_+, $\alpha_- \in \{1,\ldots, N\}$ *and* $\psi \in C(R_+, R_+)$ *such that*

$$\int^{\infty} \frac{dx}{x\psi(x)} < \infty \tag{4.11}$$

and

$$\int^{\infty} [\psi(I_l(g_{\alpha_\pm}(t), 0; p_1,\ldots, p_n))]^{-1} H_l[g_{\alpha_\pm}](t) q_{\alpha_\pm}(t) dt = \infty, \tag{4.12; l}$$

then $\mathcal{N}_l = \emptyset$ *for* (LE).

THEOREM 4.5. *Let* $(-1)^n\sigma = -1$. *If there exist* β_+, $\beta_- \in \{1,\ldots, N\}$ *such that*

$$\begin{aligned}\limsup_{t\to\infty} \int_{\tau[g_{\beta_\pm}](t)}^{t} & I_{n-i-1}(s, \tau[g_{\beta_\pm}](t); p_{n-1},\ldots, p_{i+1}) \\ & \cdot I_i(\tau[g_{\beta_\pm}](t), g_{\beta_\pm}(s); p_i,\ldots, p_1) q_{\beta_\pm}(s) ds > 1\end{aligned} \tag{4.13}$$

for some $i=0, 1,\ldots, n-1$, *then* $\mathcal{N}_0 = \emptyset$ *for* (LE).

THEOREM 4.6. *Let* $\sigma = -1$. *If there exist* γ_+, $\gamma_- \in \{1,\ldots, N\}$ *such that*

$$\begin{aligned}\limsup_{t\to\infty} \int_{t}^{\rho[g_{\gamma_\pm}](t)} & I_i(g_{\gamma_\pm}(s), \rho[g_{\gamma_\pm}](t); p_1,\ldots, p_i) \\ & \cdot I_{n-i-1}(\rho[g_{\gamma_\pm}](t), s; p_{i+1},\ldots, p_{n-1}) q_{\gamma_\pm}(s) ds > 1\end{aligned} \tag{4.14}$$

for some $i=0, 1,\ldots, n-1$, *then* $\mathcal{N}_n = \emptyset$ *for* (LE).

Applying these theorems to the particular equation

$$x^{(n)}(t) + \sigma q(t)x(g(t)) = 0, \tag{4.15}$$

we obtain following corollaries.

COROLLARY 4.1. *Let* $1 \leq l \leq n-1$ *and* $(-1)^{n-l}\sigma = -1$. *If*

$$\int^{\infty} t^{n-l-1} g^{l-\varepsilon-1}(t) g_{*}(t) q(t) dt = \infty \tag{4.16}$$

for some $\varepsilon > 0$, *then* $\mathcal{N}_l = \emptyset$ *for* (4.15).

PROOF. Put $\psi(x) = x^{\varepsilon/l}$. Then (4.11) is satisfied and it is not difficult to verify that (4.16) guarantees that (4.12; l) holds for (4.15). The conclusion follows from Theorem 4.4.

COROLLARY 4.2. *Let* $(-1)^n \sigma = -1$. *If*

$$\limsup_{t\to\infty} \int_{\tau[g](t)}^{t} \frac{\{s-\tau[g](t)\}^{n-i-1}}{(n-i-1)!} \frac{\{\tau[g](t)-g(s)\}^{i}}{i!} q(s) ds > 1 \tag{4.17}$$

for some $i = 0, 1, \ldots, n-1$, *then* $\mathcal{N}_0 = \emptyset$ *for* (4.15).

PROOF. This follows from Theorem 4.5, since

$$I_{n-i-1}(s, \tau[g](t); p_{n-1}, \ldots, p_{i+1}) = \{s-\tau[g](t)\}^{n-i-1}/(n-i-1)!$$

and

$$I_i(\tau[g](t), g(s); p_i, \ldots, p_1) = \{\tau[g](t) - g(s)\}^i/i!,$$

so that (4.17) guarantees that (4.13) holds for (4.15).

COROLLARY 4.3. *Let* $\sigma = -1$. *If*

$$\limsup_{t\to\infty} \int_{t}^{\rho[g](t)} \frac{\{g(s)-\rho[g](t)\}^{i}}{i!} \frac{\{\rho[g](t)-s\}^{n-i-1}}{(n-i-1)!} q(s) ds > 1 \tag{4.18}$$

for some $i = 0, 1, \ldots, n-1$, *then* $\mathcal{N}_n = \emptyset$ *for* (4.15).

PROOF. Note that under the hypothesis (4.14) is satisfied for (4.15), since

$$I_i(g(s), \rho[g](t); p_1, \ldots, p_i) = \{g(s) - \rho[g](t)\}^i/i!$$

and

$$I_{n-i-1}(\rho[g](t), s; p_{i+1}, \ldots, p_{n-1}) = \{\rho[g](t) - s\}^{n-i-1}/(n-i-1)!.$$

Then apply Theorem 4.6.

REMARK 4.3. If g is nondecreasing and $g(t) \leq t$, then $\tau[g](t) = g(t)$ and hence (4.17) is equivalent to

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \frac{\{s-g(t)\}^{n-i-1}}{(n-i-1)!} \frac{\{g(t)-g(s)\}^{i}}{i!} q(s) ds > 1$$

for some $i=0, 1,\dots, n-1$. On the other hand, if g is nondecreasing and $g(t)\geq t$, then $\rho[g](t)=g(t)$ and (4.18) is equivalent to

$$\limsup_{t\to\infty}\int_t^{g(t)}\frac{\{g(s)-g(t)\}^i}{i!}\frac{\{g(t)-s\}^{n-i-1}}{(n-i-1)!}q(s)ds > 1$$

for some $i=0, 1,\dots, n-1$. Therefore Corollaries 4.2 and 4.3 extend the results of Koplatadze and Chanturiya [12].

5. Oscillation. In this section we present conditions under which equation (LE) is oscillaroty or almost oscillatory.

THEOREM 5.1. *Equation* (LE) *is almost oscillatory if*:

(i) *for* $\sigma=1$ *and* n *even, there exist* $\alpha_+, \alpha_-\in\{1,\dots, N\}$ *and* $\psi\in C(R_+, R_+)$ *which satisfy* (4.11) *and* (4.12; l) ($l=1, 3,\dots, n-1$);

(ii) *for* $\sigma=1$ *and* n *odd, there exist* $\alpha_+, \alpha_-\in\{1,\dots, N\}$ *and* $\psi\in C(R_+, R_+)$ *which satisfy* (4.11) *and* (4.12; l) ($l=2, 4,\dots, n-1$), *and the condition* (3.5) *holds*;

(iii) *for* $\sigma=-1$ *and* n *even, there exist* $\alpha_+, \alpha_-\in\{1,\dots, N\}$ *and* $\psi\in C(R_+, R_+)$ *which satisfy* (4.11) *and* (4.12; l) ($l=2, 4,\dots, n-2$), *and the condition* (3.5) *and*

$$\int^{\infty}\sum_{h=1}^{N}J(g_h(t))q_h(t)dt=\infty \tag{5.1}$$

hold;

(iv) *for* $\sigma=-1$ *and* n *odd, there exist* $\alpha_+, \alpha_-\in\{1,\dots, N\}$ *and* $\psi\in C(R_+, R_+)$ *which satisfy* (4.11) *and* (4.12; l) ($l=1, 3,\dots, n-2$), *and the condition* (5.1) *holds.*

PROOF. Theorem 4.4 shows that (LE) has no solutions belonging to $\mathcal{N}_l$ ($l=1, 2,\dots, n-1$). Thus it suffices to show that $|x(t)|\downarrow 0$ as $t\uparrow\infty$ if $x\in\mathcal{N}_0$, and $|L_{n-1}x(t)|\uparrow\infty$ as $t\uparrow\infty$ if $x\in\mathcal{N}_n$. But these facts can be proved in the same manner as in the proof of Theorem 3.2.

THEOREM 5.2. *Equation* (LE) *is oscillatory if*:

(i) *for* $\sigma=1$ *and* n *odd, there exist* $\alpha_+, \alpha_-\in\{1,\dots, N\}$, $\psi\in C(R_+, R_+)$ *and* $\beta_+, \beta_-\in\{1,\dots, N\}$ *which satisfy* (4.11), (4.12; l) ($l=2, 4,\dots, n-1$) *and* (4.13), *respectively*;

(ii) *for* $\sigma=-1$ *and* n *even, there exist* $\alpha_+, \alpha_-\in\{1,\dots, N\}$, $\psi\in C(R_+, R_+)$, $\beta_+, \beta_-\in\{1,\dots, N\}$, *and* $\gamma_+, \gamma_-\in\{1,\dots, N\}$ *which satisfy* (4.11), (4.12; l) ($l=2, 4,\dots, n-2$), (4.13) *and* (4.14), *respectively*;

(iii) *for* $\sigma=-1$ *and* n *odd, there exist* $\alpha_+, \alpha_-\in\{1,\dots, N\}$, $\psi\in C(R_+, R_+)$ *and* $\gamma_+, \gamma_-\in\{1,\dots, N\}$ *which satisfy* (4.11), (4.12; l) ($l=1, 3,\dots, n-2$) *and* (4.14), *respectively.*

This theorem is an easy consequence of Theorems 4.4–4.5.

The above results specialized to the equation (4.15) are stated below.

COROLLARY 5.1. *Let $\sigma=1$ and n be even. If*

$$\int^{\infty} g_*^{n-1}(t)g^{-\varepsilon}(t)q(t)dt = \infty \qquad \textit{for some} \quad \varepsilon > 0,$$

then (4.15) *is oscillatory.*

COROLLARY 5.2. *Let $\sigma=1$ and n be odd. If*

$$\int^{\infty} g_*^{n-2}(t)g^{1-\varepsilon}(t)q(t)dt = \infty \qquad \textit{for some} \quad \varepsilon > 0$$

and

$$\int^{\infty} t^{n-1}q(t)dt = \infty,$$

then (4.15) *is almost oscillatory. If in addition*

$$\limsup_{t\to\infty} \int_{\tau[g](t)}^{t} \frac{\{s-\tau[g](t)\}^{n-i-1}}{(n-i-1)!} \frac{\{\tau[g](t)-g(s)\}^{i}}{i!} q(s)ds > 1$$

for some $i=0, 1,\dots, n-1$, then (4.15) *is oscillatory.*

COROLLARY 5.3. *Let $\sigma=-1$ and n be even. If*

$$\int^{\infty} tg_*^{n-3}(t)g^{1-\varepsilon}(t)q(t)dt = \infty \qquad \textit{for some} \quad \varepsilon > 0,$$

$$\int^{\infty} t^{n-1}q(t)dt = \infty$$

and

$$\int^{\infty} g^{n-1}(t)q(t)dt = \infty,$$

then (4.15) *is almost oscillatory. If in addition*

$$\limsup_{t\to\infty} \int_{\tau[g](t)}^{t} \frac{\{s-\tau[g](t)\}^{n-i-1}}{(n-i-1)!} \frac{\{\tau[g](t)-g(s)\}^{i}}{i!} q(s)ds > 1$$

for some $i=0, 1,\dots, n-1$, and

$$\limsup_{t\to\infty} \int_{t}^{\rho[g](t)} \frac{\{g(s)-\rho[g](t)\}^{j}}{j!} \frac{\{\rho[g](t)-s\}^{n-j-1}}{(n-j-1)!} q(s)ds > 1$$

for some $j=0, 1,\dots, n-1$, then (4.15) *is oscillatory.*

COROLLARY 5.4. *Let $\sigma=-1$ and n be odd. If*

$$\int^{\infty} tg_*^{n-2}(t)g^{-\varepsilon}(t)q(t)dt = \infty \qquad \textit{for some} \quad \varepsilon > 0$$

and

$$\int^{\infty} g^{n-1}(t)q(t)dt = \infty,$$

then (4.15) *is almost oscillatory. If in addition*

$$\limsup_{t\to\infty} \int_t^{\rho[g](t)} \frac{\{g(s)-\rho[g](t)\}^j}{j!} \frac{\{\rho[g](t)-s\}^{n-j-1}}{(n-j-1)!} q(s)ds > 1$$

for some $j=0, 1,\dots, n-1$, *then* (4.15) *is oscillatory.*

EXAMPLE 5.1. Consider the third order equation

$$(t^{-1}x''(t))' + 4t^{-4}x(t/2) = 0. \tag{5.2}$$

Since

$$\sigma = 1,\ p_1(t) = 1,\ p_2(t) = t,\ q(t) = 4t^{-4} \quad \text{and} \quad g(t) = t/2,$$

we obtain

$$\tau[g](t) = t/2$$

and we can easily calculate that

$$\begin{aligned}
&\int_{\tau[g](t)}^{t} I_2(\tau[g](t), g(s); p_2, p_1)q(s)ds \\
&\quad = [1/6 + \log 2]/12 = 0.071651 < 1, \\
&\int_{\tau[g](t)}^{t} I_1(s, \tau[g](t); p_2)I_1(\tau[g](t), g(s); p_1)q(s)ds \\
&\quad = [19/6 - 4\log 2]/4 = 0.098519 < 1
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\tau[g](t)}^{t} I_2(s, \tau[g](t); p_2, p_1)q(s)ds \\
&\quad = [16\log 2 - 29/3]/12 = 0.118641 < 1.
\end{aligned}$$

This shows that (4.13) of Theorem 4.5 is not satisfied for the equation (5.2). Indeed this equation has a solution $x(t)=t^{-1}$ which belongs to $\mathcal{N}_0$.

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