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Pettis integrability and the equality of the norms of the weak* integral and the Dunford integral

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Abstract It is shown that if (Ω, Σ, μ) is a finite measure space and X is a Banach space then X^* has the μ-Pettis Integral Property if and only if

$$\|(weak^*) - \int_{\Omega} f d\mu\| = \|(Dunford) - \int_{\Omega} f d\mu\|$$

for every bounded weakly measurable function f: $\Omega \to X^*$.

A negative answer to a question of E. Bator is also given.

1. Introduction

Let (Ω, Σ, μ) be a finite measure space. For a Banach space X we denote by $bwm(\mu; X)$ the space of all bounded and weakly measurable X-valued functions defined on Ω. X^* denotes the dual of X. B is the unit ball of X.

It is well known that if $f \in bwm(\mu; X^*)$ then for every $E \in \Sigma$ there exists $x_E^* \in X^*$ such that, for every $x \in X$,

$$x_E^*(x) = \int_E x f d\mu$$

and, for every $E \in \Sigma$ there exists $x_E^{***} \in X^{***}$ such that, for every $x^{**} \in X^{**}$,

$$x_E^{***}(x^{**}) = \int_E x^{**} f d\mu.$$

x_E^* is called the *weak* integral* of f over E, denoted by $w^* - \int_E f d\mu$, and x_E^{***} is called the *Dunford integral* of f over E, denoted by $D - \int_E f d\mu$. f is said to be *Pettis integrable* if $D - \int_E f d\mu \in X^*$ for all $E \in \Sigma$.

X is said to have the *μ-Pettis Integral Property* (μ-PIP) if every $f \in bwm(\mu; X)$ is Pettis integrable. More information on Pettis integral and μ-PIP can be found in [3].

E. M. Bator proved in [1] that X^* has μ-PIP for a perfect measure μ if and only if

$$\left\| w^* - \int_\Omega f d\mu \right\| = \left\| D - \int_\Omega f d\mu \right\|$$

for every $f \in bwm(\mu; X^*)$. The proof was based on Fremlin's subsequence theorem. It is the purpose of this note to give a short and elementary proof of the theorem of Bator in the case of arbitrary finite μ.

Moreover, assuming the existence of measurable cardinals we present a negative answer to a problem of Bator.

2. Characterization of μ-PIP of X*

THEOREM. *If* (Ω, Σ, μ) *is a finite measure space and* X *is a Banach space, then* X^* *has* μ*-PIP if and only if for every* $f \in bwm(\mu; X^*)$ *the equality*

$$\left\| w^* - \int_\Omega f d\mu \right\| = \left\| D - \int_\Omega f d\mu \right\|$$

holds.

PROOF. Noting that only one implication needs a proof let us assume that

$$\left\| w^* - \int_\Omega f d\mu \right\| = \left\| D - \int_\Omega f d\mu \right\|$$

for all $f \in bwm(\mu; X^*)$, and take an $h \in bwm(\mu; X^*)$. By the assumption, for each $E \in \Sigma$ and each $x^* \in X^*$ we have

$$\left\| w^* - \int_E (h - x^*) d\mu \right\| = \left\| D - \int_E (h - x^*) d\mu \right\|.$$

Setting $x^* = \mu(E)^{-1}\left(w^* - \int_E h d\mu \right)$ if $\mu(E) > 0$, we get at once the required equality

$$w^* - \int_E h d\mu = D - \int_E h d\mu.$$

This completes the proof.

As a consequence we get also a generalization of Corollary 6 in [1].

COROLLARY. *X^* has μ-PIP if and only if for each $f \in bwm(\mu; X^*)$ and each $x^{**} \in X^{**}$ there exists a bounded sequence $\{x_n : n = 1, \cdots\}$ in X such that $\|x_n\| \leqslant \|x^{**}\|$ for each n and $x_n f \to x^{**} f$, μ-a.e.*

PROOF. The proof of the "only if" part in [1] contains a gap (the pointwise convergence of $(x_\beta \circ f)$ does not yield, in general, the $L_1(\mu)$-convergence). We shall present here an independent proof.

Assume that X^* has μ-PIP, and take $f \in bwm\,(\mu; X^*)$. Then T_f: $X \to L_1(\mu)$ given by $T_f(x) = xf$ is weakly compact and so $T_f^{**}(X^{**}) \subset L_1(\mu)$. It follows that $T_f^{**}$is weak*-weak continuous. Applying the Goldstine theorem, we see that $T^{**}B^{**}$ is in the weak closure of $T_f^{**}B$. But $T_f^{**}B$ is convex and so the Mazur theorem yields the norm density of $T_f^{**}B$ in $T_f^{**}B^{**}$. Now, it is only sufficient to observe that the Pettis integrability of f yields the equality $T_f^{**}(x^{**}) = x^{**}f$ for all $x^{**} \in X^{**}$.

To show the reverse implication it is sufficient to observe that the convergence yields the equality $\left\| w^* - \int_\Omega f d\mu \right\| = \left\| D - \int_\Omega f d\mu \right\|$ for all $f \in bwm(\mu; X^*)$, since the main theorem can then be applied.

3. Integration of a single function

In connection with the theorem the following problem arises: Assume that $f \in bwm(\mu; X^*)$ satisfies for each $E \in \Sigma$ the equality

$$(*) \qquad \left\| D - \int_E f d\mu \right\| = \left\| w^* - \int_E f d\mu \right\|.$$

Is f Pettis integrable ?

The following example, which is a generalization of an example of Talagrand ([3], 6–2–3) shows, however, that this is not the case even if μ is perfect.

EXAMPLE. Let κ be the first real-measurable cardinal, and let μ be a universal diffused probability on $[0, \kappa]$ (i.e., defined on all subsets of $[0, \kappa]$ and vanishing on points) endowed with the order topology. Consider f: $[0, \kappa] \to C^*[0, \kappa]$ given by $f(\alpha) = \delta_\alpha$. Clearly f is weakly measurable and bounded. Using the fact that each continuous function on $[0, \kappa]$ is eventually constant (cf. [2], II.8.6.3), one can easily show that

$$w^* - \int_E f d\mu = \mu(E)\delta_\kappa$$

for all $E \subset [0, \kappa]$.

Since $\|f\| \leqslant 1$ we have $\left\| D - \int_E f d\mu \right\| \leqslant \mu(E)$ and so $(*)$ holds. Let us define now $z \in C^{**}[0, \kappa]$ by the equation $z(v) = v(\{\kappa\})$ for $v \in C^*[0, \kappa]$. It follows that for $E = [0, \kappa]$ we have

$$\left\langle z, D - \int_E f d\mu \right\rangle = 0 \text{ while } \left\langle z, w^* - \int_E f d\mu \right\rangle = 1.$$

Thus f is not Pettis integrable.

We now take κ to be the first two-valued measurable cardinal and μ to be a two-valued universal diffused probability on $[0, \kappa]$. Then we obtain the desired example involving a perfect measure.

REMARK. The above example (with the two-valued universal μ) gives also a negative answer to the following question posed in [1]: Suppose that $f \in bwm(\mu; X^*)$ and for each $x^{**} \in X^{**}$ there exists a sequence $\{x_n\}$ in X such that $x_n f \to x^{**} f$, μ-a.e. Is f Pettis integrable ?

Indeed, assume that κ is the first two-valued measurable cardinal, take f to be the non-Pettis integrable function considered in the example, and fix $x^{**} \in l_\infty[0, \kappa]$(we are applying the fact that $C^*[0, \kappa]$ is linearly isometric to $l_1[0, \kappa]$(cf. [2], V.19.7.8 or [3], 2.4.4), and so $C^{**}[0, \kappa]$ is linearly isometric to $l_\infty[0, \kappa]$). Since μ is two-valued $x^{**}f$ is μ-a.e. equal to a constant c. Setting $x_n = c$ we get a sequence of functions $x_n \in C[0, \kappa]$ such that $x_n f \to x^{**} f$, μ-a.e.

Observe yet that if κ is assumed only to be real-measurable then the function f considered in the example satisfies $(*)$ but such a sequence $\{x_n\}$ as mentioned above cannot exist in general. This follows from the fact that a functional $x^{**} \in l_\infty[0, \kappa]$ which is μ-a.e. limit of a sequence of functions $x_n \in C[0, \kappa]$ has to be constant μ-a.e.

References

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