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On the degeneration of étale $\mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}/p^2\mathbf{Z}$-torsors in equal characteristic $p > 0$

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Abstract. Let R be a complete discrete valuation ring of equal characteristic $p > 0$. In this paper we investigate finite and flat morphisms $f : Y \to X$ between formal R-schemes which have the structure of an étale $\mathbf{Z}/p^n\mathbf{Z}$-*torsor* above the *generic* fiber of X, for $n = 1, 2$, with some *extra geometric conditions* on X and Y. In the case $n = 1$, we prove that f has the structure of a torsor under a finite and flat R-group scheme of rank p and we describe the group schemes that arise as the group of the torsor. In the case $n = 2$, we describe explicitly how the Artin-Schreier-Witt equations describing f on the generic fiber, locally, degenerate. Moreover, in some cases where f has the structure of a torsor under a finite and flat R-group scheme of rank p^2, we describe the group schemes of rank p^2 which arise in this way.

Introduction

Let $p > 0$ be a prime integer. Let R be a complete discrete valuation ring of equal characteristic p, with fraction field K, and residue field k. Let X be a formal R-scheme of finite type, which is normal, connected, and flat over R. Assume that the fibers of X (over $\operatorname{Spec} R$) are geometrically integral. Let $f : Y \to X$ be a finite, and flat, cover of degree p^n, with Y normal. Assume that f has the structure of an étale torsor; with group $\mathbf{Z}/p^n\mathbf{Z}$, above the generic fiber $X_K := X \times_R K$, of X. Further, suppose that the special fiber $Y_k := Y \times_R k$, of Y, is reduced. In this paper we are interested in describing the map f, and its special fiber $f_k : Y_k \to X_k$. One of our main results is the following:

Theorem 2.2.1. *Assume that* $\deg(f) = p$ *(i.e.* $n = 1$*). Then the cover* $f : Y \to X$ *has the structure of a torsor, under a finite and flat* R*-group scheme of rank* p.

Moreover, we give an explicit description of the group schemes which appear as the group of the torsor in 2.2.1 (cf. 2.1). More precisely, we provide *integral* (local) equations for the torsor $f : Y \to X$, which also provide, by reduction,

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(local) equations for its special fiber $f_k : Y_k \to X_k$. Next, we investigate covers of degree p^2. Our main result is theorem 3.3.3. We are able in 3.3.3 to find "integral" equations for f, which provide (by reduction) equations for its special fiber $f_k : Y_k \to X_k$. In other terms, we describe how the Artin-Schreier-Witt equations of degree p^2 degenerate.

The proof of 3.3.3 is rather involved, and uses the technical lemma 3.3.2. It is based on a (non-trivial) iteration of the process used in the proof of theorem 2.2.1. This method can, in principle, be generalized to provide integral equations for p^n-cyclic covers $f : Y \to X$ as above (for $n > 2$). However, this leads to quite complicated equations, which are not so easy to write down.

In the case of covers of degree p^2 we exhibit certain cases as above, where f has the structure of a torsor, under a finite and flat R-group scheme of rank p^2 (cf. 3.3.3, and 3.3.4). In these cases we explicit the group schemes which appear as groups of the torsor. These group schemes are basically obtained by "twisting" the Artin-Schreier-Witt theory (cf. 3.2, for more details). We are also able to associate some degeneration data to the cover f, which determine explicitly the cover f_k (cf. 3.3.5).

In [S-2] we apply the results of this paper to the study of the semi-stable reduction of cyclic Galois covers, of degree p, and p^2, in equal characteristic p.

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0. Notation

In this paper we will adopt the following notations: $p > 0$ is a fixed prime integer.

For a positive integer $n > 0$, W_n denotes the fppf-sheaf which is represented by the group scheme $W_{n,\mathbf{F}_p}$, of *Witt vectors of length n*, over $\mathbf{F}_p$.

If X is a scheme, and G is a group scheme, $H^i(X, G)$ will denote the *cohomology groups*, for the fppf-*topology*, of X with values in the sheaf which is represented by G. Recall, that if G is a smooth commutative group scheme, then the $H^i(X, G)$ coincide with the cohomology groups for the étale topology.

Also, $H^i(X, W_n)$ coincides with the cohomology group for the Zariski topology, and the étale topology.

For computations, in the sheaf W_2, we will use the following notation

$$W(X, Y) = \frac{X^p + Y^p - (X+Y)^p}{p} \in \mathbf{Z}[X, Y].$$

We will frequently use the following (well-known) congruence

$$W(X, Y) \equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k} X^k Y^{p-k} \mod p.$$

1. Artin-Schreier-Witt theory of p^n-cyclic covers in characteristic p

In this section, we review the Artin-Schreier-Witt theory (first developed in [W]) which provides, in characteristic p, explicit equations for $\mathbf{Z}/p^n\mathbf{Z}$-torsors. We refer the reader to a modern treatment of the theory in [D-G]. Throughout this section X denotes a scheme of characteristic p. Also, any addition or subtraction of Witt vectors will mean the addition and subtraction in Witt theory.

1.1. We denote by $\mathbf{F}$ the *Frobenius endomorphism* of W_n, which is locally defined by

$$\mathbf{F}.(x_1, x_2, \dots, x_n) = (x_1^p, x_2^p, \dots, x_n^p),$$

and by Id the *identity automorphism* of W_n.

We have an exact sequence of group schemes over $\mathbf{F}_p$:

$$0 \longrightarrow (\mathbf{Z}/p^n\mathbf{Z}) \xrightarrow{i_n} W_n \xrightarrow{\mathbf{F}-\mathrm{Id}} W_n \longrightarrow 0, \tag{1}$$

which is exact for the étale topology on X. Here, $(\mathbf{Z}/p^n\mathbf{Z})$ denotes the constant group scheme defined by the cyclic group $(\mathbf{Z}/p^n\mathbf{Z})$, and i_n is the natural monomorphism which sends $1 \in \mathbf{Z}/p^n\mathbf{Z}$ to $1 \in W_n$ (cf [D-G], chapitre 5, 5.4). From the long cohomology exact sequence associated to (1), one deduces the following exact sequence:

$$\begin{aligned} \Gamma(X, W_n) \xrightarrow{\mathbf{F}-\mathrm{Id}} \Gamma(X, W_n) &\longrightarrow H^1(X, \mathbf{Z}/p^n\mathbf{Z}) \\ &\longrightarrow H^1(X, W_n) \xrightarrow{\mathbf{F}-\mathrm{Id}} H^1(X, W_n). \end{aligned} \tag{2}$$

Assume that $X = \operatorname{Spec} A$ is *affine*, in which case

$$H^1(X, W_n) = 0.$$

Hence, we have an isomorphism

$$H^1(\mathrm{Spec}\,A, \mathbf{Z}/p^n\mathbf{Z}) \simeq W_n(A)/\mathrm{Im}(\mathbf{F} - \mathrm{Id}).$$

The above isomorphism has the following interpretation. To an étale $\mathbf{Z}/p^n\mathbf{Z}$-torsor

$$f : Y \to X = \mathrm{Spec}\,A,$$

corresponds a Witt vector

$$(a_1, a_2, \dots, a_n) \in W_n(A),$$

of length n, which is uniquely determined, modulo addition of elements of the form

$$\mathbf{F}.(b_1, b_2, \dots, b_n) - (b_1, b_2, \dots, b_n).$$

Further, the equations

$$\mathbf{F}.(x_1, x_2, \dots, x_n) - (x_1, x_2, \dots, x_n) = (a_1, a_2, \dots, a_n),$$

where the x_i are *indeterminate*, are equations for the torsor f. More precisely, there is a canonical factorization of f as

$$Y = Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Y_0 = X,$$

where each

$$Y_i = \mathrm{Spec}\,B_i,$$

is affine, and

$$f_i : Y_i := \mathrm{Spec}\,B_i \to Y_{i-1} := \mathrm{Spec}\,B_{i-1},$$

is the étale $\mathbf{Z}/p\mathbf{Z}$-torsor corresponding to the algebra extension $B_{i-1} \to B_i$, where

$$B_i := B_{i-1}[x_i].$$

In the general case, where $H^1(X, W_n) \neq 0$, the above equations provide *local equations* for an étale $\mathbf{Z}/p^n\mathbf{Z}$-torsor, in characteristic p.

1.2. Examples. We follow the notations in 1.1.

1.2.1. $\mathbf{Z}/p\mathbf{Z}$-Torsors. Let

$$f : Y \to X$$

be an étale $\mathbf{Z}/p\mathbf{Z}$-torsor. Then f is locally given by an equation

$$x^p - x = a,$$

where a is a regular function on X, which is uniquely determined up to addition of elements of the form $b^p - b$.

1.2.2. $\mathbf{Z}/p^2\mathbf{Z}$-Torsors. Let

$$f : Y \to X,$$

be an étale $\mathbf{Z}/p^2\mathbf{Z}$-torsor. We have a canonical factorization of f as

$$Y_2 := Y \xrightarrow{f_2} Y_1 \xrightarrow{f_1} X,$$

where f_2, and f_1, are étale $\mathbf{Z}/p\mathbf{Z}$-torsors. The torsor f is locally given, if $p \neq 2$, by equations of the form

$$\mathbf{F}.(x_1, x_2) - (x_1, x_2) := (x_1^p - x_1, x_2^p - x_2 + W(x_1^p, -x_1)) = (a_1, a_2),$$

which can be rewritten as

$$\mathbf{F}.(x_1, x_2) - (x_1, x_2) = \left(x_1^p - x_1, x_2^p - x_2 - \sum_{k=1}^{p-1} \frac{1}{k} x_1^{pk+p-k} \right) = (a_1, a_2),$$

resp.

$$\mathbf{F}.(x_1, x_2) - (x_1, x_2) = (x_1^2 - x_1, x_2^2 - x_2 + x_1^3 - x_1^2) = (a_1, a_2),$$

if $p = 2$; for some regular functions a_1 and a_2 on X.

Moreover, the Witt vector

$$(a_1, a_2),$$

is uniquely determined, up to addition (in the Witt theory) of vectors of the form

$$(b_1^p, b_2^p) - (b_1, b_2),$$

which if $p \neq 2$ equals

$$(b_1^p - b_1, b_2^p - b_2 + W(b_1^p, -b_1)),$$

resp. equals

$$(b_1^2 - b_1, b_2^2 - b_2 + b_1^3 - b_1^2),$$

if $p = 2$. Thus, locally, the torsor f_1 is defined by the equation

$$x_1^p - x_1 = a_1,$$

and the torsor f_2 by the equation

$$x_2^p - x_2 = a_2 - W(x_1^p, -x_1) = a_2 + \sum_{k=1}^{p-1} \frac{1}{k} x_1^{pk+p-k},$$

if $p \neq 2$, resp.

$$x_2^2 - x_2 = a_2 - x_1^3 + x_1^2,$$

if $p = 2$. Moreover, if we replace the vector

$$(a_1, a_2),$$

by the vector

$$(a_1, a_2) + (b_1^p, b_2^p) - (b_1, b_2),$$

the above equations are replaced by

$$x_1^p - x_1 = a_1 + b_1^p - b_1$$

and

$$x_2^p - x_2 = a_2 + b_2^p - b_2 + \sum_{k=1}^{p-1} \frac{1}{k} x_1^{pk+p-k} - \sum_{k=1}^{p-1} \frac{1}{k} b_1^{pk+p-k}$$
$$- \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} (a_1)^k (b_1^p - b_1)^{p-k}$$

if $p \neq 2$, resp.

$$x_2^2 - x_2 = x_1^2 - x_1^3 + a_2 + b_2^2 - b_2 + b_1^3 - b_1^2 - a_1(b_1^2 - b_1),$$

if $p = 2$.

2. Degeneration of p-cyclic covers in equal characteristic $p > 0$

In this section we use the following notations: R is a complete discrete valuation ring of **equal characteristic** $p > 0$, with *perfect residue field* k, and fraction field $K := \text{Fr}\, R$. We denote by π a uniformising parameter of R.

2.1. The group schemes $\mathcal{M}_n$ (cf. also [M], 3.2). Let $n \geq 0$ be an integer, and let $\mathbf{G}_{a,R} = \text{Spec}\, R[T]$ be the *additive group scheme* over R. The map

$$\phi_n : \mathbf{G}_{a,R} \to \mathbf{G}_{a,R},$$

given by

$$T \mapsto T^p - \pi^{(p-1)n} T,$$

is an isogeny of group schemes. The kernel of ϕ_n is denoted by $\mathcal{M}_{n,R}$, or simply $\mathcal{M}_n$, if no confusion occurs. Thus,

$$\mathcal{M}_n := \operatorname{Spec} R[T]/(T^p - \pi^{(p-1)n}T),$$

and $\mathcal{M}_n$ is a finite and flat R-group scheme of rank p. Further, the following sequence is exact:

$$0 \to \mathcal{M}_n \to \mathbf{G}_{a,R} \xrightarrow{\phi_n} \mathbf{G}_{a,R} \to 0. \tag{3}$$

If $n = 0$, the sequence (3) is the Artin-Schreier sequence, and $\mathcal{M}_0$ is the étale constant group scheme $(\mathbf{Z}/p\mathbf{Z})_R$. If $n > 0$, the sequence (3) has a generic fiber which is isomorphic to the étale Artin-Schreier sequence, and a special fiber isomorphic to the radicial exact sequence

$$0 \to \alpha_p \to \mathbf{G}_{a,k} \xrightarrow{\mathbf{F}} \mathbf{G}_{a,k} \to 0. \tag{4}$$

Thus, if $n > 0$, the group scheme $\mathcal{M}_n$ has a generic fiber which is étale, isomorphic to $(\mathbf{Z}/p\mathbf{Z})_K$, and its special fiber is isomorphic to the *infinitesimal group scheme* $\alpha_{p,k}$.

Let X be an R-scheme. The sequence (3) induces a long cohomology exact sequence

$$\Gamma(X,\mathcal{O}_X) \xrightarrow{\phi_n} \Gamma(X,\mathcal{O}_X) \to H^1(X,\mathcal{M}_n) \to H^1(X,\mathcal{O}_X) \xrightarrow{\phi_n} H^1(X,\mathcal{O}_X). \tag{5}$$

The cohomology group

$$H^1(X,\mathcal{M}_n)$$

classifies the isomorphism classes of fppf-torsors with group $\mathcal{M}_n$, above X. The exact sequence (5) allows the following description of $\mathcal{M}_n$-torsors. Locally, a torsor

$$f : Y \to X$$

under the group scheme $\mathcal{M}_n$, is given by an equation

$$T^p - \pi^{(p-1)n}T = a,$$

where T is an indeterminate, and a is a regular function on X which is uniquely determined, up to addition of elements of the form $b^p - \pi^{(p-1)n}b$ (for some regular function b). In particular, if $H^1(X,\mathcal{O}_X) = 0$ (e.g. if X is affine), then an $\mathcal{M}_n$-torsor above X is globally defined by an equation as above.

2.2. Degeneration of étale $\mathbf{Z}/p\mathbf{Z}$-torsors. In what follows let X be a *formal R-scheme* of finite type which is normal, connected, and flat over R. Let $X_K := X \times_R K$ (resp. $X_k := X \times_R k$) be the *generic* (resp. *special*) fiber of X. By "generic fiber" of X we mean the associated *K-rigid space* (cf. [B-L]).

We assume that the special fiber X_k is **integral**. Let η be the generic point of the special fiber X_k, and let $\mathcal{O}_\eta$ be the local ring of X at η, which is a discrete valuation ring with fraction field $K(X) :=$ the function field of X. Let

$$f_K : Y_K \to X_K,$$

be a non-trivial étale $\mathbf{Z}/p\mathbf{Z}$-torsor, with Y_K geometrically connected. Let

$$K(X) \to L,$$

be the corresponding extension of function fields. The main result of this section is the following.

2.2.1. THEOREM. *Assume that the ramification index above $\mathcal{O}_\eta$, in the extension $K(X) \to L$, equals 1. Then the torsor $f_K : Y_K \to X_K$ extends to a torsor $f : Y \to X$ under a finite and flat R-group scheme of rank p, with Y normal.*

Let δ be the degree of the different above η, in the extension $K(X) \to L$. Then the following cases occur:

a) $\delta = 0$. *In which case f is an étale torsor under the group scheme $\mathcal{M}_0$, and $f_k : Y_k \to X_k$ is an étale $\mathbf{Z}/p\mathbf{Z}$-torsor.*

b) $\delta > 0$. *In which case $\delta = n(p-1)$, for a certain integer $n \geq 1$, and f is a torsor under the group scheme $\mathcal{M}_n$. Further, in this case $f_k : Y_k \to X_k$ is a non-trivial radicial torsor under the k-group scheme α_p.*

Note that starting from a torsor $f_K : Y_K \to X_K$, as in 2.2.1, the condition that the ramification index above $\mathcal{O}_\eta$ equals 1 is always satisfied, after possibly a finite extension of R (cf. e.g. [E]).

PROOF. We denote by v the discrete valuation of $K(X)$ corresponding to the valuation ring $\mathcal{O}_\eta$, which is normalized by $v(\pi) = 1$. Note that π is a uniformiser of $\mathcal{O}_\eta$. We first start with the special case where $H^1(X_K, \mathcal{O}_{X_K}) = 0$. The torsor f_K is then given by an Artin-Schreier equation of the form $T^p - T = a_K$, where a_K is a regular function on X_K. We have $a_K = \pi^m a$, where $m \in \mathbf{Z}$ is an integer, and a is a regular function on X, with $v(a) = 0$.

First, note that necessarily $m \leq 0$. For if $m > 0$, then $a_K = b^p - b$, where $b = a\pi^m + (a\pi^m)^p + (a\pi^m)^{p^2} + \cdots + (a\pi^m)^{p^i} + \cdots$ (the sum converges, since X_K is complete for the π-adic topology). But this contradicts the fact that f_K is a non-trivial torsor.

If $m = 0$, the equation $T^p - T = a$ defines an étale $\mathbf{Z}/p\mathbf{Z}$-torsor $f : Y \to X$ above X, which coincides with f_K on the generic fiber, and we are in the case a). In this case the étale torsor $f_k : Y_k \to X_k$ is given by the Artin-Schreier equation $T^p - T = \bar{a}$, where $\bar{a}$ is the image of a modulo π.

Next, we treat the case where $m < 0$. In this case m is necessarily divisible by p. For otherwise, the extension $K(X) \to L$ is totally ramified above $\mathcal{O}_\eta$. Write $-m = np$. Assume first that the image $\bar{a}$ of a modulo π, via the canonical map $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)/\pi\Gamma(X, \mathcal{O}_X)$, is not a p-power. Consider the cover $f : Y \to X$ given by the equation $\tilde{T}^p - \pi^{n(p-1)}\tilde{T} = a$. Then f is an fppf-torsor under the group scheme $\mathcal{M}_n$, which coincides with f_K on the generic fiber (consider the change of variables $T := \tilde{T}/\pi^n$). Its special fiber $f_k : Y_k \to X_k$, is the α_p-torsor given by the equation $t^p = \bar{a}$.

In the case where $\bar{a}$ is a p-power, the following two cases occur.

First: $\bar{a}$ is a p^s-power for every integer s, which implies necessarily that $\bar{a} \in k$. In this case, and after some modifications (allowed by the Artin-Schreier theory) which do not change the torsor f_K, we can reduce to an equation of the above form, where $\bar{a}$ doesn't belong to k. To explain this, assume for simplicity that $n = 1$. Then $a = a'^p + \pi^\alpha b$, where $b \in \Gamma(X, \mathcal{O}_X)$, and $a' \in R$. Thus, the equation defining f_K is $T^p - T = a'^p/\pi^p + \pi^\alpha b/\pi^p$, which after some modifications (which are allowed by the Artin-Schreier theory) can be written as $T^p - T = a'/\pi + \pi^\alpha b/\pi^p$. But this equation ramifies above π, which is not the case by assumption. Thus the first case doesn't occur and we are lead to the second case.

There exists a positive integer r such that $\bar{a}$ is a p^r-power but not a p^{r+1}-power. We assume for simplicity that $r = 1$ (the general case $r > 1$ is treated in a similar way, and is left to the reader). Let $\bar{a} = \bar{b}^p$, so that $a = b^p + \pi\tilde{b}$, where b and $\tilde{b}$ are functions on X, and b reduces to $\bar{b}$ modulo π. Our equation is then of the form $T^p - T = (b/\pi^n)^p + \tilde{b}/\pi^{(pn-1)}$. After adding $(b/\pi^n) - (b/\pi^n)^p$ to the right hand side, which doesn't change the torsor f_K, we get the equation $T^p - T = (b/\pi^n) + \tilde{b}/\pi^{(pn-1)}$, which can also be written in the form $T^p - T = (b/\pi^n) + b'/\pi^{n'}$, where b' is a function with $v(b') = 0$, and $n' \le pn - 1$. If $n > n'$, then n is necessarily divisible by p, by the above argument. Write $n = ps$. The equation $\tilde{T}^p - \pi^{s(p-1)}\tilde{T} = b + \pi^{n-n'}b'$ defines a torsor $f : Y \to X$ under the group scheme $\mathcal{M}_s$, which coincides with f_K on the generic fiber. Its special fiber $f_k : Y_k \to X_k$ is the α_p-torsor given by the equation $\tilde{t}^p = \bar{b}$. In the case where $n' \ge n$, n' is necessarily divisible by p. Write $n' = s'p$. In this case if $\bar{b}'$ (resp. $\bar{b}' + \bar{b}$ in case $n' = n$) is not a p-power (where $\bar{b}$, and $\bar{b}'$, denote the reduction of b, resp. b', modulo π), then the equation $\tilde{T}^p - \pi^{s'(p-1)}\tilde{T} = \pi^{n'-n}b + b'$ defines a torsor $f : Y \to X$, under the group scheme $\mathcal{M}_{s'}$, which coincides with f_K on the generic fiber. Its special fiber $f_k : Y_k \to X_k$ is the α_p-torsor given by the equation $\tilde{t}^p = \bar{b}'$ (resp. $\tilde{t}^p = \bar{b}' + \bar{b}$, in the case $n = n'$). Otherwise, if $\bar{b}'$ (or $\bar{b}' + \bar{b}$ in case $n = n'$) is a p-power, then we repeat the same procedure as above. Since n and n' decrease at each step this process must stop at some finite stage, and we end up with an equation of the form $\tilde{T}^p - \pi^{r(p-1)}\tilde{T} = \tilde{b}$, where $\tilde{b}$ is a function whose reduction modulo π is not a p-power, for some positive integer r. Hence the required result. Observe that in the above case $m < 0$, the α_p-torsor $f_k : Y_k \to X_k$ that we obtain above is non-trivial, since the ramification index above $\mathcal{O}_\eta$, in the extension $K(X) \to L$, equals 1.

The argument in the general case, where $H^1(X_K, \mathcal{O}_{X_K}) \neq 0$, is similar to the one used in [S], proof of 2.4. More precisely, in general there exists an open covering $(U_i)_i$ of X, and regular functions $\tilde{a}_i \in \Gamma(U_{i,K}, \mathcal{O}_X)$ (where $U_{i,K} := U_i \times_R K$, and the $\tilde{a}_i$ are defined up to addition of functions of the form $b_i^p - b_i$), such that the torsor f_K is defined above $U_{i,K}$ by the equation $T_i^p - T_i = \tilde{a}_i$. Now the above discussion shows that after some modifications (of the type used above) the torsor f_K can be defined above each open $U_{i,K}$ by an equation $\tilde{T}_i - \pi^{n_i(p-1)}\tilde{T} = a_i$, for some (uniquely determined) integer $n_i \geq 0$, such that if $n_i > 0$ the image $\bar{a}_i$ of a_i, modulo π, is not a p-power. Moreover, the degree of the different δ_i above the generic point η of $U_{i,k} := U_i \times_R k$ equals $n_i(p-1)$. From this we deduce that all n_i are equal. Write $n := n_i$. Then the $\mathcal{M}_n$-torsor $f : Y \to X$, which is locally given by the equation $\tilde{T}_i - \pi^{n(p-1)}\tilde{T}_i = a_i$, above the open U_i, coincides on the generic fiber with the torsor f_K.

2.2.2. It follows from 2.2.1 that an étale $\mathbf{Z}/p\mathbf{Z}$-torsor above the generic fiber X_K of X induces canonically a *degeneration data*, which consists of a torsor above the special fiber X_k of X, under a finite and flat k-group scheme which is either étale or of type α_p. Reciprocally, we have the following result of *lifting* of such a *degeneration data*.

2.2.3. Proposition. *Assume that X is* **affine**. *Let $f_k : Y_k \to X_k$ be a torsor under a finite and flat k-group scheme, which is étale (resp. of type α_p). Then f_k can be lifted to a torsor $f : Y \to X$, under a finite and flat R-group scheme of rank p, which is étale (resp. isomorphic to $\mathcal{M}_n$, for an integer $n > 0$).*

Proof. Since X is affine, the torsor f_k is given by an equation $x^p - x = \bar{a}$, where $\bar{a}$ is a regular function on X_k (resp. an equation $x^p = \bar{a}$, where $\bar{a}$ is a regular function on X_k). Let a be a regular function on X which reduces to $\bar{a}$ modulo π. The equation $X^p - X = a$ (resp. $X^p - \pi^{n(p-1)}X = a$, where $n > 0$ is an integer) defines a cover $f : Y \to X$ above X, which has the structure of a torsor under the étale group scheme $(\mathbf{Z}/p\mathbf{Z})_R$ (resp. under the group scheme $\mathcal{M}_n$), and which clearly induces the torsor f_k above the special fiber X_k.

2.2.4. Remark. If X is not affine, one can find examples of an α_p-torsor above the special fiber X_k of X, which cannot be lifted to a torsor above X, under a finite and flat R-group scheme of rank p, which is étale above the generic fiber of X. This is indeed the case if X is a proper and smooth R-curve, whose generic fiber is *ordinary*, and whose special fiber has a jacobian which is isogenuous to a product of *supersingular* elliptic curves. However, for a proper and smooth R-curve X, the same arguments used in [S-1], 4.7, show that it is always possible to lift an α_p-torsor above the special fiber X_k of X, after possibly replacing X by another R-curve X' which lifts X_k.

3. Degeneration of p^2-cyclic covers in equal characteristic $p > 0$

Throughout this section we use the same notations as in section 2.

3.1. The group schemes W_{m_1,m_2}. Let m_1 and m_2 be non-negative integers, such that $m_2 - pm_1 \geq 0$. We define the *twisted R-Witt group scheme*

$$W_{m_1,m_2},$$

of length two, as follows. Scheme theoretically

$$W_{m_1,m_2} \simeq \mathbf{G}_{a,R}^2,$$

and the group law is defined by

$$(x_1, x_2) + (y_1, y_2) := (x_1 + y_1, x_2 + y_2 + \pi^{m_2 - pm_1} W(x_1, y_1)).$$

Note, that if $p = 2$, then the subtraction in W_{m_1,m_2} is given by

$$(x_1, x_2) - (y_1, y_2) = (x_1 - y_1, x_2 - y_2 + \pi^{m_2 - 2m_1}(x_1 y_1 - y_1^2)).$$

The generic fiber $(W_{m_1,m_2})_K$, of W_{m_1,m_2}, is isomorphic to the Witt group scheme $W_{2,K} := W_2 \times_{\mathbf{F}_p} K$, via the map

$$(W_{m_1,m_2})_K \to W_{2,K}$$
$$(x_1, x_2) \mapsto (x_1/\pi^{m_1}, x_2/\pi^{m_2}).$$

Its special fiber $(W_{m_1,m_2})_k$ is isomorphic either to the Witt group scheme $W_{2,k} := W_2 \times_{\mathbf{F}_p} k$, if $m_2 - pm_1 = 0$, or to the group scheme $\mathbf{G}_{a,k}^2$, otherwise. Note that we have an exact sequence

$$0 \to \mathbf{G}_a \xrightarrow{V} W_{m_1,m_2} \xrightarrow{R} \mathbf{G}_a \to 0,$$

where

$$V : \mathbf{G}_a \to W_{m_1,m_2},$$

is the *Vershiebung* homomorphism defined by

$$V(x) = (0, x),$$

and

$$R : W_{m_1,m_2} \to \mathbf{G}_a,$$

is the projection

$$R(x_1, x_2) = x_1.$$

3.2. The group schemes $\mathscr{H}_{m_1,m_2}$. We use the same notations as in 3.1. The following maps I_{m_1,m_2}, and $\mathbf{F}$, are group scheme homomorphisms:

$$I_{m_1,m_2} : W_{m_1,m_2} \longrightarrow W_{pm_1,pm_2},$$
$$(x_1, x_2) \to (\pi^{m_1(p-1)}x_1, \pi^{m_2(p-1)}x_2),$$

and

$$\mathbf{F} : W_{m_1,m_2} \longrightarrow W_{pm_1,pm_2},$$
$$(x_1, x_2) \to (x_1^p, x_2^p).$$

Consider the following isogeny:

$$\varphi_{m_1,m_2} := \mathbf{F} - I_{m_1,m_2} : W_{m_1,m_2} \longrightarrow W_{pm_1,pm_2}$$
$$(x_1, x_2) \mapsto (x_1^p, x_2^p) - (\pi^{m_1(p-1)}x_1, \pi^{m_2(p-1)}x_2)$$

which, if $p \neq 2$, is given by

$$(x_1, x_2) \mapsto (x_1^p - \pi^{m_1(p-1)}x_1, x_2^p - \pi^{m_2(p-1)}x_2 + \pi^{pm_2 - p^2m_1} W(x_1^p, -\pi^{m_1(p-1)}x_1));$$

which can be rewritten as

$$(x_1, x_2) \mapsto \left(x_1^p - \pi^{m_1(p-1)}x_1, x_2^p - \pi^{m_2(p-1)}x_2 - \sum_{k=1}^{p-1} \frac{\pi^{m_2p - m_1(pk+p-k)}}{k} x_1^{p+(p-1)k} \right),$$

and if $p = 2$, is given by

$$(x_1, x_2) \mapsto \left(x_1^2 - \pi^{m_1}x_1, x_2^2 - \pi^{m_2}x_2 + \pi^{2m_2}\left(\frac{x_1^3}{\pi^{3m_1}} - \frac{x_1^2}{\pi^{2m_1}} \right) \right).$$

We define the group scheme

$$\mathscr{H}_{m_1,m_2},$$

to be the kernel of the above isogeny. Thus, we have an exact sequence:

$$0 \longrightarrow \mathscr{H}_{m_1,m_2} \longrightarrow W_{m_1,m_2} \xrightarrow{\mathbf{F}-I_{m_1,m_2}} W_{pm_1,pm_2} \longrightarrow 0, \tag{6}$$

and $\mathscr{H}_{m_1,m_2}$ is a finite and flat commutative R-group scheme of rank p^2. Further, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{M}_{m_2} & \longrightarrow & \mathbf{G}_a & \xrightarrow{\phi_{m_2}} & \mathbf{G}_a & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{H}_{m_1,m_2} & \longrightarrow & W_{m_1,m_2} & \xrightarrow{\varphi_{m_1,m_2}} & W_{pm_1,pm_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R} & & \\
0 & \longrightarrow & \mathscr{M}_{m_1} & \longrightarrow & \mathbf{G}_a & \xrightarrow{\phi_{m_1}} & \mathbf{G}_a & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The group scheme $\mathscr{H}_{m_1,m_2}$ is an extension of the group scheme $\mathscr{M}_{m_1}$ by $\mathscr{M}_{m_2}$. Its generic fiber $(\mathscr{H}_{m_1,m_2})_K$ is isomorphic to the étale constant group scheme $\mathbf{Z}/p^2\mathbf{Z}$. Its special fiber $(\mathscr{H}_{m_1,m_2})_k$ is either isomorphic to the product $\mathbf{Z}/p\mathbf{Z} \times \alpha_p$, if $m_1 = 0$, and $m_2 > 0$; in which case we denote it by H_k. Or, is isomorphic to the product $\alpha_p \times \alpha_p$, if $m_1 > 0$; in which case we denote it by G_k. We have the following exact sequences:

$$0 \longrightarrow H_k \longrightarrow \mathbf{G}_{a,k}^2 \xrightarrow{(\mathbf{F}-\mathrm{Id})\times\mathbf{F}} \mathbf{G}_{a,k}^2 \longrightarrow 0, \tag{7}$$

and

$$0 \longrightarrow G_k \longrightarrow \mathbf{G}_{a,k}^2 \xrightarrow{\mathbf{F}\times\mathbf{F}} \mathbf{G}_{a,k}^2 \longrightarrow 0. \tag{8}$$

Let X be an R-scheme. The sequence (6) induces a long cohomology exact sequence

$$\begin{aligned} \Gamma(X, W_{m_1,m_2}) \xrightarrow{\varphi_{m_1,m_2}} \Gamma(X, W_{pm_1,pm_2}) \longrightarrow H^1(X, \mathscr{H}_{m_1,m_2}) \\ \longrightarrow H^1(X, W_{m_1,m_2}) \xrightarrow{\varphi_{m_1,m_2}} H^1(X, W_{pm_1,pm_2}). \end{aligned} \tag{9}$$

The cohomology group

$$H^1(X, \mathscr{H}_{m_1,m_2}),$$

classifies the isomorphism classes of fppf-torsors, with group $\mathscr{H}_{m_1,m_2}$, above X. The above exact sequence (9) allows the following description of $\mathscr{H}_{m_1,m_2}$-torsors. Locally, a torsor

$$f : Y \to X,$$

under the group scheme $\mathscr{H}_{m_1,m_2}$, is given by the equations

$$T_1^p - \pi^{m_1(p-1)}T_1 = a_1,$$

and

$$T_2^p - \pi^{m_2(p-1)}T_2 = a_2 - \pi^{pm_2-p^2m_1}W(T_1^p, -\pi^{m_1(p-1)}T_1),$$

which can be rewritten as:

$$T_2^p - \pi^{m_2(p-1)}T_2 = a_2 + \sum_{k=1}^{p-1}\frac{\pi^{m_2p-m_1(pk+p-k)}}{k}T_1^{p+k(p-1)},$$

if $p \neq 2$, resp.

$$T_2^p - \pi^{m_2}T_2 = a_2 + \pi^{2m_2}\left(\frac{T_1^2}{\pi^{2m_1}} - \frac{T_1^3}{\pi^{3m_1}}\right),$$

if $p = 2$; where T_1, T_2, are indeterminates, and a_1, a_2, are regular functions on X. Its special fiber is either the H_k-torsor given by the equations

$$t_1^p - t_1 = \bar{a}_1,$$

and

$$t_2^p = \bar{a}_2,$$

if $m_1 = 0$. Or, the G_k-torsor given by the equations

$$t_1^p = \bar{a}_1,$$

and

$$t_2^p = \bar{a}_2,$$

otherwise. Here, $\bar{a}_1$ (resp. $\bar{a}_2$) is the image of a_1 (resp. a_2) modulo π. In particular, if $H^1(X, \mathcal{O}_X) = 0$ (e.g. if X is affine), then an $\mathscr{H}_{m_1,m_2}$-torsor above X is globally defined by an equation as above.

3.3. Degeneration of étale $\mathbf{Z}/p^2\mathbf{Z}$-torsors. In this section we use the same notations as in 2.2. Our aim is to describe explicitly the degeneration of étale $\mathbf{Z}/p^2\mathbf{Z}$-torsors.

Let

$$f_K : Y_K \to X_K,$$

be a non-trivial étale $\mathbf{Z}/p^2\mathbf{Z}$-torsor. Let

$$K(X) \to L,$$

be the (degree p^2) cyclic extension of function fields, corresponding to the torsor f_K, which canonically factorizes as

$$K(X) \to L_1 \to L_2 := L,$$

where

$$K(X) \to L_1,$$

is a cyclic extension of degree p. *We assume that the ramification index above the generic point η of X_k, in the extension $K(X) \to L$, equals* 1. We have a canonical factorization

$$Y_K = Y_{2,K} \xrightarrow{f_{2,K}} Y_{1,K} \xrightarrow{f_{1,K}} X_K,$$

of f_K, where $f_{i,K}$ is a $\mathbf{Z}/p\mathbf{Z}$-torsor, $i \in \{1,2\}$. Moreover, by 2.2.1, the torsor $f_{2,K}$ (resp. $f_{1,K}$) extends to a torsor $f_2 : Y_2 \to Y_1$ (resp. $f_1 : Y_1 \to X$) under a finite and flat R-group scheme of rank p. The composite $f := f_1 \circ f_2$ is a finite and flat cover which coincides, on the generic fiber, with f_K. We assume that the special fiber $Y_{2,k}$, of Y_2, is **irreducible**. In particular, above the generic point η, there exists a unique generic point η_1 in Y_1, which lies above η. We denote by δ (resp. δ_1, and δ_2) the degree of different in the extension L above the point η (resp. the degree of different in the extension L_1, above the point η, and that of the different in the extension L_2, above the point η_1). Note that $\delta = \delta_1 + \delta_2$.

3.3.1. We start with the following lemma 3.3.2, which will be used in the proof of 3.3.3. In what follows we assume that

$$X = \operatorname{Spf} A,$$

is **affine**, and that

$$f_1 : Y_1 := \operatorname{Spf} B \to X,$$

is a torsor under the group scheme $\mathscr{M}_n$, for some integer $n > 0$ (cf. 2.1). Thus, f_1 is given by an equation

$$T^p - \pi^{n(p-1)} T = v,$$

where $v \in A$ is such that its image $\bar{v} \in \bar{A} := A/\pi A$ is not a p-power. In particular, the special fiber

$$\bar{f}_1 : Y_{1,k} = \operatorname{Spec} \bar{B} \to X_k = \operatorname{Spec} \bar{A},$$

of the torsor f_1, where $\bar{B} := B/\pi B$ (resp. $\bar{A} := A/\pi A$), is the α_p-torsor given by the equation

$$t^p = \bar{v}.$$

Further, $\bar{B}$ is a *free* $\bar{A}$-algebra with basis

$$\{1, t, t^2, \dots, t^{p-1}\}.$$

We need to characterize elements of A which become p-powers, modulo π, in $\bar{B}$, but are not necessarily p-powers, modulo π, in $\bar{A}$.

3.3.2. LEMMA. *Let $u \in A$. Assume that the image $\bar{u}$, of u, is a p-power in $\bar{B}$.*
Then $u = f(v) + \pi u'$, where $u' \in A$, and $f(v)$ belongs to the additive subgroup

$$A_v := A^p \oplus A^p.v \oplus \cdots \oplus A^p.v^{p-1}$$

of A. Moreover, let

$$f(v) := a_0^p + a_1^p v + \cdots + a_{p-1}^p v^{p-1} \in A_v,$$

and let $m > 0$ be an integer. Consider the element $g := f(v)\pi^{-pm} \in A_K$. Then

$$g = \pi^{-pm}(a_0^p + a_1^p(T^p - \pi^{n(p-1)}T) + \cdots + a_{p-1}^p(T^p - \pi^{n(p-1)}T)^{p-1})$$

in B_K, and after addition of elements of B_K, of the form $b^p - b$, one can transform g in

$$\tilde{g} = \pi^{-m}(a_0 + a_1 T + \cdots + a_{p-1}T^{p-1}) + \pi^{-(pm-n(p-1))}\left(-\sum_{j=1}^{p-1} j a_j^p T^{p(j-1)+1}\right)$$
$$+ \pi^{-(pm-2n(p-1))}h(T),$$

where $h(T) \in B$. Moreover, the image of

$$-\sum_{j=1}^{p-1} j a_j^p T^{p(j-1)+1} = -a_1^p T - 2a_2^p T^{p+1} - \cdots - (p-1)a_{p-1}^p T^{p(p-2)+1},$$

in $\bar{B}$, is not a p-power.

PROOF. We have $\bar{B} = \bar{A} \oplus \bar{A}.t \oplus \cdots \oplus \bar{A}.t^{p-1}$. Hence, $\bar{B}^p = \bar{A}^p \oplus \bar{A}^p.\bar{v} \oplus \cdots \oplus \bar{A}^p.\bar{v}^{p-1}$, and the first assertion of the lemma follows. Let $g := (a_0^p + a_1^p v + \cdots + a_{p-1}^p v^{p-1})\pi^{-pm} \in B_K$. Since $T^p - \pi^{n(p-1)}T = v$ in B_K, we can write $g = \pi^{-pm}(a_0^p + a_1^p(T^p - \pi^{n(p-1)}T) + \cdots + a_{p-1}^p(T^p - \pi^{n(p-1)}T)^{p-1})$ in B_K. After developing the terms $(T^p - \pi^{n(p-1)}T)^j$, for $j \in \{1, p-1\}$; according to the binomial expansion, and putting together the terms with the same power of π we get that

$$
\begin{aligned}
g = {} & \pi^{-pm}(a_0^p + a_1^p T^p + \cdots + a_{p-1}^p T^{p(p-1)}) \\
& + \pi^{-(pm-n(p-1))}(-a_1^p T - 2a_2^p T^{p+1} - \cdots - (p-1)a_{p-1}^p T^{p(p-2)+1}) \\
& + \pi^{-(pm-2n(p-1))} h(T),
\end{aligned}
$$

where $h(T) \in B$. Finally, after adding $(a_0 + a_1 T + \cdots + a_{p-1} T^{p-1})/\pi^m - (a_0^p + a_1^p T^p + \cdots + a_{p-1}^p T^{p(p-1)})/\pi^{pm}$ to the right hand side of the above equality, we get the desired expression for $\tilde{g}$.

The next theorem is the main result of this section. It describes locally (and explicitly) the degeneration of étale $\mathbf{Z}/p^2\mathbf{Z}$-torsors. More precisely, we are able to find "canonical integral equations" which describe the reduction of p^2-cyclic covers, in equal characteristic p.

3.3.3. Theorem. *We use the same notations as in* 3.3. *Assume that* $X =$ Spf A *is* **affine**. *Then the torsor* f_K *can be described by an equation of the form*

$$(T_1^p, T_2^p) - (T_1, T_2) = (\pi^{m_1} a_1, \pi^{m_2} a_2)$$

where a_1, a_2, *are regular functions on* X, *with* $v(a_1) = v(a_2) = 0$, $m_1 \leq 0$, $m_2 \in \mathbf{Z}$ *is an integer. Moreover, the following cases occur:*

a) $m_1 = 0$, *and* $m_2 \geq 0$. *In this case* f *is an étale* $\mathbf{Z}/p^2\mathbf{Z}$*-torsor above* X, *given by the equations*

$$(T_1^p, T_2^p) - (T_1, T_2) = (a_1, \pi^{m_2} a_2).$$

Its special fiber $f_k : Y_k \to X_k$, *is the étale* $\mathbf{Z}/p^2\mathbf{Z}$*-torsor given by the equations*

$$(t_1^p, t_2^p) - (t_1, t_2) = (\bar{a}_1, \overline{\pi^{m_2} a_2}),$$

and $\delta = \delta_1 = \delta_2 = 0$ *(here,* $\bar{a}_1$ *(resp.* $\overline{\pi^{m_2} a_2}$*) denotes the image of* a_1 *(resp.* $\pi^{m_2} a_2$*) modulo* π*).*

b) $m_1 = 0$, $m_2 < 0$ *is divisible by* p, *and* a_2 *is not a* p*-power modulo* π. *In this case* f *is a torsor under the* R*-group scheme* $\mathscr{H}_{0,m_2'}$; *where* $m_2' := \frac{-m_2}{p}$ *(cf.* 3.2*), and is given by the equations*

$$T_1^p - T_1 = a_1,$$

and

$$\tilde{T}_2^p - \pi^{m_2'(p-1)} \tilde{T}_2 = a_2 - \pi^{-m_2} W(T_1^p, -T_1) = a_2 + \pi^{-m_2} \sum_{k=1}^{p-1} \frac{1}{k} T_1^{p+(p-1)k},$$

if $p \neq 2$, *resp.*

$$\tilde{T}_2^2 - \pi^{m_2'} \tilde{T}_2 = a_2 - \pi^{-m_2}(T_1^3 - T_1^2)$$

if $p = 2$. *Its special fiber is the torsor under the* k*-group scheme* $(\mathscr{H}_{0,m_2'})_k \simeq H_k$, *given by the equations*

$$t_1^p - t_1 = \bar{a}_1,$$

and

$$\tilde{t}_2^p = \bar{a}_2,$$

where $\bar{a}_1$ *(resp.* $\bar{a}_2$*) is the image of* a_1 *(resp. of* a_2*) modulo* π*. In this case* $\delta_1 = 0$*, and* $\delta = \delta_2 = m_2'(p-1)$*.*

c) $m_1 < 0$ *is divisible by* p*, and the image* $\bar{a}_1$*, of* a_1 *modulo* π*, is not a* p*-power. Write* $m_1 = -pm_1'$*. In this case* f_1 *is an* $\mathcal{M}_{m_1'}$*-torsor, given by the equation*

$$\tilde{T}_1^p - \pi^{m_1'(p-1)}\tilde{T}_1 = a_1.$$

Its special fiber $f_{1,k} : Y_{1,k} \to X_k$ *is the* α_p*-torsor given by the equation*

$$\tilde{t}_1^p = \bar{a}_1.$$

We have $\delta_1 = m_1'(p-1)$*. As for* f_2*, the following cases occur:*

c-1) $m_1'(p(p-1)+1) > -m_2$ *(resp.* $m_1'(p(p-1)+1) = -m_2$*). In this case* m_1' *is necessarily divisible by* p*. Write* $m_1' = pm_1''$*. If* $\tilde{m}_1 := m_1''(p(p-1)+1)$*, then* f_2 *is a torsor under* $\mathcal{M}_{\tilde{m}_1,R}$ *given by the equation:*

$$\begin{aligned}\tilde{T}_2^p - \pi^{\tilde{m}_1(p-1)}\tilde{T}_2 &= \pi^{\tilde{m}_1 p + m_2}a_2 - \pi^{m_1'(p(p-1)+1)}W(\pi^{-m_1'p}\tilde{T}_1^p, -\pi^{-m_1'}\tilde{T}_1)\\ &= \pi^{\tilde{m}_1 p + m_2}a_2 + \sum_{k=1}^{p-1}\frac{\pi^{m_1'((p-1)^2-(p-1)k)}}{k}\tilde{T}_1^{p+(p-1)k}\end{aligned}$$

if $p \neq 2$*, resp.*

$$\tilde{T}_2^2 - \pi^{\tilde{m}_1}\tilde{T}_2 = \pi^{2\tilde{m}_1+m_2}a_2 + \pi^{m_1'}\tilde{T}_1^2 - \tilde{T}_1^3.$$

Its special fiber is the α_p*-torsor given by the equation*

$$\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}$$

resp.

$$\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1} + \bar{a}_2.$$

Otherwise, $-m_2 > m_1'(p(p-1)+1)$*, in which case* m_2 *is necessarily divisible by* p*. Write* $-m_2 = pm_2'$*. We have the following description for* $\pi^{m_2}a_2$*:*

$$\begin{aligned}\pi^{m_2}a_2 = f_1(a_1)/\pi^{pm_2'} + f_2(a_1)/\pi^{pm_2'-t_1} + \cdots + f_r(a_1)/\pi^{pm_2'-t_1-\cdots-t_{r-1}}\\ + g/\pi^{pm_2'-t_1-\cdots-t_r}\end{aligned}$$

where $f_i(a_1)$ *belongs to the subgroup* A_{a_1} *of* A *(cf.* 3.3.2*),* $g \in A$*, and the* t_i *are positive integers (note that* g *and the* f_i *can be* 0*). Moreover, the torsor* $f_{2,K}$ *is given by the equation*

$$T_2^p - T_2 = f_1(a_1)/\pi^{pm_2'} + f_2(a_1)/\pi^{pm_2'-t_1} + \cdots + f_r(a_1)/\pi^{pm_2'-t_1-\cdots-t_{r-1}}$$

$$+ g/\pi^{pm_2'-t_1-\cdots-t_r} + \sum_{k=1}^{p-1} \frac{\pi^{-m_1'(pk+p-k)}}{k} \tilde{T}_1^{p+(p-1)k},$$

if $p \neq 2$, *and*

$$T_2^2 - T_2 = f_1(a_1)/\pi^{2m_2'} + f_2(a_1)/\pi^{2m_2'-t_1} + \cdots + f_r(a_1)/\pi^{2m_2'-t_1-\cdots-t_{r-1}}$$

$$+ g/\pi^{2m_2'-t_1-\cdots-t_r} + \frac{\tilde{T}_1^2}{\pi^{2m_1'}} - \frac{\tilde{T}_1^3}{\pi^{3m_1'}},$$

if $p = 2$. *And the following distinct cases occur:*

c-2) $pm_2' - (p-1)m_1' > \sup(m_1'(p(p-1)+1), pm_2' - t_1 - \cdots - t_r)$ *(resp.* $pm_2' - (p-1)m_1' = m_1'(p(p-1)+1) > pm_2' - t_1 - \cdots - t_r$*). In this case* $pm_2' - m_1'(p-1)$ *is divisible by* p*, and* $\delta_2 = m_2''(p-1)$*; where* $m_2'' := (pm_2' - m_1'(p-1))/p$*. Let* $f_1(a_1) := c_0^p + c_1^p a_1 + \cdots + c_{p-1}^p a_1^{p-1}$*. Then* f_2 *is a torsor under* $\mathcal{M}_{m_2''}$*, and its special fiber is the* α_p*-torsor given by the equation*

$$\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - \cdots - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1}$$

resp.

$$\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - \cdots - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1} - \tilde{t}_1^{p(p-1)+1},$$

where $\bar{c}_i$ *is the image of* c_i *modulo* π.

c-3) $pm_2' - t_1 - \cdots - t_r > \sup(pm_2' - (p-1)m_1', m_1'(p(p-1)+1))$ *(resp.* $pm_2' - (p-1)m_1' = pm_2' - t_1 - \cdots - t_r > m_1'(p(p-1)+1)$*), and the image of* g *modulo* π *is not a* p*-power in* $\mathcal{O}(Y_{1,k})$*. In this case* $pm_2' - t_1 - \cdots - t_r$ *is divisible by* p*,* $\delta_2 = m_2''(p-1)$*; where* $m_2'' := (pm_2' - t_1 - \cdots - t_r)/p$*, and* f_2 *is an* $\mathcal{M}_{m_2''}$*-torsor. Its special fiber is the* α_p*-torsor given by the equation*

$$\tilde{t}_2^p = \bar{g}$$

resp.

$$\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - \cdots - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1} + \bar{g}.$$

c-4) $m_1'(p(p-1)+1) > \sup(pm_2' - t_1 - \cdots - t_r, pm_2' - (p-1)m_1')$ *(resp.* $pm_2' - t_1 - \cdots - t_r = m_1'(p(p-1)+1) > pm_2' - (p-1)m_1'$*). In this case* m_1' *is divisible by* p*, and if* $\tilde{m}_1 := m_1''(p(p-1)+1)$*; where* $m_1'' := \frac{m_1'}{p}$*, then* f_2 *is a torsor under* $\mathcal{M}_{\tilde{m}_1,R}$*. Its special fiber is the* α_p*-torsor given by the equation*

$$\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}$$

resp.

$$\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1} + \bar{g}.$$

c-5) $pm_2' - t_1 - \cdots - t_r = m_1'(p(p-1)+1) = pm_2' - (p-1)m_1'$. *In this case* m_1' *is divisible by* p, *and if* $\tilde{m}_1 := m_1''(p(p-1)+1)$*; where* $m_1'' := \frac{m_1'}{p}$, *then* f_2 *is a torsor under* $\mathcal{M}_{\tilde{m}_1,R}$. *Its special fiber is the* α_p*-torsor given by the equation*

$$\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - \cdots - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1} - \tilde{t}_1^{p(p-1)+1} + \bar{g}.$$

Further, in all the above cases, if f_1 *(resp.* f_2*) is a torsor under the group scheme* $\mathcal{M}_{\tilde{m}_1}$ *(resp.* $\mathcal{M}_{\tilde{m}_2}$*), then necessarily* $\tilde{m}_2 \geq \tilde{m}_1(p(p-1)+1)/p$. *Moreover, in all the cases* c-2, c-3, c-4, *and* c-5 *above the functions* $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_{p-1}$ *(resp.* $\bar{g}$*) are uniquely determined (resp. is uniquely determined up to addition of elements of the form* $\bar{h}^p$*, where* $\bar{h}$ *is a regular function on* X_k*). In the case* c-1 *the function* $\bar{a}_2$ *is uniquely determined up to addition of* $\bar{b}^p$*, where* $\bar{b}$ *is a regular function on* X_k.

PROOF. The torsor f_K is given, by the Artin-Schreier-Witt theory, by an equation of the form

$$(T_1^p, T_2^p) - (T_1, T_2) = (\tilde{a}_1, \tilde{a}_2),$$

where $\tilde{a}_1$, and $\tilde{a}_2$, are regular functions on X_K. We can write $\tilde{a}_1 = \pi^{m_1} a_1$ (resp. $\tilde{a}_2 = \pi^{m_2} a_2$), where a_1, and a_2, are regular functions on X, with $v(a_1) = v(a_2) = 0$. Also, it follows from 2.2.1 that $m_1 \leq 0$. If $m_1 = 0$, and $m_2 \geq 0$, then we are in case a), and our assertion there is then clear.

Assume that $m_1 = 0$, and $m_2 < 0$. Then it follows from 2.2.1 that m_2 is necessarily divisible by p, and after possibly some modifications (as in the proof of 2.2.1) we may assume that a_2 is not a p-power modulo π (here one uses the fact that a regular function u on X, which is not a p-power modulo π in X_k, cannot become a p-power in $Y_{1,k}$, since $f_{1,k}$ is an étale torsor, hence is not radicial). Write $m_2 = -pm_2'$. Then f is defined by the equations

$$T_1^p - T_1 = a_1$$

and

$$\tilde{T}_2^p - \pi^{m_2'(p-1)} \tilde{T}_2 = a_2 - \pi^{-m_2} W(T_1^p, -T_1)$$

if $p \neq 2$, resp.

$$\tilde{T}_2^2 - \pi^{m_2'} \tilde{T}_2 = a_2 - \pi^{-m_2}(T_1^3 - T_1^2)$$

if $p = 2$, where $\tilde{T}_2 := \pi^{m_2'} T_2$. The rest of the assertion in case b) follows then easily. Assume now that $m_1 < 0$. Then the assertion concerning f_1 follows from 2.2.1. Assume first that $m_2 \geq 0$. The assertion concerning f_2 follows then easily after adapting the equation defining the torsor $f_{2,K}$ to the change of variables $T_1 = \frac{\tilde{T}_1}{\pi^{m_1'}}$, and we are in the case c-1). In this case m_1' is divisible by p, and the cover f is given by the equations:

$$\tilde{T}_1^p - \pi^{m_1'(p-1)}\tilde{T}_1 = a_1$$

and

$$\tilde{T}_2^p - \pi^{\tilde{m}_1(p-1)}\tilde{T}_2 = \pi^{\tilde{m}_1 p + m_2} a_2 - \pi^{\tilde{m}_1 p} W(\pi^{-m_1' p}\tilde{T}_1^p, -\pi^{-m_1'}\tilde{T}_1)$$

$$= \pi^{\tilde{m}_1 p + m_2} a_2 + \sum_{k=1}^{p-1} \frac{\pi^{(\tilde{m}_1 - m_1')p - m_1'(p-1)k}}{k} \tilde{T}_1^{p+(p-1)k}$$

if $p \neq 2$, resp.

$$\tilde{T}_2^2 - \pi^{3m_1''}\tilde{T}_2 = \pi^{2\tilde{m}_1 + m_2} a_2 + \pi^{m_1'}\tilde{T}_1^2 - \tilde{T}_1^3$$

if $p = 2$; where $m_1'' := m_1'/p$, and $\tilde{m}_1 := m_1''(p(p-1)+1)$. From this we deduce that the special fiber of the cover f is given by the equations

$$\tilde{t}_1^p = \bar{a}_1,$$

and

$$\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1},$$

where $\tilde{t}_2$ (resp. $\tilde{t}_1$) is the image of $\tilde{T}_2$ (resp. image of $\tilde{T}_1$) modulo π.

Finally, we assume that $m_2 < 0$. Then the torsor $f_{2,K}$ is given by the equation

$$T_2^p - T_2 = \pi^{m_2} a_2 - W(\pi^{-m_1' p}\tilde{T}_1^p, -\pi^{-m_1'}\tilde{T}_1)$$

$$= \pi^{m_2} a_2 + \sum_{k=1}^{p-1} \frac{\pi^{-m_1'(p+(p-1)k)}}{k} \tilde{T}_1^{p+(p-1)k},$$

if $p \neq 2$, resp.

$$T_2^2 - T_2 = \pi^{m_2} a_2 + \frac{\tilde{T}_1^2}{\pi^{2m_1'}} - \frac{\tilde{T}_1^3}{\pi^{3m_1'}},$$

if $p = 2$. The highest power of π in the denominators of the summand

$$\sum_{k=1}^{p-1} \frac{\pi^{-m_1'(pk+p-k)}}{k} \tilde{T}_1^{p+(p-1)k},$$

resp.

$$\frac{\tilde{T}_1^2}{\pi^{2m_1'}} - \frac{\tilde{T}_1^3}{\pi^{3m_1'}},$$

is $m_1'(p(p-1)+1)$, and in order to understand the reduction of the torsor f_2 we have to compare this to m_2. Assume first that $m_1'(p(p-1)+1) > -m_2$. Then it follows from 2.2.1 that m_1' must be divisible by p. Write $m_1' = m_1''p$, and let $\tilde{m}_1 := m_1''(p(p-1)+1)$. Then we are in the case c-1), and f_2 is a torsor under the group scheme $\mathcal{M}_{\tilde{m}_1,R}$ defined by the equation

$$\begin{aligned}\tilde{T}_2^p - \pi^{\tilde{m}_1(p-1)}\tilde{T}_2 &= \pi^{\tilde{m}_1 p+m_2}a_2 - \pi^{\tilde{m}_1 p}W(\pi^{-m_1'p}\tilde{T}_1^p, -\pi^{-m_1'}\tilde{T}_1) \\ &= \pi^{\tilde{m}_1 p+m_2}a_2 + \sum_{k=1}^{p-1}\frac{\pi^{(\tilde{m}_1-m_1')p-m_1'(p-1)k}}{k}\tilde{T}_1^{p+(p-1)k}\end{aligned}$$

if $p \neq 2$, resp.

$$\tilde{T}_2^2 - \pi^{3m_1''}\tilde{T}_2 = \pi^{2\tilde{m}_1+m_2}a_2 + \pi^{m_1'}\tilde{T}_1^2 - \tilde{T}_1^3,$$

if $p = 2$. Its special fiber is the α_p-torsor given by the equation

$$\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}.$$

Assume next that $m_1'(p(p-1)+1) < -m_2$ (the case where $m_1'(p(p-1)+1) = -m_2$ is easily treated, and is left to the reader). Then it follows from 2.2.1 that $m_2 = -pm_2'$ is divisible by p, and two cases occur, depending on whether or not the image $\bar{a}_2$, of a_2 modulo π is, or is not, a p-power in $\mathcal{O}(Y_{1,k})$. If $\bar{a}_2$ is not a p-power in $\mathcal{O}(Y_{1,k})$, then f_2 is a torsor under $\mathcal{M}_{m_2',R}$ given by the equation

$$\begin{aligned}\tilde{T}_2^p - \pi^{m_2'(p-1)}\tilde{T}_2 &= a_2 - \pi^{pm_2'}W(\pi^{-m_1'p}\tilde{T}_1^p, -\pi^{-m_1'}\tilde{T}_1) \\ &= a_2 + \sum_{k=1}^{p-1}\frac{\pi^{(m_2'-m_1')p-m_1'(p-1)k}}{k}\tilde{T}_1^{p+(p-1)k},\end{aligned}$$

if $p \neq 2$, resp.

$$\tilde{T}_2^2 - \pi^{m_2'}\tilde{T}_2 = a_2 + \pi^{2m_2'-2m_1'}\tilde{T}_1^2 - \pi^{2m_2'-3m_1'}\tilde{T}_1^3,$$

if $p = 2$; where $m_2' := \frac{-m_2}{p}$. Its special fiber is the α_p-torsor given by the equation

$$\tilde{t}_2^p = \bar{a}_2,$$

and we are in the case c-3). Assume that $\bar{a}_2$ is a p-power in $\mathcal{O}(Y_{1,k})$. Then either a_2 is already a p-power in $\mathcal{O}(X_k)$, in which case we can transform (using the kind of transformations used in the proof of 2.2.1) the term $\pi^{m_2}a_2$ into $\pi^{\tilde{m}_2}\tilde{a}_2$, where $0 > \tilde{m}_2 > m_2$, and $\tilde{a}_2 \in A$. Or, a_2 is not a p-power in $\mathcal{O}(X_{1,k})$, but becomes a p-power in $\mathcal{O}(Y_{1,k})$. In the latter case it follows from 3.3.2 that $a_2 = f_1(a_1) + \pi^{t_1} g_1$, where $f_1(a_1) := c_0^p + c_1^p a_1 + \cdots + c_{p-1}^p a_1^{p-1}$ belongs to the subgroup A_{a_1} of A, $t_1 > 0$, and $g_1 \in A$. Moreover, the term $\pi^{m_2} a_2 = f_1(a_1)/\pi^{pm_2'} + g_1/\pi^{pm_2'-t_1}$ can be transformed to $\tilde{f}_1(T_1)/\pi^{pm_2'-m_1'(p-1)} + g_1/\pi^{pm_2'-t_1}$, where the image $\bar{\tilde{f}}_1(T_1) := -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - \cdots - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1}$, of $\tilde{f}_1(T_1)$ modulo π, is not a p-power (cf. loc. cit.). At this point we can repeat the same argument as above. Namely if in the first case the image $\bar{\tilde{a}}_2$, of $\tilde{a}_2$ modulo π, is not a p-power in $\mathcal{O}(Y_{1,k})$, then we conclude as above that we are either in case c-3), if $\tilde{m}_2 \geq m_1'(p(p-1)+1)$. In this case $\tilde{m}_2$ is divisible by p, and f_2 is a torsor under $\mathcal{M}_{m_2'',R}$; where $m_2'' := \tilde{m}_2/p$, whose special fiber is the α_p-torsor given by the equation

$$\tilde{t}_2^p = \bar{\tilde{a}}_2.$$

Otherwise, we repeat the same process as above. And in the second case if $pm_2' - (p-1)m_1' > \sup(pm_2' - t_1, m_1'(p(p-1)+1))$, then $pm_2' - (p-1)m_1'$ is divisible by p, and f_2 is a torsor under the group scheme $\mathcal{M}_{m_2'',R}$; where $m_2'' := (pm_2' - (p-1)m_1')/p$, defined by the equation

$$\begin{aligned}\tilde{T}_2^p - \pi^{m_2''(p-1)} \tilde{T}_2 &= \tilde{f}_1(T_1) + \pi^{pm_2''-pm_2'+t_1} g_1 - \pi^{pm_2''} W(\pi^{-m_1'p} \tilde{T}_1^p, -\pi^{-m_1'} \tilde{T}_1) \\ &= \tilde{f}_1(T_1) + \pi^{pm_2''-pm_2'+t_1} g_1 + \sum_{k=1}^{p-1} \frac{\pi^{(m_2''-m_1')p-m_1'(p-1)k}}{k} \tilde{T}_1^{p+(p-1)k}\end{aligned}$$

if $p \neq 2$, resp.

$$\tilde{T}_2^2 - \pi^{m_2''} \tilde{T}_2 = \tilde{f}_1(T_1) + \pi^{2m_2''-2m_2'+t_1} g_1 + \pi^{2m_2''} \left(\frac{\tilde{T}_1^2}{\pi^{2m_1'}} - \frac{\tilde{T}_1^3}{\pi^{3m_1'}} \right),$$

if $p = 2$. Its special fiber is the α_p-torsor given by the equation

$$\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - \cdots - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1},$$

and we are in the case c-2). If $m_1'(p(p-1)+1) > \sup(pm_2' - t_1, pm_2' - (p-1)m_1')$, then $m_1'(p(p-1)+1)$ is divisible by p, f_2 is a torsor under the group scheme $\mathcal{M}_{(m_1'(p(p-1)+1))/p,R}$. Its special fiber is the α_p-torsor given by the equation

$$\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1},$$

and we are in the case c-4). If $pm_2' - t_1 > \sup(m_1'(p(p-1)+1), pm_2' - (p-1)m_1')$, and the image $\bar{g}_1$ of g_1 in $\mathcal{O}(Y_{1,k})$ is not a p-power, then $pm_2' - t_1$ is divisible by p, $pm_2' - t_1 =: pm_2''$ is divisible by p, and f_2 is a torsor under the group scheme $\mathcal{M}_{m_2'',R}$; where $m_2'' := (pm_2' - t_1)/p$, defined by the equation

$$\tilde{T}_2^p - \pi^{m_2''(p-1)}\tilde{T}_2 = \pi^{\tilde{m}_2}\tilde{f}_1(T_1) + g_1 - \pi^{pm_2''}W(\pi^{-m_1'p}\tilde{T}_1^p, -\pi^{-m_1'}\tilde{T}_1)$$
$$= \pi^{\tilde{m}_2}\tilde{f}_1(T_1) + g_1 + \sum_{k=1}^{p-1}\frac{\pi^{(m_2''-m_1')p-m_1'(p-1)k}}{k}\tilde{T}_1^{p+(p-1)k}$$

if $p \neq 2$, resp.

$$\tilde{T}_2^2 - \pi^{m_2''}\tilde{T}_2 = \pi^{\tilde{m}_2}\tilde{f}_1(T_1) + g_1 + \pi^{2m_2''}\left(\frac{\tilde{T}_1^2}{\pi^{2m_1'}} - \frac{\tilde{T}_1^3}{\pi^{3m_1'}}\right),$$

if $p = 2$; where $\tilde{m}_2 := pm_2'' - pm_2' + (p-1)m_1'$. Its special fiber is the α_p-torsor given by the equation

$$\tilde{t}_2^p = \bar{g}_1,$$

and we are in the case c-3).

Finally, in the general case, we repeat the same argument as above if in the first case the image $\bar{\tilde{a}}_2$, of $\tilde{a}_2$ modulo π, is a p-power. Or, if in the second case $pm_2' - t_1 > \sup(m_1'(p(p-1)+1), pm_2' - (p-1)m_1')$, and the image $\bar{g}_1$ of g_1 in $\mathcal{O}(Y_{1,k})$ is a p-power. As the π-exponent of the denominators in the equation defining $f_{2,K}$ decreases at each step, we conclude that this process must stop after finitely many steps, and we end up with an equation as claimed in the statement c). The rest of the conclusion follows then easily.

3.3.4. Remark. Assume that we are in the case c-3) of 3.3.3, that $t_1 = \cdots = t_r = 0$, and $f_1 = \cdots = f_r = 0$. Then f is a torsor under the R-group scheme $\mathcal{H}_{m_1', m_2'}$ given by the equations

$$\tilde{T}_1^p - \pi^{m_1'(p-1)}\tilde{T}_1 = a_1$$

and

$$\tilde{T}_2^p - \pi^{m_2'(p-1)}\tilde{T}_2 = g - \pi^{pm_2'}W(\pi^{-m_1'p}\tilde{T}_1^p, -\pi^{-m_1'}\tilde{T}_1)$$
$$= g + \sum_{k=1}^{p-1}\frac{\pi^{(m_2'-m_1')p-m_1'(p-1)k}}{k}\tilde{T}_1^{p+(p-1)k},$$

if $p \neq 2$, resp.

$$\tilde{T}_2^2 - \pi^{m_2'}\tilde{T}_2 = g + \pi^{2m_2'}\left(\frac{\tilde{T}_1^2}{\pi^{2m_1'}} - \frac{\tilde{T}_1^3}{\pi^{3m_1'}}\right),$$

if $p = 2$. Its special fiber is the $(\mathcal{H}_{m_1', m_2'})_k \simeq H_k$-torsor given by the equations

$$\tilde{t}_1^p = \bar{a}_1$$

and

$$\tilde{t}_2^p = \bar{g}.$$

Next, we define the "*degeneration data*" arising from the reduction of an étale $\mathbf{Z}/p^2\mathbf{Z}$-torsor.

3.3.5. Definition. Let $f_K : Y_K \to X_K$ be an étale $\mathbf{Z}/p^2\mathbf{Z}$-torsor, with $X = \operatorname{Spf} A$ affine as in 3.3.3. Then we define the *degeneration type* of the torsor f_K as follows: f_K has a degeneration of type A, or of type $\{\text{etale}, \text{etale}\}$, if we are in the case a) of 3.3.3, a degeneration of type B, or of type $\{\text{etale}, \text{radicial}\}$, if we are in the case b) of 3.3.3 and a degeneration of type C, or of type $\{\text{radicial}, \text{radicial}\}$, if we are in the case c) of 3.3.3. Further, we define the *degeneration data* associated to a degeneration type as follows:

a) A degeneration data of type A consists of an element of $H^1(X_k, \mathbf{Z}/p^2\mathbf{Z})$.

b) A degeneration data of type B consists of an element of $H^1(X_k, G_k)$, where $G_k \simeq \mathbf{Z}/p\mathbf{Z} \times \alpha_p$, is defined in 3.2.

c) A degeneration data of type C consists of an element of $H^1(X_k, H_k) \oplus \Gamma(X_k, \mathcal{O}_{X_k})^{p-1}$, where $H_k \simeq \alpha_p \times \alpha_p$ is defined in 3.2.

A $\mathbf{Z}/p^2\mathbf{Z}$-torsor $f_K : Y_K \to X_K$ as above gives rise naturally, via 3.3.3, to a degeneration data as in 3.3.5. More precisely we have the following.

3.3.6. Proposition. *Assume that X is affine as in 3.3.3. Let $f_K : Y_K \to X_K$ be an étale $\mathbf{Z}/p^2\mathbf{Z}$-torsor which has a degeneration of type A (resp. B, or C). Then f_K induces canonically a degeneration data of type A (resp. of type B, or C).*

Proof. This is a direct consequence of 3.3.3. More precisely, let $f : Y \to X$ be the finite cover that we obtain in the proof of 3.3.3, and which extends the torsor f_K. If f_K has a degeneration of type A, then the special fiber f_k of f is an étale $\mathbf{Z}/p^2\mathbf{Z}$-torsor, and the assertion follows in this case. Assume that f_K has a degeneration of type B. Then the special fiber f_k of f is canonically a G_k-torsor, and the assertion follows in this case too. Finally, assume that the torsor f_K has a degeneration of type C. Then it follows from 3.3.3 that the special fiber f_k of the cover f is defined by the equations: $\tilde{t}_1^p = \bar{a}_1$ and $\tilde{t}_2^p = -\bar{c}_1^p\tilde{t}_1 - 2\bar{c}_2^p\tilde{t}_1^{p+1} - \cdots - (p-1)\bar{c}_{p-1}^p\tilde{t}_1^{p(p-2)+1} - \tilde{t}_1^{p(p-1)+1} + \bar{g}$ (resp. $\tilde{t}_1^p = \bar{a}_1$, and $\tilde{t}_2^p = -\bar{c}_1^p\tilde{t}_1 - 2\bar{c}_2^p\tilde{t}_1^{p+1} - \cdots - (p-1)\bar{c}_{p-1}^p\tilde{t}_1^{p(p-2)+1} + \bar{g}$, or $\tilde{t}_1^p = \bar{a}_1$, and $\tilde{t}_2^p = \bar{g}$) where $\bar{c}_1, \ldots, \bar{c}_{p-1}$ (resp. $\bar{g}$) are functions on X_k (eventually equal to 0) which are uniquely determined (resp. determined up to addition of element of the form $\bar{h}^p$, where $\bar{h}$ is a function on X_k). The pair $(\bar{a}_1, \bar{g})$ defines then canonically an element of $H^1_{\text{fppf}}(X_k, H_k)$, and the tuple $(\bar{c}_1, \ldots, \bar{c}_{p-1})$ an element of $\Gamma(X_k, \mathcal{O}_{X_k})^{p-1}$. Thus we get canonically, in this case, an element of $H^1(X_k, H_k) \oplus \Gamma(X_k, \mathcal{O}_{X_k})^{p-1}$ associated to f_K.

3.3.7. It follows from 3.3.6 that an étale $\mathbf{Z}/p^2\mathbf{Z}$-torsor above the generic fiber X_K of X induces canonically a *degeneration data* of type either A, B, or C. Reciprocally, we have the following result of *lifting* of such a degeneration data.

3.3.8. PROPOSITION. *Assume given a degeneration data, say $\mathscr{D}$, of type either A, B or C, as in* 3.3.5. *Then there exists a $\mathbf{Z}/p^2\mathbf{Z}$-torsor $f_K : Y_K \to X_K$ such that the degeneration data associated to f_K, via* 3.3.6, *equals $\mathscr{D}$.*

PROOF. The proof in the case where the degeneration data is of type A, or B, is similar to the proof in 2.2.3, and is left to the reader. Assume that the degeneration data is of type C, and consists of the pair $(\bar{a}_1, \bar{a}_2)$, where $\bar{a}_1$, and $\bar{a}_2$, are functions on X_k which are not p-powers, and the tuple of functions $(\bar{c}_1, \dots, \bar{c}_{p-1})$. Let a_1, and a_2 (resp. $c_1, \dots, c_{p-1}$) be regular functions on X which lift $\bar{a}_1$ and $\bar{a}_2$ (resp. which lifts $\bar{c}_1, \dots, \bar{c}_{p-1}$). Let $n = pn' = p^2 n'' > 0$ be an integer. Consider the $\mathbf{Z}/p^2\mathbf{Z}$-torsor $f_K : Y_K \to X_K$ given by the equations: $(T_1^p, T_2^p) - (T_1, T_2) = (a_1 \pi^{-n'p}, f(a_1)\pi^{-pm} + a_2 \pi^{-pm + n'(p-1)})$, where $f(a_1) = c_1 a_1 + \cdots + c_{p-1} a_1^{p-1}$, and $m = n'p$. Then it follows easily from the proof of 3.3.3 that the degeneration data associated to f_K, via 3.3.6, equals $\mathscr{D}$. Moreover, in this case we have $\delta_1 = n'(p-1)$, and $\delta_2 = n''(p(p-1)+1) \cdot (p-1)$. We have also the following possibility for such a lifting. Namely, consider the $\mathbf{Z}/p^2\mathbf{Z}$-torsor given by the equations: $(T_1^p, T_2^p) - (T_1, T_2) = (a_1 \pi^{-n'p}, f(a_1)\pi^{-pm} + g\pi^{-pm+n'(p-1)})$, where m is a positive integer such that $mp - n'(p-1) > n'(p(p-1)+1)$, and $mp - n'(p-1) = pm'$. In this latter case we have $\delta_1 = n'(p-1)$, and $\delta_2 = m'(p-1)$.

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