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A complete characterization of local martingales which are functions of Brownian motion and its maximum

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We prove the max-martingale conjecture of Obłój and Yor. We show that for a continuous local martingale $(N_t : t \geqslant 0)$ and a function $H : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, $H(N_t, \sup_{s \leqslant t} N_s)$ is a local martingale if and only if there exists a locally integrable function f such that $H(x, y) = \int_0^y f(s)\mathrm{d}s - f(y)(x - y) + H(0, 0)$. This readily implies, via Lévy's equivalence theorem, an analogous result with the maximum process replaced by the local time at 0.

Keywords: Azéma–Yor martingales; continuous martingales; maximum process; max-martingales; Motoo's theorem

1. Introduction

In recent work with Marc Yor (Obłój and Yor 2006), we argued for the importance of a class of local martingales which are functions of a continuous local martingale and its one-sided maximum process. We called them *max-martingales* or simply *M-martingales*. Such processes were first introduced by Azéma and Yor (1979), who described a family of such martingales, often referred to as *Azéma–Yor martingales*. Obłój and Yor (2006) gave a complete description of this family. These martingales have a remarkably simple form, yet they proved to be a useful tool in various problems. In the same paper Obłój and Yor (2006) assembled applications including the Skorokhod embedding problem (cf. Obłój 2004), a simple proof of Doob's maximal and L^p- inequalities, as well as a derivation of bounds on the possible laws of the maximum or the local time at 0 of a continuous, uniformly integrable martingale, and links with Brownian penalization problems (cf. Roynette *et al.* 2006).

In this paper, we obtain a complete characterization of max-martingales, which was conjectured by Obłój and Yor (2006). Let (N_t) be a continuous local martingale and $\overline{N}_t = \sup_{s \leqslant t} N_s$. We show that the process $H(N_t, \overline{N}_t)$ is a local martingale if and only if there exists a locally integrable function f such that $H(x, y) = H(0, 0) + \int_0^y f(s)\mathrm{d}s - f(y)(y - x)$. Put differently, we show that Azéma–Yor martingales are the only max-martingales. We will thus use both terms interchangeably. We note that this result, under the additional assumption that $H \in C^2$, can be obtained via Itô's formula and has been known for a long time. The general case, however, has remained an open problem (cf. Revuz and Yor 1999: 279).

Let us give some motivation, apart from pure mathematical curiosity, as to why the question of fully characterizing local martingales of the form $H(N_t, \overline{N}_t)$ is interesting. Our first observation is that any such martingale is basically equivalent to an equality $\mathbb{E}H(B_T, \overline{B}_T) = H(0, 0)$ for stopping times T, where (B_t) is a Brownian motion. A prime example of such equality is Doob's maximal equality: $\lambda\mathbb{P}(\overline{B}_T \geqslant \lambda) = \mathbb{E}B_T\mathbf{1}_{\overline{B}_T\geqslant\lambda}$. Thus, looking for a particular kind of equalities is the same as seeking a max-martingale of a particular form.

Our second observation is that max-martingales appear naturally in penalizations of the Wiener measure. We describe this briefly here and refer to the original work of Roynette *et al.* (2006) for details. Solely for the purpose of this example we use the canonical space and coordinate process notation. Let (X_t) be the coordinate process on the space of continuous functions $\Omega = C(\mathbb{R}_+, \mathbb{R})$ which is a Brownian motion under the Wiener measure $\mathbb{W}$. Denote its natural filtration by $(\mathcal{F}_t)$. For f a probability density on $\mathbb{R}_+$, we define a family of measures $(\mathbb{W}_t^f : t \geqslant 0)$ via

$$\mathbb{W}_t^f = \frac{f(\overline{X}_t)}{\mathbb{E}_{\mathbb{W}} f(\overline{X}_t)} \cdot \mathbb{W}$$

and ask if $\mathbb{W}_t^f$ converges to $\mathbb{W}_\infty^f$, as $t \to \infty$, in the sense that for $\Gamma_s \in \mathcal{F}_s$, $\mathbb{W}_t^f(\Gamma_s) \to \mathbb{W}_\infty^f(\Gamma_s)$. This is equivalent to the following question: does

$$M_s = \lim_{t\to\infty} \mathbb{E}_{\mathbb{W}}\left[\frac{f(\overline{X}_t)}{\mathbb{E}_{\mathbb{W}} f(\overline{X}_t)}\middle|\mathcal{F}_s\right]$$

exist and define a martingale? Using the Markov property of $(X_t, \overline{X}_t)$, we can see that if M_s exists, it has to be of the form $M_s = H(s, X_s, \overline{X}_s)$ and a further argument shows that actually we have to have $M_s = H(X_s, \overline{X}_s)$, thus a max-martingale. Roynette *et al.* (2006) show that $\mathbb{W}_\infty^f$ and M_s do indeed exist and the latter is an Azéma–Yor martingale associated with f.

The rest of the paper is organized as follows. Section 2 contains our main theorem, its corollaries and a complementary result. All proofs are gathered in the subsequent Section 3. Section 4 contains some arguments based on the optional stopping theorem which are very different from the arguments used in the proofs in Section 3, and hopefully will give the reader some additional insight.

2. Main results

Throughout, $N = (N_t : t \geqslant 0)$ denotes a continuous local martingale with $N_0 = 0$ and $\langle N\rangle_\infty = \infty$ almost surely. Extensions of our results to local martingales with arbitrary N_0 are immediate. The maximum and minimum processes are denoted respectively by $\overline{N}_t = \sup_{s\leqslant t} N_s$ and $\underline{N}_t = -\inf_{s\leqslant t} N_s$. Note that $\underline{N}_t = \sup_{s\leqslant t}(-N_s)$ and thus all our results about the maximum translate into results about the minimum by simply considering $-N$ instead of N. The local time at zero of N is denoted by L_t^N. Filtrations considered are always taken to be complete and right-continuous. $B = (B_t : t \geqslant 0)$ denotes a real-valued Brownian motion. The following theorem is the main result of this paper.

Theorem 1. *Let* $\mathbf{D} = \{(x, y) \in \mathbb{R}^2 : y \geqslant x \vee 0\}$, $H : \mathbf{D} \to \mathbb{R}$ *be a Borel function, and* $N = (N_t : t \geqslant 0)$ *be a continuous local martingale with* $N_0 = 0$ *and* $\langle N \rangle_\infty = \infty$ *a.s. Then* $(H(N_t, \overline{N}_t) : t \geqslant 0)$ *is a right-continuous local martingale, in the natural filtration of N, if and only if there exists* $f : \mathbb{R}_+ \to \mathbb{R}$*, a locally integrable function, such that a.s., for all* $t \geqslant 0$,

$$H(N_t, \overline{N}_t) = F(\overline{N}_t) - f(\overline{N}_t)(\overline{N}_t - N_t) + H(0, 0) \tag{1}$$

$$= \int_0^t f(\overline{N}_s) \mathrm{d}N_s + H(0, 0), \tag{2}$$

where $F(y) = \int_0^\infty f(x)\mathrm{d}x$.

Furthermore, if $f \geqslant 0$ *and* $\int_0^\infty f(x)\mathrm{d}x < \infty$, *then* $(H(N_t, \overline{N}_t) : t \geqslant 0)$ *given in* (1) *converges a.s., as* $t \to \infty$, *to* $F(\infty) + H(0, 0)$. *If, moreover,* $(N_t : t \geqslant 0)$ *is a martingale with* $\mathbb{E}\sup_{s \leqslant t}|N_s| < \infty$, $t > 0$, *then the local martingale in* (1) *is a martingale.*

Note that, in particular, if $H(N_t, \overline{N}_t)_{t \geqslant 0}$ is a right-continuous local martingale, then it is in fact a continuous local martingale and $H(\cdot, y)$ is a linear function for almost all $y > 0$. We also can specify the maximum process of this local martingale as $\sup_{s \leqslant t} H(N_s, \overline{N}_s) = F(\overline{N}_t) + H(0, 0)$, when $f \geqslant 0$. Indeed, from (1) it is clear that the left-hand side is less than or equal to the right, and the right-hand side is attained by considering the last time the maximum of N is reached before time t. Note that the formula can also be obtained with Skorokhod's lemma (cf. Revuz and Yor 1999: Exercise VI.4.24) and that it is no longer true for arbitrary f.

The martingale property stated in the theorem was observed by Roynette *et al.* (2006). We stress here that the local martingales we obtain have some interesting properties. If $f \geqslant 0$ and $F(\infty) < \infty$ then the process $M_t^f = F(\overline{N}_t) - f(\overline{N}_t)(\overline{N}_t - N_t)$ provides an example of a local martingale which converges a.s. to its maximum:

$$M_t^f \underset{t \to \infty}{\longrightarrow} F(\infty) = \overline{M^f}_\infty.$$

Equivalently, $f(\overline{N}_t)(\overline{N}_t - N_t)$ is a local submartingale, zero at zero, which converges a.s. to zero as $t \to \infty$.

Theorem 1 tells us that a local martingale $H(N_t, \overline{N}_t)$ is entirely characterized by its initial value $H(0, 0)$ and by a locally integrable function f such that (1) holds. We point out that we can recover this function from the process $H(N_t, \overline{N}_t)$ itself. Indeed, from (2) we have that $\langle N, H(N, \overline{N}) \rangle_t = \int_0^t f(\overline{N}_s)\mathrm{d}\langle N \rangle_s$. Thus the measure $\mathrm{d}\langle N, H(N, \overline{N}) \rangle_t$ is absolutely continuous with respect to $\mathrm{d}\langle N \rangle_t$, and the density is given by $f(\overline{N}_t)$. This yields

$$f(x) = \left. \frac{\mathrm{d}\langle N, H(N, \overline{N}) \rangle_t}{\mathrm{d}\langle N \rangle_t} \right|_{t = T_x}, \tag{3}$$

where $T_x = \inf\{t : N_t = x\}$, $x > 0$. It is easier to obtain $F(x)$. Indeed, as $N_{T_x} = \overline{N}_{T_x} = x$, from (1) we have $F(x) = H(\mathrm{N}_{T_x}, \overline{N}_{T_x}) - H(0, 0) = H(x, x) - H(0, 0)$. The function f can then be obtained as the derivative of F.

As indicated at the beginning, replacing N with $-N$ gives an immediate analogue of Theorem 1 with the maximum $\overline{N}_t$ replaced by the minimum $\underline{N}_t$. More precisely, $H(N_t, \underline{N}_t)_{t\geqslant 0}$ is a right-continuous local martingale if and only if there exists a locally integrable function f such that $H(N_t, \underline{N}_t) = F(\underline{N}_t) - f(\underline{N}_t)(\underline{N}_t + N_t) + H(0,0)$ a.s.

Thanks to Lévy's equivalence theorem, we can rephrase Theorem 1 also in terms of the local time at zero instead of the maximum process:[1]

Theorem 2. *Let* $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *be a Borel function and* $N = (N_t : t \geqslant 0)$ *be a continuous local martingale with* $N_0 = 0$ *and* $\langle N \rangle_\infty = \infty$ *a.s. Then* $(H(|N_t|, L_t^N) : t \geqslant 0)$ *is a right-continuous local martingale, in the natural filtration of N, if and only if there exists* $g : \mathbb{R}_+ \to \mathbb{R}$ *a locally integrable function such that a.s., for all* $t \geqslant 0$,

$$H(|N_t|, L_t^N) = G(L_t^N) - g(L_t^N)|N_t| + H(0, 0) \tag{4}$$

$$= -\int_0^t g(L_s^N)\mathrm{sgn}(N_s)\mathrm{d}N_s + H(0, 0), \tag{5}$$

where $G(y) = \int_0^y g(x)\mathrm{d}x$.

Furthermore, if $g \geqslant 0$ *and* $\int_0^\infty g(x)\mathrm{d}x < \infty$, *then* $(H(|N_t|, L_t^N) : t \geqslant 0)$ *given in* (1) *converges a.s., as* $t \to \infty$, *to* $G(\infty) + H(0, 0)$*. If, moreover,* $(N_t : t \geqslant 0)$ *is a martingale with* $\mathbb{E}\sup_{s\leqslant t}|N_s| < \infty$, $t > 0$, *then the local martingale in* (1) *is a martingale.*

Note that if $H(|N_t|, L_t^N)$ is a continuous local martingale then we can recover from it, in a similar manner to (3), the function g such that (4) holds.

Theorem 1 allows us also to consider local martingales of the form $H(N_t^+, \overline{N}_t)$, where H is a Borel function. Indeed, $H(N_t^+, \overline{N}_t)$ can be written as $G(N_t, \overline{N}_t)$ with $G(x, y) = H(x \vee 0, y)$, and we can then apply Theorem 1. This yields the following theorem.

Theorem 3. *Let* $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *be a Borel function. Let* $(N_t : t \geqslant 0)$ *be a continuous local martingale with* $N_0 = 0$ *and* $\langle N \rangle_\infty = \infty$ *a.s. and put* $N_t^+ = \max\{N_t, 0\}$, $N_t^- = \max\{-N_t, 0\}$ *and* $N_t^* = \sup_{s\leqslant t}|N_s|$*. Then the processes* $(H(N_t^+, \overline{N}_t) : t \geqslant 0)$, $(H(N_t^-, \underline{N}_t) : t \geqslant 0)$ *and* $(H(|N_t|, N_t^*) : t \geqslant 0)$ *are right-continuous local martingales, in the natural filtration of N, if and only if they are a.s. constant.*

Note that it is not true that all martingales in the natural filtration of N^+ are also martingales in the natural filtration of N. In fact, the former also admits discontinuous martingales, unlike the latter. From the proof it will be clear that the theorem remains true if we replace $N_t^+ = \max\{N_t, 0\}$ with some more complicated, appropriate function of N_t

[1] After submitting this paper we realized that by using techniques similar to those in the proof of Theorem 1 we can also characterize the class of local martingales of the form $H(N_t, L_t)$ which is larger than that described in Theorem 2. This will be included in our next work on the subject.

and work with local martingales in the natural filtration of N. For example we can easily see that if A is an interval and $(-\infty, 0) \not\subseteq A$, then $(H(N_t \mathbf{1}_{N_t \in A}, \overline{N}_t) : t \geqslant 0)$, is a right-continuous local martingale, in the natural filtration of N, if and only if it is constant a.s.

Similar arguments can be naturally developed for the local time L^N in place of the maximum $\overline{N}$.

Finally, we present a deterministic description of Hs such that (1) holds. This is very close to studying the fine topology for the process $(B_t, \overline{B}_t)$.

Proposition 4. *In the set-up of Theorem 1,* (1) *holds if and only if there exists a Borel set* $\Gamma \subset \mathbf{D}$*, such that*

$$H(x, y) = F(y) - f(y)(y - x) + H(0, 0), \qquad \forall (x, y) \in \mathbf{D} \backslash \Gamma, \tag{6}$$

and $\Gamma_2 = \{y : \exists x, (x, y) \in \Gamma\}$ *has Lebesgue measure zero, and* $\{(y, y) : y \geqslant 0\} \cap \Gamma = \varnothing$.

An analoguous result for the function G satisfying (4) follows.

3. Proofs

We now present proofs of the results given in Section 2. Whenever possible we split the proof into steps or remarks and indicate clearly what is being proved. We start with a simplifying technical remark.

Remark 1. It suffices to prove Theorems 1, 2, 3 and Proposition 4 for $N = B$, a standard real-valued Brownian motion, as then by virtue of the Dambis–Dubins–Schwarz theorem (Revuz and Yor 1999: 181) it extends to any continuous local martingale N with $N_0 = 0$ and $\langle N \rangle_\infty = \infty$.

Proof of Remark 1. We know that if τ_u is the right-continuous inverse of $\langle N \rangle_t$ then the process $\beta_u = N_{\tau_u}$ is a Brownian motion and $N_t = \beta_{\langle N \rangle_t}$. It follows that $\overline{N}_t = \overline{\beta}_{\langle N \rangle_t}$.

Denote by $(\mathcal{F}^N_t)$ the natural filtration of N and by $\mathcal{G}_u = \mathcal{F}^N_{\tau_u}$, $\mathcal{F}^\beta_u = \sigma(\beta_s : s \leqslant u)$ two filtrations with respect to which (β_u) is a Brownian motion. Naturally $\mathcal{F}^\beta_u \subset \mathcal{G}_u$ but in fact the smaller filtration is immersed in the larger, meaning that all $(\mathcal{F}^\beta_u)$-local martingales are also $(\mathcal{G}_u)$-local martingales. This follows readily from the representation of $(\mathcal{F}^\beta_u)$-local martingales as stochastic integrals with respect to β and thus $(\mathcal{G}_u)$-local martingales (Yor 1979). This entails that $H(\beta_u, \overline{\beta}_u)$, which is $(\mathcal{F}^\beta_u)$-measurable, is a $(\mathcal{G}_u)$-local martingale if and only if it is also an $(\mathcal{F}^\beta_u)$-local martingale.

Thus if $H(\beta_u, \overline{\beta}_u)$ is an $(\mathcal{F}^\beta_u)$-local martingale, it is a $(\mathcal{G}_u)$-local martingale and therefore its time-changed version $H(\beta_{\langle N \rangle_t}, \overline{\beta}_{\langle N \rangle_t}) = H(N_t, \overline{N}_t)$ is an $(\mathcal{F}^N_t)$-local martingale (note that the time change is continuous). Conversely, as N is constant on the jumps of (τ_u), we have $\overline{\beta}_u = \overline{N}_{\tau_u}$ and if $H(N_t, \overline{N}_t)$ is an $(\mathcal{F}^N_t)$-local martingale, then its time-changed version $H(\beta_u, \overline{\beta}_u)$ is a $(\mathcal{G}_u)$-local martingale (cf. Revuz and Yor 1999: Proposition V.1.5) and thus an $(\mathcal{F}^\beta_u)$-local martingale. Analogous arguments apply to local martingales considered in Theorem 3.

Likewise, as $L_t^N = \lim_{\epsilon\to 0}(2\epsilon)^{-1}\int_0^t \mathbf{1}_{|N_s|\leqslant\epsilon}\mathrm{d}\langle N\rangle_s$ (Revuz and Yor 1999: 227), it is easy to see that $L^{\beta}_{\langle N\rangle_t} = L_t^N$, $L_u^{\beta} = L_{\tau_u}^N$ and that Theorem 2 holds for an arbitrary continuous local martingale N, with $N_0 = 0$ and $\langle N\rangle_\infty = \infty$ a.s., if and only if it holds for Brownian motion. □

In view of Remark 1, in proving Theorems 2 and 3 it suffices to consider $N = B$.

Proof of Theorem 2. This follows from Theorem 1 with Lévy's equivalence theorem, which asserts that the processes $((B_t, \overline{B}_t) : t \geqslant 0)$ and $((L_t^B - |B_t|, L_t^B) : t \geqslant 0)$ have the same distribution. □

Proof of Theorem 3. Suppose that $(H(B_t^+, \overline{B}_t) : t \geqslant 0)$ is a right-continuous local martingale in the natural filtration of B. Then $(G(B_t, \overline{B}_t) : t \geqslant 0)$ is also a right-continuous local martingale, where $G(x, y) = H(x \vee 0, y)$. By Theorem 1, there exists a locally integrable function f such that $G(x, y) = F(y) - f(y)(y - x) + G(0, 0)$ almost everywhere, where $F(y) = \int_0^y f(x)\mathrm{d}x$. However, for any fixed $y \in \mathbb{R}_+$, G is constant for $x < 0$ which means that $f(x) = 0$ a.e. (cf. the proof of Proposition 4 below) and thus $H(B_t^+, \overline{B}_t) = H(0, 0)$ a.s. An analogous result for $H(B_t^-, \underline{B}_t)$ follows.

Finally, we prove that local martingales of the form $H(|B_t|, B_t^*)$ are constant. To this end, let $A_t^+ = \int_0^t \mathbf{1}_{B_s\geqslant 0}\,\mathrm{d}s$ and α_u^+ be its right-continuous inverse. Then $W_u = B^+_{\alpha_u^+}$ is a reflected Brownian motion in the filtration $\mathcal{G}_u = \mathcal{F}_{\alpha_u^+}$, where $(\mathcal{F}_t)$ is the natural filtration of B, and $\overline{W}_u = \overline{B}_{\alpha_u^+}$. If we write $(\mathcal{F}_u^W)$ for the natural filtration of W, then $H(W_u, \overline{W}_u)$ is a $(\mathcal{G}_u)$-local martingale if and only if it is also an $(\mathcal{F}_u^W)$-local martingale. This follows from our discussion above and the fact that W can be written as $W_u = \beta_u + L_u^{\beta}$, where β_u is a $(\mathcal{G}_u)$-Brownian motion and the natural filtrations of W and β are equal (Yor 1977).

Suppose now that $H(W_u, \overline{W}_u)$ is an $(\mathcal{F}_u^W)$ right-continuous local martingale and thus a $(\mathcal{G}_u)$ right-continuous local martingale. As the time change A_t^+ is continuous, the time-changed version $H(W_{A_t^+}, \overline{W}_{A_t^+}) = H(B_t^+, \overline{B}_t)$ is an $(\mathcal{F}_t)$ right-continuous local martingale and thus is constant a.s. This completes the proof of Theorem 3. □

Proof of Proposition 4. As the law of $(B_t, \overline{B}_t)$ is equivalent to the Lebesgue measure on $\mathbf{D}$, it is clear that (1) implies that (6) holds for $(x, y) \in \mathbf{D}\backslash\Gamma$, for some Borel set Γ of two-dimensional Lebesgue measure zero. However, we know also that $H(B_t, \overline{B}_t)$ is a.s. continuous and this yields more constraints on Γ. Conversely, if (6) holds for a two-dimensional Lebesgue null set Γ, small enough so that $H(B_t, \overline{B}_t)$ is a.s. continuous, then (1) holds.

For a set $\Gamma \subset \mathbf{D}$ define $\Gamma_2 := \{y : \exists x, (x, y) \in \Gamma\}$ and, for $y \in \Gamma_2$, $x_y := \sup\{x : (x, y) \in \Gamma\}$. Let $\Gamma_2^+ = \{y \in \Gamma_2 : x_y < y\}$, $\Gamma_2^- = \Gamma_2\backslash\Gamma_2^+$ and $\Gamma^+ = \{(x, y) \in \Gamma : y \in \Gamma_2^+\}$, $\Gamma^- = \Gamma\backslash\Gamma^+$.

First of all, note that upon stopping at $T_y = \inf\{t \geqslant 0 : B_t = y\}$ we have that $H(y, y) = F(y) + H(0, 0)$ which means that Γ cannot contain points from the diagonal in $\mathbb{R}_+^2$.

Let $\mathcal{R}$ be the range of the process $(B_t, \overline{B}_t)$, $\mathcal{R}(\omega) = \{(B_t(\omega), \overline{B}_t(\omega)) : t \geqslant 0\}$. The restriction we have to impose on Γ is that $\mathbb{P}(\mathcal{R} \cap \Gamma = \varnothing) = 1$. Notice, however, due to the continuity of sample paths of B, that if $(x, y) \in \mathcal{R} \cap \Gamma$ then $(x_y, y) \in \mathcal{R}$. With Lévy's equivalence theorem we know that the process $\gamma_t = (\overline{B}_t - B_t)$ is a reflected standard Brownian motion and the stretches $[x, y] \times \{y\}$ in $\mathcal{R}$ correspond to its excursions, which form a Poisson point process on the time scale given by $\overline{B}_t$. Thus the process of extremal values of these excursions, $(\bar{e}_y : y \geqslant 0) = (\sup_{T_y \leqslant s \leqslant T_{y+}}(y - B_s) : y \geqslant 0)$, is also a Poisson point process and its intensity measure is $\mathrm{d}v/v^2$ (Revuz and Yor 1999: Exercise XII.2.10). Then we have

$$\mathbb{P}(\mathcal{R} \cap \Gamma^+ = \varnothing) = \exp\left(-\int_{\Gamma_2^+} \frac{\mathrm{d}y}{y - x_y}\right),$$

which is equal to one if and only if $|\Gamma_2^+| = 0$, where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

The probability $\mathbb{P}(\mathcal{R} \cap \Gamma^- = \varnothing)$ is just the probability that the Poisson point process $(\bar{e}_y : y \geqslant 0)$ has no jumps for $y \in \Gamma_2^-$, and this probability is zero if and only if $|\Gamma_2^-| = 0$. Indeed if, for $A \subset \mathbb{R}_+^2$, we denote by $N(A)$ the cardinality of $\{y : (y, \bar{e}_y) \in A\}$ then we have $\mathbb{P}(\mathcal{R} \cap \Gamma^- = \varnothing) = \mathbb{P}(N(\Gamma_2^- \times (0, \infty)) = 0)$ and

$$\mathbb{P}(N(\Gamma_2^- \times (0, \infty)) > 0) = \lim_{h \searrow 0} \mathbb{P}(N(\Gamma_2^- \times [h, \infty)) > 0) = \lim_{h \searrow 0} |\Gamma_2^-|/h.$$

The limit is zero if and only if $|\Gamma_2^-| = 0$, which justifies our claim.[2]

As $\mathbb{P}(\mathcal{R} \cap \Gamma = \varnothing) = \mathbb{P}(\mathcal{R} \cap \Gamma^+ = \varnothing) + \mathbb{P}(\mathcal{R} \cap \Gamma^- = \varnothing)$ we need to impose on Γ the requirement that $|\Gamma_2| = 0$. This ends the proof of Proposition 4. □

Proof of Theorem 1. The rest of this section is devoted to the proof of Theorem 1 for Brownian motion. The proof is divided into two parts. In Part 1 we show that if $f : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function and H is given by (6) then $(H(B_t, \overline{B}_t) : t \geqslant 0)$ is a local martingale, and (2) holds; this was proved by Obłój and Yor (2006) but we include it here for the sake of completeness. In Part 2 we show the converse.

Part 1. Suppose $f \in C^1$ and H is given through (6), so that (1) holds by Proposition 4. We can apply Itô's formula to obtain:

$$\begin{aligned} H(B_t, \overline{B}_t) &= H(0, 0) + \int_0^t f(\overline{B}_s)\mathrm{d}B_s + \int_0^t f'(\overline{B}_s)(\overline{B}_s - B_s)\mathrm{d}\overline{B}_s \\ &= H(0, 0) + \int_0^t f(\overline{B}_s)\mathrm{d}B_s, \end{aligned}$$

since $\mathrm{d}\overline{B}_s$-a.e. $B_s = \overline{B}_s$. We have thus established formula (2) for f of class C^1. Thus if we can show that the quantities given in (1) and (2) are well defined and finite for any locally integrable f on $[0, \infty)$, then the formula (1)–(2) extends to such functions through the monotone class theorem. In particular, we see that $(H(B_t, \overline{B}_t) : t \geqslant 0)$, for H given by (6), is a local martingale, as it is a stochastic integral with respect to Brownian motion. For f a locally integrable function, $F(x)$ is well defined and finite, so all we need to show is that

[2] We wish to thank Victor Rivero for his helpful remarks.

$\int_0^t f(\overline{B}_s)\mathrm{d}B_s$ is well defined and finite a.s. for all $t > 0$. The proof of Part 1 thus concludes with the following crucial remark:

Remark 2. The stochastic integral $\int_0^t f(\overline{B}_s)\mathrm{d}B_s$ is well defined and finite a.s. for all $t > 0$ if and only if f is a locally integrable function.

This is equivalent to asking when $\int_0^t (f(\overline{B}_s))^2\mathrm{d}s < \infty$ a.s., for all $t > 0$, and we now show that it is necessary and sufficient to impose local integrability of f.

Write $\tau_x = \inf\{t \geqslant 0 : B_t > x\}$ for the first hitting time of (x, ∞), which is a well defined, a.s. finite, stopping time. The integrals in question are finite, $\int_0^t (f(\overline{B}_s))^2\mathrm{d}s < \infty$ a.s., for all $t > 0$, if and only if, for all $x > 0$, $\int_0^{\tau_x}(f(\overline{B}_s))^2\mathrm{d}s < \infty$. However, the last integral can be rewritten as

$$\int_0^{\tau_x} \mathrm{d}s(f(\overline{B}_s))^2 = \sum_{0\leqslant u\leqslant x} \int_{\tau_{u-}}^{\tau_u} \mathrm{d}s(f(\overline{B}_s))^2$$

$$= \sum_{0\leqslant u\leqslant x} f^2(u)(\tau_u - \tau_{u-}) = \int_0^x f^2(u)\mathrm{d}\tau_u. \tag{7}$$

Now it suffices to note that[3]

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^x f^2(u)\mathrm{d}\tau_u\right)\right] = \exp\left(-\int_0^x |f(u)|\mathrm{d}u\right), \tag{8}$$

to see that the last integral in (7) is finite if and only if $\int_0^x |f(u)|\mathrm{d}u < \infty$, which is precisely our hypothesis on f. Finally, note that the function H given by (6) is locally integrable as both $x \to f(x)$ and $x \to xf(x)$ are locally integrable.

Part 2. In this part we show the converse to the first part. That is, we show that if $H : \mathbf{D} \to \mathbb{R}$ is a Borel function such that $(H(B_t, \overline{B}_t) : t \geqslant 0)$ is a right-continuous local martingale, then there exists a locally integrable function $f : \mathbb{R}_+ \to \mathbb{R}$ such that (1) holds. Equation (2) then holds by Part 1 of the proof above, and H is described by Proposition 4. Let $T_r = \inf\{t \geqslant 0 : B_t = r\}$. We start with a useful lemma.

Lemma 5. *Let $r > 0$ and $K : (-\infty, r] \to \mathbb{R}$ be a Borel function, such that $(K(B_{t\wedge T_r}) : t \geqslant 0)$ is a right-continuous local martingale. Then there exists a constant α such that $K(x) = \alpha x + K(0)$ for $x \in (-\infty, r]$ and $(K(B_{t\wedge T_r}) : t \geqslant 0)$ is a martingale.*

Proof of Lemma 5. This lemma essentially says that the scale functions for Brownian motion are the affine functions. This is a well-known fact; however, for the sake of completeness, we provide a short proof.

We know that a right-continuous local martingale has actually a.s. cadlag paths. Furthermore, as $K_t = K(B_{t\wedge T_r})$ is a local martingale with respect to the Brownian filtration generated by B, it actually has a continuous version. As the laws in the space of cadlag

[3]Recall that $(\tau_x : x \geqslant 0)$ is a $\frac{1}{2}$-stable subordinator. Equality (8) is easily established for simple functions and passage to the limit (Revuz and Yor 1999: 72, 107; Exercise III.4.5).

functions are determined by finite-dimensional projections, K_t is a.s. continuous, which implies that $K(\cdot)$ is continuous on $(-\infty, r]$.[4]

Let $T_{a,b} = \inf\{t \geqslant 0 : B_t \notin [a, b]\}$. Then, as K is bounded on compact sets, for any $0 < x < r$ the local martingale $K(B_{t\wedge T_{-1,x}\wedge T_r}) = K(B_{t\wedge T_{-1,x}})$ is bounded and hence it is a uniformly integrable martingale. Applying the optional stopping theorem, we obtain $\mathbb{E}K(B_{T_{-1,x}}) = K(0)$ and thus $K(x) = x(K(0) - K(-1)) + K(0)$. Similarly, for $x < 0$, we can apply the optional stopping theorem to see that $\mathbb{E}K(B_{T_{x,r/2}}) = K(0)$, which yields

$$K(x) = x\frac{2K(r/2) - 2K(0)}{r} + K(0).$$

Expressing $K(r/2)$ with the formula above, we conclude that K is an affine function on $(-\infty, r]$.[5] □

We now return to the proof of the theorem. We will show how it reduces to Lemma 5. With no loss of generality we can assume that $H(0, 0) = 0$. The proof is carried out in five steps:

1. For almost all y, the function $H(\cdot, y)$ is continuous.
2. For all $y > x \vee 0$ and suitable random variables ξ independent of Brownian motion β, $(H(x + \beta_{t\wedge R_{y-x}}, \xi) : t \geqslant 0)$ is a local martingale (where $R_u = \inf\{t : \beta_t = u\}$) in the filtration of β initially enlarged with ξ.
3. For almost all z, $z > y > x \vee 0$, actually $(H(x + \beta_{t\wedge R_{y-x}}, z) : t \geqslant 0)$ is a local martingale in the natural filtration of β.
4. Apply Lemma 5 to obtain (1).
5. Proof of the martingale property.

Step 1. As in the proof of Lemma 5, we can argue that $(H(B_t, \overline{B}_t) : t \geqslant 0)$ is a continuous local martingale. From the proof of Proposition 4, in particular from the discussion of the range of the process $(B, \overline{B})$, it follows that for almost all $z \geqslant 0$, $H(\cdot, z)$ is a continuous function on $(-\infty, z]$. As we wish to prove the almost sure representation given by (1) we know, by Proposition 4, that we can change H on any set of the form $\cup_{z\in A}(-\infty, z) \times \{z\}$ with A of zero Lebesgue measure, and so we can and will assume that $H(\cdot, z)$ is continuous on $(-\infty, z]$ for all $z \geqslant 0$.

Let $y > x \vee 0$ and $T_y = \inf\{t \geqslant 0 : B_t = y\}$, and $T_x^y = \inf\{t > T_y : B_t = x\}$, two a.s. finite stopping times. Denote $\xi = \overline{B}_{T_x^y}$, which is a random variable with an absolutely continuous distribution on $[y, \infty)$. We denote its density by ρ, $\mathbb{P}(\xi \in \mathrm{d}u) = \rho(u)\mathbf{1}_{u\geqslant y}\mathrm{d}u$. We will need this notation in due course. Note that we could also derive the continuity properties of H by analysing the behaviour of $H(B_t, \overline{B}_t)$ for t between the last visit to y before T_x^y and T_x^y.

Step 2. Using the representation theorem for Brownian martingales, we know that there exists a predictable process $(h_s : s \geqslant 0)$ such that $H(B_t, \overline{B}_t) = \int_0^t h_s\,\mathrm{d}B_s$ a.s. Let

[4] We could also just say that the right continuity of $K(B_t)$ implies fine continuity of K (Blumenthal and Getoor 1968: Theorem II.4.8) and the fine topology for real-valued Brownian motion is the ordinary topology.
[5] We thank Goran Peskir and Dmitry Kramkov for their remarks, which simplified our earlier proof of the lemma.

$(\theta_t : t \geqslant 0)$ be the standard shift operator for the two-dimensional Markov process $((B_t, \overline{B}_t) : t \geqslant 0)$. Obviously, for $t, s > 0$, we have

$$H(B_{t+s}, \overline{B}_{t+s}) - H(B_t, \overline{B}_t) = (H(B_s, \overline{B}_s) - H(B_0, \overline{B}_0)) \circ \theta_t.$$

If we rewrite this using the integral representation, we see that

$$\int_0^s h_{u+t} \mathrm{d}B_{u+t} = \int_0^s (h_u \circ \theta_t) \mathrm{d}B_{u+t} \qquad \text{a.s.,} \tag{9}$$

which implies that $h_{u+t} = h_u \circ \theta_t$ for $u > 0$ a.s. This reasoning remains true if we replace t by an arbitrary, a.s. finite, stopping time T. This in turn means that the process $\langle H(B, \overline{B}), B\rangle_t = \int_0^t h_s \, \mathrm{d}s$ is a signed (strong) additive functional of the process $((B_t, \overline{B}_t) : t \geqslant 0)$. To each of the strong additive functionals $\int_0^t (h_s \vee 0)\mathrm{d}s$ and $\int_0^t (-h_s \vee 0)\mathrm{d}s$ we can apply Motoo's theorem (cf. Sharpe 1988: 309; see also Meyer 1967: 122; Revuz and Yor 1999: Exercise X.2.25) to see that there exists a measurable function $h : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$H(B_t, \overline{B}_t) = \int_0^t h(B_s, \overline{B}_s) \mathrm{d}B_s, \qquad t \geqslant 0, \text{ a.s.}$$

Let $y > x \vee 0$ and recall T_y, T_x^y and $\xi = \overline{B}_{T_x^y}$, as defined in Step 1 above. These objects are represented in Figure 1. An application of the strong Markov property at the stopping time T_x^y yields that the process

$$\begin{aligned} H(x + \beta_t, \xi \vee (x + \overline{\beta}_t)) - H(x, \xi) &= H(B_{T_x^y + t}, \overline{B}_{T_x^y + t}) - H(B_{T_x^y}, \overline{B}_{T_x^y}) \\ &= \int_0^t h(x + \beta_s, \xi \vee (x + \overline{\beta}_s)) \mathrm{d}\beta_s \end{aligned} \tag{10}$$

is a local martingale in the enlarged filtration $\mathcal{G}_t = \sigma(\xi, \beta_s : s \leqslant t)$, where $\beta_s = B_{T_x^y + s} - B_{T_x^y}$ is a new Brownian motion independent of $(B_u : u \leqslant T_x^y)$. Furthermore, as on the interval $[0, R_{y-x}]$, where $R_{y-x} = \inf\{s \geqslant 0 : \beta_s = y - x\}$, we have $\xi \vee (x + \overline{\beta}_s) = \xi$ (cf. Figure 1), the stopped local martingale can be written as

$$H(x + \beta_{t \wedge R_{y-x}}, \xi) = H(x, \xi) + \int_0^{t \wedge R_{y-x}} h(x + \beta_t, \xi) \mathrm{d}\beta_s. \tag{11}$$

Step 3. We wish to show that actually, for almost all $z \in (y, \infty)$, $(H(x + \beta_{t \wedge R_{y-x}}, z) : t \geqslant 0)$ is a local martingale in the natural filtration of β.

Let $\tilde{H}(x, z) = (H(x, z) - H(0, z))\mathbf{1}_{x \leqslant z} + (H(z, z) - H(0, z))\mathbf{1}_{x > z}$, which is a measurable function, continuous in the first coordinate. Fix $K > 0$ and define the function $\epsilon : (0, \infty) \to [0, 1]$ via

$$\epsilon(z) = \sup\{0 \leqslant \delta \leqslant 1 : |\tilde{H}(x, z)| \leqslant K \text{ for } x \in [-\delta K, \delta K]\} \tag{12}$$

$$= \sup\{0 \leqslant \delta \leqslant 1 : |\tilde{H}(x, z)| \leqslant K \text{ for } x \in [-\delta K, \delta K] \cap \mathbb{Q}\}, \tag{13}$$

where the equality follows from continuity properties of $\tilde{H}$. We now show that $\epsilon(\cdot)$ is a measurable function. To this end, let $\delta \in (0, 1]$ and write

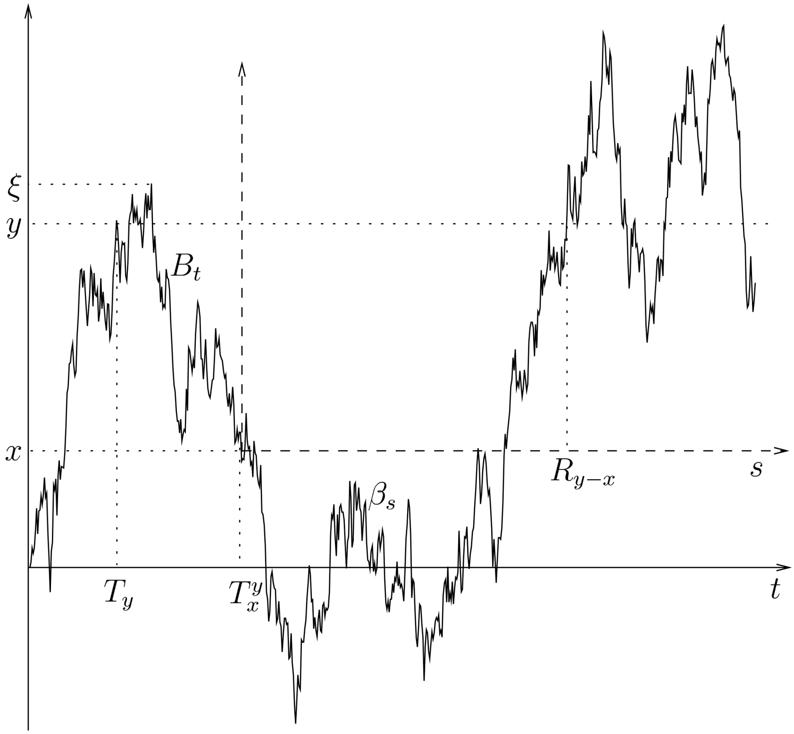

Figure 1. Quantities used in Step 2.

$$\{z : \epsilon(z) < \delta\} = g\{z : \sup\{|\tilde{H}(x, z)| : x \in [-\delta K, \delta K] \cap \mathbb{Q}\} > K\}$$

$$= \bigcup_{x \in [-\delta K, \delta K] \cap \mathbb{Q}} \{z : |\tilde{H}(x, z)| > K\}.$$

Measurability of ϵ follows as

$$\{z : |\tilde{H}(x, z)| > K\} = (\{x\} \times \mathbb{R}_+) \cap \tilde{H}^{-1}[(-\infty, K) \cup (K, \infty)],$$

is a Borel set.

We defined ϵ so that $|\tilde{H}(x, z)| \leqslant K$ on $[-\epsilon(z)K, \epsilon(z)K]$ (and it is the largest such interval). Note that, since a continuous function is bounded on compact intervals, we have $\epsilon(z)K \to \infty$ as $K \to \infty$. Let T_K be a stopping time in the filtration $(\mathcal{G}_t)$ defined by $T_K = T_K(\beta, \xi) = \inf\{t \geqslant 0 : |\beta_t| \geqslant \epsilon(\xi)K\}$, and write T_K^z for $T_K(\beta, z)$, which is a stopping time in the natural filtration of β. Then by (11) we see that $(H(x + \beta_{t \wedge R_{y-x} \wedge T_K}, \xi) - H(x, \xi) : t \geqslant 0)$ is an a.s. bounded local martingale, hence a martingale. Recall that ρ is the density function of the distribution of ξ, $\mathbb{P}(\xi \in \mathrm{d}z) = \rho(z)\mathbf{1}_{z \geqslant y}\mathrm{d}z$, and let $b : \mathbb{R} \to \mathbb{R}$ be a Borel, bounded function. Put $x = 0$. Then the process $(M_t^b = b(\xi)H(\beta_{t \wedge R_y \wedge T_K}, \xi) : t \geqslant 0)$ is a $(\mathcal{G}_t)$-martingale. We wish to show that the process

$H(\beta_{t\wedge T_K^z\wedge R_y}, z)$ is a martingale for almost all $z > y$. As we deal with continuous, a.s. bounded processes in a continuous filtration, it suffices to verify the martingale property for rational times. Fix $t, s \in \mathbb{Q}_+$. For any $A \in \mathcal{F}_t^\beta = \sigma(\beta_s : s \leqslant t)$, by the martingale property of M^b, we have

$$\mathbb{E}[\mathbf{1}_A b(\xi) H(\beta_{(t+s)\wedge R_y\wedge T_K}, \xi)] = \mathbb{E}[\mathbf{1}_A b(\xi) H(\beta_{t\wedge R_y\wedge T_K}, \xi)]$$

$$\int_y^\infty \mathrm{d}z\rho(z)b(z)\mathbb{E}[\mathbf{1}_A H(\beta_{(t+s)\wedge R_y\wedge T_K^z}, z)]$$

$$= \int_y^\infty \mathrm{d}z\rho(z)b(z)\mathbb{E}[\mathbf{1}_A H(\beta_{t\wedge R_y\wedge T_K^z}, z)]$$

and, as b was arbitrary,

$$\mathbb{E}[\mathbf{1}_A H(\beta_{(t+s)\wedge R_y\wedge T_K^z}, z)] = \mathbb{E}[\mathbf{1}_A H(\beta_{t\wedge R_y\wedge T_K^z}, z)], \text{ d}z\text{-a.e.} \tag{14}$$

We will now argue that the above actually holds dz-a.e. for all $t, s \in \mathbb{Q}$ and all $A \in \mathcal{F}_t^\beta$. Let $\Pi \subset \mathcal{F}_t^\beta$ be a countable π-system which generates $\mathcal{F}_t^\beta$ (Revuz and Yor 1999: Exercise I.4.21). We can thus choose a set $\Gamma_{t,s} \subset (y, \infty)$ of full Lebesgue measure, such that for any $A \in \Pi$, (14) holds for all $z \in \Gamma_{t,s}$. As sets A which satisfy (14) for all $z \in \Gamma_{t,s}$ form a λ-system, it follows that (14) holds for any $A \in \mathcal{F}_t^\beta$ for all $z \in \Gamma_{t,s}$. Letting $\Gamma = \cap_{t,s\in\mathbb{Q}+}\Gamma_{t,s}$, we see that $(H(\beta_{t\wedge T_K^z\wedge R_y}, z) : t \geqslant 0)$ is a martingale for all $z \in \Gamma$, and Γ is of full Lebesgue measure. This implies the local martingale property for $(H(\beta_{t\wedge R_y}, z) : t \geqslant 0)$ since $(T_K^z : K \in \mathbb{N})$ is a good localizing sequence. Indeed, for almost all $z > y$, $\epsilon(z)K \to \infty$ as $K \to \infty$, and so $T_K^z \to \infty$ a.s., as $K \to \infty$.

Step 4. We thus know that for almost all $z > y$, $(H(\beta_{t\wedge R_y}, z) : t \geqslant 0)$ is a local martingale with respect to the natural filtration of β. We can thus apply Lemma 5 to see that $K(b) = H(b, z)$ is a linear function on $(-\infty, y]$, for almost all $z > y$. Thus $H(\beta_{t\wedge R_y}, \xi) = \alpha(\xi)\beta_{t\wedge R_y} + H(0, \xi)$ a.s. Comparing this with (11) and using uniqueness of representation for stochastic integrals, we deduce that $h(b, z)$ does not depend on b for $b \in (-\infty, y]$, $h(b, z) = h(z)$ for almost all $z > y$. As $y > 0$ was arbitrary, taking $y \in \mathbb{Q}$ and $y \to 0$, we see that $h(b, z) = h(z)$ for almost all $z > 0$, and therefore $H(B_t, \overline{B}_t) = \int_0^t h(\overline{B}_u)\mathrm{d}B_u$ a.s., and we put $f = h$. From the Part 1 of the proof we know that if $\int_0^t f(\overline{B}_s)\mathrm{d}B_s$ is well defined and finite then f is locally integrable and (1) holds.

Step 5. We turn now to the proof of the last statement in Theorem 1. Let f be a Borel, positive function in L^1, $\|f\| = \int_0^\infty f(x)\mathrm{d}x$. Define $H(x, y) = \|f\| - F(y) + f(y)(y - x)$. The process $H(N_t, \overline{N}_t)$ is a local martingale as in (1). Furthermore, it is a positive process and we can apply Fatou's lemma to see that it is a positive supermartingale and thus converges a.s., as $t \to \infty$. As $\langle N\rangle_\infty = \infty$ we know that $\overline{N}_\infty = \infty$ a.s. This entails that

$$\|f\| - F(\overline{N}_t) = \int_{\overline{N}_t}^\infty f(x)\mathrm{d}x \underset{t\to\infty}{\longrightarrow} 0 \text{ a.s.}$$

and thus $f(\overline{N}_t)(\overline{N} - N_t)$ also converges a.s. as $t \to \infty$. However, it can only converge to zero since $\overline{N}_\infty = \infty$ a.s. and thus $\overline{N}_t - N_t$ has zeros for arbitrary large t. The convergences stated in Theorem 1 and in the remarks which follow it are immediate. To establish the martingale property it suffices to see that the expectation of the positive supermartingale $H(N_t, \overline{N}_t)$ is constant in time and equal to $H(0, 0) = \|f\|$. If f is bounded and $\mathbb{E}(\sup_{s\leqslant t}|N_s|) < \infty$, this follows readily from Lebesgue's dominated convergence theorem. The general case follows from the monotone convergence theorem by replacing f with $\min\{f, n\}$ and taking the limit as $n \to \infty$. This concludes the proof of Theorem 1. □

4. Optional stopping arguments

In the previous section we proved Theorem 1. Here we wish to present some alternative arguments which rely on the optional stopping theorem. Our goal is to provide the reader with additional intuition. The reasoning below can be rigorously justified; however, parts of it rely on our results and an independent proof would require fresh arguments. We comment on this at the end.

We place ourselves in the Brownian motion set-up, that is, $N = B$ is a real-valued Brownian motion. As above, we may assume $H(0, 0) = 0$. Let $T_{x,y} = \inf\{t \geqslant 0 : B_t \notin [x, y]\}$. Relying on the first part of the proof of Theorem 1, which grants that the processes defined via (1) are local martingales, one can verify the well-known fact that the law of $\overline{B}_{T_{-x,y}}\mathbf{1}_{\{B_{T_{-x,y}}=-x\}}$ is given by

$$\mathbb{P}(\overline{B}_{T_{-x,y}}\mathbf{1}_{\{B_{T_{-x,y}}=-x\}} \in \mathrm{d}s) = \frac{x\mathrm{d}s}{(s+x)^2}\mathbf{1}_{0\leqslant s\leqslant y}.$$

Suppose we can apply the optional stopping theorem to $H(B_t, \overline{B}_t)$ at $T_{-x,y}$. Then we obtain:

$$0 = \frac{x}{x+y}H(y, y) + x\int_0^y \frac{H(-x, s)}{(x+s)^2}\,\mathrm{d}s, \tag{15}$$

and thus

$$H(y, y) = -(x+y)\int_0^y \frac{H(-x, s)}{(x+s)^2}\,\mathrm{d}s. \tag{16}$$

Note that the left-hand side does not depend on x. Taking $x \to 0$ on the right-hand side (formally) yields

$$y\int_0^y \frac{H(0, s)}{s^2}\,\mathrm{d}s = -H(y, y) = (x+y)\int_0^y \frac{H(-x, s)}{(x+s)^2}\,\mathrm{d}s, \tag{17}$$

for any $x, y > 0$. Differentiating the extreme expressions in y, we obtain

$$\int_0^y \frac{H(0, s)}{s^2}\,\mathrm{d}s + \frac{H(0, y)}{y} = \int_0^y \frac{H(-x, s)}{(x+s)^2}\,\mathrm{d}s + \frac{H(-x, y)}{x+y}$$

and, by (17),

$$\frac{-H(y, y) + H(0, y)}{y} = \frac{-H(y, y) + H(-x, y)}{x+y},$$

which leads directly to

$$H(-x, y) = H(y, y) - (a + y)\frac{H(y, y) - H(0, 0)}{y}. \tag{18}$$

To obtain (6) we write $H(y, y) = F(y)$. It suffices to use (17) to see that

$$\frac{H(y, y) - H(0, y)}{y} = -\int_0^y \frac{H(0, s)}{s^2}\,\mathrm{d}s - \frac{H(0, y)}{y} = F'(y).$$

The above provides arguments analogous to those used in the proof that Brownian scale functions are affine, presented in the proof of Lemma 5. However, it appears that to make the above reasoning into a complete proof of Theorem 1 one would need the following two facts. We formulate them as propositions and prove them using Theorem 1. An independent proof would require fresh arguments which are not apparent.

Proposition 6. *Let $x, y > 0$ and $H_t = H(B_t, \overline{B}_t)$ be a right-continuous local martingale with $H_0 = 0$. Then one can apply the optional stopping theorem at $T_{-x,y}$* to (H_t)*, that is, $\mathbb{E}H(B_{T_{-x,y}}, \overline{B}_{T_{-x,y}}) = 0$.*

Proof. Theorem 1 implies that H_t is of the form (1) for some locally integrable function f. Let g_n be f bounded to take values in $[-n, n]$, $g_n(x) = (f(x) \wedge n) \vee (-n)$, $G_n(y) = \int_0^y g_n(x)\mathrm{d}x$ and $H_n(x, y) = G_n(y) - g_n(y)(y - x)$. Then the local martingale $H_n(B_t, \overline{B}_t)$ for $t \leqslant T_{-x,y}$ is bounded by $(2y + x)n$ and is therefore a uniformly integrable martingale. We thus have, for $n > 0$,

$$\mathbb{E}G_n(\overline{B}_{T_{-x,y}}) = \mathbb{E}g_n(\overline{B}_{T_{-x,y}})(\overline{B}_{T_{-x,y}} - B_{T_{-x,y}}) = x\int_0^y \frac{g_n(s)}{s + x}\,\mathrm{d}s.$$

As $|f|$ is integrable on $[0, y]$, by the dominated convergence theorem, the right-hand side converges to $x\int_0^y f(s)/(x + s)\mathrm{d}s$ and the left-hand side converges to $\mathbb{E}F(\overline{B}_{T_{-x,y}})$. This yields the desired result. □

Proposition 7. *Let $H(B_t, \overline{B}_t)$ be a right-continuous local martingale. Then for any $x \geqslant 0$, $H(x + B_t, x + \overline{B}_t)$ is a continuous local martingale.*

Proof. Theorem 1 implies that $H(B_t, \overline{B}_t)$ is of the form (1) for some locally integrable function f. Let $g(u) = f(x + u)$ and $G(y) = \int_0^y g(u)\mathrm{d}u = F(y + x) - F(x)$. Then $H(x + B_t, x + \overline{B}_t) = G(\overline{B}_t) - g(\overline{B}_t)(\overline{B}_t - B_t) + F(x)$ and is a continuous local martingale by Theorem 1. □

5. Closing remarks

In the Introduction we indicated two motivations for developing a complete characterization of max-martingales. Actually similar questions motivate a much larger project of describing all local martingales which are functions of Brownian motion and its maximum, minimum and local time processes. Therefore, we see this work as the first step in the general project of describing local martingales which are functions of Brownian motion and some adapted, $\mathbb{R}^d$-valued, process with 'small support' (such as the maximum, minimum and local time processes). Such martingales, for functions which are sufficiently regular, can be described entirely via Itô's formula as shown by Obłój (2005: Chapter 7). However, a complete characterization for arbitrary functions is more delicate. We believe that the methodology developed in our proof of Theorem 1 will be useful for this purpose. We plan to develop these issues in subsequent papers.

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