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Regular variation in the mean and stable limits for Poisson shot noise

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Poisson shot noise is a natural generalization of a compound Poisson process when the summands are stochastic processes starting at the points of the underlying Poisson process. We study the limiting behaviour of Poisson shot noise when the limits are infinite-variance stable processes. In this context a sufficient condition for this convergence turns up which is closely related to multivariate regular variation – we call it regular variation in the mean. We also show that the latter condition is necessary and sufficient for the weak convergence of the point processes constructed from the normalized noise sequence and also for the weak convergence of its extremes.

Keywords: extremes; infinitely divisible distribution; multivariate regular variation; Poisson random measure; self-similar process; stable process; weak convergence

1. Introduction

The compound Poisson process

$$S(t) = \sum_{i=1}^{N(t)} X_i, \qquad t \geqslant 0, \tag{1.1}$$

occurs in various applied probability contexts. Here (X_i) is a sequence of independent and identically distributed (i.i.d.) random variables, independent of the homogeneous Poisson process N with points $0 < T_1 < T_2 < \ldots$. In what follows, we also assume without loss of generality that N is unit rate. For example, $S(t)$ is a natural model for the total amount of claims in an insurance portfolio which have been accumulated in $[0, t]$. However, the model (1.1) implies that claims are paid at the same time as they occur. This is an assumption which is hardly realistic, and therefore the following is a natural generalization.

Let (X_i) be a sequence of i.i.d. stochastic processes on $\mathbb{R}$ such that $X_i(t) = 0$ for negative t. The process

$$S(t) = \sum_{i=1}^{N(t)} X_i(t - T_i), \qquad t \geqslant 0,$$

is called a *Poisson shot noise process*. In an insurance context, for example, X_i would be a process with non-decreasing sample paths representing the pay-off for the ith claim in the portfolio in the period $[0, t]$. With this application in mind, Klüppelberg and Mikosch (1995a; 1995b) studied the weak limit behaviour with Gaussian limits. Traditionally, the shot noise process has been considered with sample paths decreasing to zero, possibly allowing for a stationary version of S; see, for example, Bondesson (1988; 1992) and Parzen (1962). Shot noise processes have been used to model bunching in traffic (Bartlett 1963), computer failure times (Lewis 1964) and earthquake aftershocks (Vere-Jones 1970). But they have recently also been used in other contexts, including applications to workload input models and teletraffic (Konstantopoulos and Lin 1998; Kurtz 1997; Maulik *et al.* 2000; Maulik and Resnick 2001), finance (Samorodnitsky 1996) and physics (Giraitis *et al.* 1993). The continuing interest in the field is also shown by an unsophisticated search for the keyword 'shot noise' in *Mathematical Reviews* which found 71 publications, the majority appearing over the last 15 years.

The aim of this paper is to continue the investigations started in Klüppelberg and Mikosch (1995a; 1995b) which are in line with other work on the asymptotic behaviour of shot noise processes such as Lane (1984; 1987) or Heinrich and Schmidt (1985). Motivated by insurance applications, we are mainly interested in the asymptotic behaviour of shot noise processes in the 'explosive' case, that is, when the noise processes do not die out sufficiently fast so that no stationary version of the shot noise process exists, in particular when the noise processes have 'very heavy tails'. Since the 'tail' of such a process needs to be defined, we borrow from the notion of multivariate regular variation of random vectors which occurs as a necessary and sufficient domain condition for, *inter alia*, sums of i.i.d. random vectors with infinite-variance stable limits (Rvačeva 1962) and componentwise maxima for i.i.d. random vectors (see Resnick 1987, Chapter 5); see Remark 2.4 below for more details. The resulting condition is of the form

$$\nu \int_0^1 P((X_1(\nu(z+s_1)), \ldots, X_1(\nu(z+s_k)))/\sigma(\nu) \in A(r, S))\, dz \to \mu(A(r, S)) \tag{1.2}$$

for all continuity sets $A(r, S) = \{\mathbf{x} : |\mathbf{x}| > r, \mathbf{x}/|\mathbf{x}| \in S\}$ of the limiting measure μ, where r is any positive number, S is a subset of the k-dimensional unit sphere, s_i are any non-negative numbers, $\sigma(\nu)$ is a normalizing function and, most importantly, the measure μ has to satisfy the homogeneity condition

$$\mu(A(r, S)) = r^{-\alpha}\mu(A(1, S)), \qquad r > 0,$$

for some $\alpha > 0$. If the vector in (1.2) does not depend on z the condition degenerates to standard multivariate regular variation. We call condition (1.2) *regular variation in the mean*. Under the natural condition that σ is regularly varying, (1.2) turns out to be the crucial condition for Poisson shot noise to converge weakly to an infinite-variance stable process. In Section 3.8 this condition again occurs as a necessary and sufficient condition for the weak convergence of the maxima of the noise processes towards a Fréchet distribution.

For $\alpha < 2$, condition (1.2) can be considered as a *domain-of-attraction condition* for an infinite-variance stable limiting process. The various examples in Section 3 show that these domains are quite rich. In contrast to the compound Poisson process (1.1) where convergence to a stable process is possible only if X_1 has a distribution with regularly varying tails of order α, the stable domains of attraction for Poisson shot noise contain large classes of stochastic processes ('noise processes') which include compound Poisson processes, various stable processes, processes with 'long-range dependence' and many more. In this sense, shot noise is a class of processes which, from a modelling perspective, is much more flexible than compound Poisson processes.

Our paper is organized as follows. In Section 2 we give necessary and sufficient conditions for the normalized and centred shot noise process to converge weakly to an infinite-variance stable process. This supplements our results for the Gaussian case; see Klüppelberg and Mikosch (1995a; 1995b). As in the latter case, the limit is an unfamiliar self-similar process. Before this result appears, we explain the dependence structure of Poisson shot noise (Section 2.1), and consider the infinite divisibility of $S(t)$ (Section 2.2) and weak limits of infinitely divisible distributions (Section 2.3). Multivariate stable distributions and stable processes appear in Section 2.4. Finally, in Section 2.5 necessary and sufficient conditions for the convergence of normalized and centred shot noise to an infinite-variance stable distribution are given (Corollary 2.7), where the regular variation condition in the mean will play a major role. In Section 3 we apply Corollary 2.7 in different situations:

- X_1 degenerates on the positive real line to a positive regularly varying random variable with index α, that is, S is a compound Poisson process.
- X_1 is an α-stable Lévy motion.
- Multiplicative noise processes of the form $X_i(t) = Y_i f(t)$, where Y_i are i.i.d. regularly varying random variables with index $\alpha \in (0, 2)$ and f is a regularly varying deterministic function with positive index.
- Shots are of the form $X_i(t) = Y_i B_H(t)$, where Y_i are i.i.d. regularly varying random variables with index $\alpha \in (0, 2)$ and B_H is an H-fractional Brownian motion.
- X_1 is a compound Poisson process with infinite-variance summands.
- A heavy-tailed workload process as used for modelling teletraffic.
- A shot noise process with a slowly varying normalizing function.
- Point-process convergence of the normalized noise processes $X_i(t - T_i)$, which turns out to be equivalent to regular variation in the mean as mentioned above.

These examples, in particular, show that the domains of attraction of α-stable processes for shot noise are quite rich and contain various interesting noise processes which also deserve attention in applications, for example in insurance and in telecommunications. Moreover, we intend to convince the reader that our approach to the weak convergence of shot noise processes via the convergence of the underlying triplets (see Sections 2.2 and 2.3) characterizing infinitely divisible processes is a relatively simple way of checking the convergence of the finite-dimensional distributions in the case of α-stable limits. In this sense, our paper can be understood as trying to explain the methodology of convergence rather than providing spectacular new limit results. Although possible in some cases, we refrain from proving functional central limit theorems which would lead to checking the usual tightness conditions and would make the paper more technical. Finally, we mention that the methodology of this paper could be used to verify the weak convergence of Poisson shot noise processes towards more general Lévy or infinitely divisible processes.

2. Necessary and sufficient conditions for convergence to a stable law

2.1. Preliminaries on the shot noise process

Consider the *Poisson shot noise process*

$$S(t) = \sum_{n=1}^{N(t)} X_n(t - T_n), \qquad t \geq 0,$$

where (X_n) are i.i.d. stochastic processes on $\mathbb{R}$ with cadlag sample paths and such that $X_n(s) = 0$, $s \leq 0$, independent of the homogeneous Poisson process N on $[0, \infty)$ with points T_n and intensity 1. (The restriction to unit rate involves no loss of generality.)

We aim to find conditions under which the finite-dimensional distributions of the process S (provided the process is properly normalized and centred) converge to an infinite-variance stable process.

In this context the following simple decomposition of the process S at the instants $0 \leq t_1 < \ldots < t_k$ is crucial:

$$\begin{aligned}
S(t_1) &= \sum_{n=1}^{N(t_1)} X_n(t_1 - T_n),\\
S(t_2) &= \sum_{n=1}^{N(t_1)} X_n(t_2 - T_n) + \sum_{n=N(t_1)+1}^{N(t_2)} X_n(t_2 - T_n),\\
&\vdots\\
S(t_k) &= \sum_{n=1}^{N(t_1)} X_n(t_k - T_n) + \sum_{n=N(t_1)+1}^{N(t_2)} X_n(t_k - T_n) + \ldots + \sum_{n=N(t_{k-1})+1}^{N(t_k)} X_n(t_k - T_n).
\end{aligned}$$

In this decomposition, by virtue of the regenerative property of the Poisson process and the i.i.d. property of the processes X_n, the terms in different columns of the display are independent. Hence, we have following identity in law:

$$(S(t_1), S(t_2), \ldots, S(t_k))^{\mathrm{T}}$$

$$\stackrel{d}{=} \begin{pmatrix} \sum_{n=1}^{N^{(1)}(t_1)} X_n^{(1)}(t_1 - T_n^{(1)}) \\ \sum_{n=1}^{N^{(1)}(t_1)} X_n^{(1)}(t_2 - T_n^{(1)}) + \sum_{n=1}^{N^{(2)}(t_2-t_1)} X_n^{(2)}((t_2 - t_1) - T_n^{(2)}) \\ \vdots \qquad\qquad \vdots \\ \sum_{n=1}^{N^{(1)}(t_1)} X_n^{(1)}(t_k - T_n^{(1)}) + \sum_{n=1}^{N^{(2)}(t_2-t_1)} X_n^{(2)}((t_k - t_1) - T_n^{(2)}) + \ldots + \\ \sum_{n=1}^{N^{(k)}(t_k-t_{k-1})} X_n^{(k)}((t_k - t_{k-1}) - T_n^{(k)}) \end{pmatrix},$$

where the processes $N^{(i)}$ are i.i.d. copies of N with corresponding points $T_n^{(i)}$, independent of the i.i.d. processes $X_n^{(j)}$ with the same distribution as X_1. By virtue of the order statistics property of the Poisson process, we immediately obtain for the latter relation the following identity in law:

$$(S(t_1), S(t_2), \ldots, S(t_k))^{\mathrm{T}}$$

$$\stackrel{d}{=} \begin{pmatrix} \sum_{n=1}^{N^{(1)}(t_1)} X_n^{(1)}(t_1 U_n^{(1)}) \\ \sum_{n=1}^{N^{(1)}(t_1)} X_n^{(1)}((t_2 - t_1) + t_1 U_n^{(1)}) + \sum_{n=1}^{N^{(2)}(t_2-t_1)} X_n^{(2)}((t_2 - t_1) U_n^{(2)}) \\ \vdots \qquad\qquad \vdots \\ \sum_{n=1}^{N^{(1)}(t_1)} X_n^{(1)}((t_k - t_1) + t_1 U_n^{(1)}) + \sum_{n=1}^{N^{(2)}(t_2-t_1)} X_n^{(2)}((t_k - t_2) + (t_2 - t_1) U_n^{(2)}) + \ldots + \\ \sum_{n=1}^{N^{(k)}(t_k-t_{k-1})} X_n^{(k)}((t_k - t_{k-1}) U_n^{(k)}) \end{pmatrix},$$

where $(U_n^{(i)})$ are i.i.d. copies of a sequence (U_n) of random variables i.i.d. uniform on (0, 1). Notice that the terms in different columns of the above display are mutually independent, and therefore it suffices to study the convergence of the finite-dimensional distributions of the (normalized and centred) processes

$$\tilde{S}(\nu t, \nu s) = \sum_{n=1}^{N(\nu t)} X_n(\nu t U_n + \nu s), \qquad s \geqslant 0,$$

as $\nu \to \infty$, for every fixed $t > 0$. Moreover, we will assume that the normalizing constants

$\sigma(\nu) > 0$ for such a convergence result are regularly varying with a non-negative index, that is, there exists $\alpha \geqslant 0$ such that

$$\lim_{\nu\to\infty} \frac{\sigma(c\nu)}{\sigma(\nu)} = c^\alpha, \qquad \text{for all } c > 0.$$

Then notice that, for appropriate centring constants $b(\nu t, \nu s)$,

$$[\tilde{S}(\nu t, \nu s) - b(\nu t, \nu s)]/\sigma(\nu) \sim t^\alpha[\tilde{S}(\tilde{\nu}, \tilde{\nu}s/t) - b(\tilde{\nu}, \tilde{\nu}s/t)]/\sigma(\tilde{\nu}),$$

where $\tilde{\nu} = \nu t$. The limits of the processes $\tilde{S}(\nu t, \nu\cdot) - b(\nu t, \nu\cdot)$ then only differ by a power of t, and therefore it suffices to study the case $t = 1$. For ease of presentation, we write

$$\tilde{S}(\nu s) = \tilde{S}(\nu, \nu s), \qquad s \geqslant 0.$$

2.2. Infinite divisibility of the shot noise process

The distribution of $\tilde{S}(\nu s)$ is infinitely divisible. This follows from the fact that $\tilde{S}(\nu s)$ is a compound Poisson sum. The same applies to any linear combination of the $\tilde{S}(\nu s_i)$, $s_0 = 0 < s_1 < \ldots < s_k$, $k \geqslant 1$. (In what follows, $\mathbf{s} = (s_0, s_1, \ldots, s_k)$ is a fixed multi-index and therefore we suppress the dependence on $\mathbf{s}$ in the notation wherever possible.) This can be seen from the form of the logarithm of the characteristic function of the vector

$$\tilde{\mathbf{S}}_k(\nu) = (\tilde{S}(\nu s_0), \ldots, \tilde{S}(\nu s_k))^{\mathrm{T}}$$

given by

$$\log \mathrm{E} \exp\{(\boldsymbol{\theta}, \tilde{\mathbf{S}}_k(\nu))\} = \nu \int_0^1 (\mathrm{E} \exp\{\mathrm{i}(\boldsymbol{\theta}, \mathbf{X}_k(\nu, \nu z))\} - 1)\, \mathrm{d}z, \qquad \boldsymbol{\theta} \in \mathbb{R}^{k+1}, \tag{2.1}$$

where

$$\mathbf{X}_k(\nu, \nu z) = (X_1(\nu z), X_1(\nu(z + s_1)), \ldots, X_1(\nu(z + s_k)))^{\mathrm{T}}.$$

After renormalizing $\tilde{\mathbf{S}}_k(\nu)$ with positive constants $\sigma(\nu)$ (to be determined later), we can rewrite the right-hand expression in (2.1) as

$$\begin{aligned} &\int_{\mathbb{R}^{k+1}} \left(\mathrm{e}^{\mathrm{i}(\boldsymbol{\theta},\mathbf{x})} - 1 - \frac{\mathrm{i}(\boldsymbol{\theta}, \mathbf{x})}{1 + |\mathbf{x}|^2} \right) \mu(\nu, \mathrm{d}\mathbf{x}) + \mathrm{i}(\boldsymbol{\theta}, \boldsymbol{\gamma}(\nu)) \\ &\quad = -\frac{1}{2} Q(\nu, \boldsymbol{\theta}) + \int_{\mathbb{R}^{k+1}\setminus\{\mathbf{0}\}} \left(\mathrm{e}^{\mathrm{i}(\boldsymbol{\theta},\mathbf{x})} - 1 - \frac{\mathrm{i}(\boldsymbol{\theta}, \mathbf{x})}{1 + |\mathbf{x}|^2} \right) \mu(\nu, \mathrm{d}\mathbf{x}) + \mathrm{i}(\boldsymbol{\theta}, \boldsymbol{\gamma}(\nu)), \end{aligned} \tag{2.2}$$

where

$$\mu(\nu, \cdot) = \nu \int_0^1 P(\mathbf{X}_k(\nu, \nu z)/\sigma(\nu) \in \cdot)\, \mathrm{d}z$$

is a measure on $\mathbb{R}^{k+1}$,

$$\boldsymbol{\gamma}(\nu) = \int_{\mathbb{R}^{k+1}} \frac{\mathbf{x}}{1+|\mathbf{x}|^2}\mu(\nu, \mathrm{d}\mathbf{x}),$$

and

$$Q(\nu, \boldsymbol{\theta}) = \lim_{\epsilon \downarrow 0} \int_{|\mathbf{x}|<\epsilon} (\boldsymbol{\theta}, \mathbf{x})^2 \mu(\nu, \mathrm{d}\mathbf{x}) \tag{2.3}$$

which limit exists and is finite.

More generally, if we replace in (2.2) the triple $(\mu(\nu, \cdot), \boldsymbol{\gamma}(\nu), Q(\nu, \cdot))$ by the triple $(\mu(\cdot), \boldsymbol{\gamma}, Q)$, where $\boldsymbol{\gamma}$ is a constant vector in $\mathbb{R}^d$, μ is a measure on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} \frac{|\mathbf{x}|^2}{1+|\mathbf{x}|^2}\mu(\mathrm{d}\mathbf{x}) < \infty, \tag{2.4}$$

and Q is defined analogously to (2.3), then we obtain the so-called *Lévy representation* of the logarithm of the characteristic function of an infinitely divisible distribution. A measure with the property (2.4) is called a *Lévy measure*. The distribution of any infinitely divisible vector is uniquely determined by the triple $(\mu, \boldsymbol{\gamma}, Q)$. See Sato (1999) for an encyclopedic treatment of infinitely divisible distributions and processes.

2.3. Weak limits of infinitely divisible distributions

It is well known that the weak limits of infinitely divisible distributions are infinitely divisible. Hence the weak limits of the finite-dimensional distributions of a Poisson shot noise process must be infinitely divisible. According to Rvačeva (1962, Theorem 1.2),

$$[\tilde{\mathbf{S}}_k(\nu) - \mathbf{b}(\nu)]/\sigma(\nu) \Rightarrow \mathbf{Z}_k$$

for some infinitely divisible vector $\mathbf{Z}_k$ with triple $(\mu, \boldsymbol{\gamma}, Q)$ in the Lévy representation and appropriate normalizing constants $\sigma(\nu) > 0$ and centring constants $\mathbf{b}(\nu)$ if and only if the following three relations hold:

1. $\mu(\nu, A(r, S)) \to \mu(A(r, S))$ for all continuity sets $A(r, S)$ of μ of the form

$$A(r, S) = \{\mathbf{x} : |\mathbf{x}| > r, \tilde{\mathbf{x}} \in S\}, \tag{2.5}$$

where

$$\tilde{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|, \qquad \mathbf{x} \neq \mathbf{0},$$

and S is any Borel subset of the unit sphere $\mathbb{S}^k$ of $\mathbb{R}^{k+1}$.

2. $\boldsymbol{\gamma}(\nu) - \mathbf{b}(\nu)/\sigma(\nu) \to \boldsymbol{\gamma}$.
3. $\lim_{\epsilon\downarrow 0} \lim_{\nu\to\infty} \int_{|\mathbf{x}|<\epsilon} (\boldsymbol{\theta}, \mathbf{x})^2 \mu(\nu, \mathrm{d}\mathbf{x}) = Q(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \mathbb{R}^{k+1}$.

In what follows, we use both symbols $\tilde{\mathbf{x}}$ and $\mathbf{x}^{\sim}$ for $\mathbf{x}/|\mathbf{x}|$.

2.4. Multivariate stable distributions

Multivariate stable distributions are particular infinitely divisible distributions; see Sato (1999) for the general case of infinitely divisible distributions and Samorodnitsky and Taqqu (1994) for an encyclopedic treatment of stable distributions and processes. The characteristic function of a stable random vector $\mathbf{X}$ with values in $\mathbb{R}^d$ and index $\alpha \in (0, 2]$ is characterized by the triple $(\mu, \boldsymbol{\gamma}, Q)$ in the Lévy representation:

$\alpha = 2$. μ is the null measure on $\mathbb{R}^{k+1}\backslash\{\mathbf{0}\}$ and Q is a non-negative definite quadratic form with non-null coefficient matrix. In this case, $\mathbf{X}$ has a multivariate Gaussian distribution.

$\alpha \in (0, 2)$. $Q \equiv 0$ and μ is homogeneous of order $-\alpha$, that is, for any set $A(r, S)$ given in (2.5),

$$\mu(A(r, S)) = r^{-\alpha}\mu(A(1, S)), \qquad r > 0. \tag{2.6}$$

Remark 2.1. The Lévy representation of an infinite-variance stable distribution (i.e. if $\alpha < 2$) can be given in a more appealing form, involving the index α and a uniquely determined *spectral measure* on the unit sphere $\mathbb{S}^{d-1}$ which, up to a constant multiple, is the spherical part of the measure μ. See Samorodnitsky and Taqqu (1994, Theorem 2.3.1) for this representation, and see Kuelbs (1973) for a proof of it which, in combination with Gnedenko and Kolmogorov (1954, Chapter 7), proves that the spectral measure and the spherical part of μ are identical up to a constant multiple. See also Remark 3 in Samorodnitsky and Taqqu (1994, p. 66).

Finally, we say that a stochastic process $(\xi(t), t \geqslant 0)$ is α-stable if all its finite-dimensional distributions are α-stable in the sense defined above. A particular case is *α-stable Lévy motion*. This is defined as a process with stationary independent α-stable increments and cadlag sample paths. Independent α-stable Lévy motions will constitute the noise processes in Section 3.2.

2.5. Convergence of the finite-dimensional distributions of the shot noise to an infinite variance stable distribution

The following is our main result on convergence of the finite-dimensional distributions of a Poisson shot noise process to an infinite-variance stable distribution. (Recall that all random vectors depend on the multi-index $\mathbf{s}$.)

Theorem 2.2. *Consider the Poisson shot noise process as introduced in Section 2.1. Assume $\alpha \in (0, 2)$. There exist a normalizing function $\sigma(\nu) > 0$ and a centring function $\mathbf{b}(\nu)$ such that*

$$[\tilde{\mathbf{S}}_k(\nu) - \mathbf{b}(\nu)]/\sigma(\nu) \Rightarrow \mathbf{Z}_k, \tag{2.7}$$

for some α-stable random vector $\mathbf{Z}_k$ *with values in* $\mathbb{R}^{k+1}$ *which is characterized by the triple* $(\mu, \boldsymbol{\gamma}, 0)$ *in the Lévy representation, if and only if*

(i)

$$\mu(\nu, A(r, S)) = \nu \int_0^1 P(\mathbf{X}_k(\nu, \nu z)/\sigma(\nu) \in A(r, S))\, \mathrm{d}z \to \mu(A(r, S)) \tag{2.8}$$

for all continuity sets $A(r, S)$ *of a measure* μ *satisfying the homogeneity condition* (2.6)*;*

(ii) $\gamma(\nu) - \mathbf{b}(\nu)/\sigma(\nu) \to \gamma$*;*

(iii) $\lim_{\epsilon \downarrow 0} \limsup_{\nu \to \infty} \int_{|\mathbf{x}|<\epsilon} (\boldsymbol{\theta}, \mathbf{x})^2 \, \mathrm{d}\mu(\nu, \mathbf{x}) = 0$ *for all* $\boldsymbol{\theta} \in \mathbb{R}^{k+1}$.

Proof. The proof follows from Rvačeva's result (Section 2.3) and the definition of a multivariate stable distribution (Section 2.4) by observing that $Q \equiv 0$ for $\alpha < 2$. □

Remark 2.3. Observe that, for any $\delta > 0$,

$$\gamma(\nu) - \int_{|\mathbf{x}| \leq \delta} \mathbf{x} \mu(\nu, \mathrm{d}\mathbf{x}) = \int_{|\mathbf{x}|>\delta} \frac{\mathbf{x}}{1 + |\mathbf{x}|^2} \mu(\nu, \mathrm{d}\mathbf{x}) - \int_{|\mathbf{x}| \leq \delta} \mathbf{x} \frac{|\mathbf{x}|^2}{1 + |\mathbf{x}|^2} \mu(\nu, \mathrm{d}\mathbf{x}). \tag{2.9}$$

The integrands on the right-hand side are bounded continuous functions in their domains. Therefore, since $\mu(\nu, \cdot)$ converges vaguely to $\mu(\cdot)$ on $\overline{\mathbb{R}}^{k+1} \backslash \{\mathbf{0}\}$ and $\mu(\{\mathbf{x} : |\mathbf{x}| = \delta\}) = 0$ for every positive δ, the right-hand side of (2.9) converges to

$$\int_{|\mathbf{x}|>\delta} \frac{\mathbf{x}}{1 + |\mathbf{x}|^2} \mu(\mathrm{d}\mathbf{x}) - \int_{|\mathbf{x}| \leq \delta} \mathbf{x} \frac{|\mathbf{x}|^2}{1 + |\mathbf{x}|^2} \mu(\mathrm{d}\mathbf{x}),$$

and the limits are finite. Therefore possible centring constants in (2.7) (the constant $\boldsymbol{\gamma}$ has to be suitably chosen) are given by

$$\mathbf{b}(\nu) = \sigma(\nu) \int_{|\mathbf{x}| \leq \delta} \mathbf{x} \mu(\nu, \mathrm{d}\mathbf{x}) \tag{2.10}$$

for any choice of $\delta > 0$.

Remark 2.4. Recall, for example from Resnick (1986) or Resnick (1987, Chapter 5), that the random vector $\mathbf{X}$ with values in $\mathbb{R}^d$ is *regularly varying with index* $\alpha \geq 0$ *and spectral (probability) distribution* P_s on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$ if there exist positive constants c and $\sigma_n > 0$ such that

$$\mu_n(A(r, S)) = nP(\sigma_n^{-1} \mathbf{X} \in A(r, S)) \to \mu(A(r, S)) = cr^{-\alpha} P_s(S) \tag{2.11}$$

for all continuity sets S of $\mathbb{S}^{d-1}$. Equivalently, $\mu_n \xrightarrow{v} \mu$, where $\xrightarrow{v}$ denotes vague convergence on the Borel σ-field of $\overline{\mathbb{R}}^d \backslash \{\mathbf{0}\}$. (The measures μ_n and μ are well defined on all Borel sets through their values on the sets $A(r, S)$.) Multivariate regular variation for some $\alpha \in (0, 2)$ is necessary and sufficient for the distribution of $\mathbf{X}$ to belong to the domain of attraction of a stable distribution with index α; see Rvačeva (1962). This means that for i.i.d. copies $\mathbf{X}_i$ of the vector $\mathbf{X}$ and suitable centring constants $\mathbf{b}_n$, the relation

$$(\mathbf{X}_1 + \ldots + \mathbf{X}_n - \mathbf{b}_n)/\sigma_n \Rightarrow \mathbf{Z}$$

holds, where $\mathbf{Z}$ is a d-dimensional α-stable random vector whose spectral measure, up to a constant multiple, is the spherical part of μ.

In this sense, assumption (2.8) can be understood as a *multivariate regular variation condition in the mean*. In particular, if $(\mathbf{X}_1(t),\ t \geqslant 0)$ degenerates to a random vector $\mathbf{X}_1$ then (2.8) is nothing but the regular variation condition (2.11).

Remark 2.5. In the case of random vectors mentioned in Remark 2.4, it follows from the definition of regular variation that a possible choice for the normalizing constants σ_n is given by the asymptotic relation

$$P(|\mathbf{X}| > \sigma_n) \sim n^{-1}, \tag{2.12}$$

or one can choose σ_n as the $(1 - n^{-1})$ quantile of the distribution of $|\mathbf{X}|$. Since $|\mathbf{X}|$ is regularly varying in $\mathbb{R}$ with index α, if $\alpha > 0$, the sequence σ_n is regularly varying with index $1/\alpha$, i.e. $\sigma_{[nc]}/\sigma_n \to c^{1/\alpha}$ for every $c > 0$. By choosing $A(1, S)$ with $S = \mathbb{S}^k$, we conclude from (2.8) that the normalizing constants $\sigma(\nu)$ satisfy the condition

$$\int_0^1 P(|\mathbf{X}_k(\nu, \nu z)| > \sigma(\nu) r)\,\mathrm{d}z \sim \nu^{-1} r^{-\alpha} \mu(A(1, \mathbb{S}^k)) \tag{2.13}$$

for any $r > 0$, which is similar to (2.12) and, again, can be interpreted as a regular variation condition in the mean; cf. Remark 2.4.

We finally mention that condition (iii) of Theorem 2.2 follows from (i) if the stochastic process $\mathbf{X}_1$ degenerates to a random vector; see Rvačeva (1962).

Remark 2.6. In some cases of interest (see Sections 3.2 and 3.3) a possible choice of the normalizing constants $\sigma(\nu)$ is given by

$$P(|X_1(\nu)| > \sigma(\nu)) \sim \nu^{-1}. \tag{2.14}$$

This is similar to the case where the process $X_1(t) \equiv X_1$ for $t \geqslant 0$. In general, such a simple relation for $\sigma(\nu)$ cannot be expected – that is to say, condition (2.13), which is necessary for convergence of the centred and normalized shot noise process to a stable limit, is not equivalent to (2.14).

In Theorem 2.2 we suppressed the dependence of $\mu(\nu, \cdot)$, μ, $\mathbf{X}_k$, etc., on the choice of the index $\mathbf{s} = (s_0, \ldots, s_k)$ with $0 = s_0 < \ldots < s_k$; see Section 2.2 for details. In what follows, we indicate this dependence by adding the corresponding subscripts to the symbols – for example $\mu_{\mathbf{s}}(\nu, \cdot)$, $\mu_{\mathbf{s}}$. As a matter of fact the normalizing constants $\sigma(\nu)$ would also depend on $\mathbf{s}$. However, since we choose $\sigma(\nu)$ to be regularly varying, the corresponding normalizing constants would only differ by positive constants. This explains the appearance of the factors Δ_{ij}^{β} in part (b) of Corollary 2.7 below.

The following result summarizes our findings about the convergence of the finite-dimensional distributions of the Poisson shot noise process to a stable process (see Section 2.4 for its definition).

Corollary 2.7. (a) *Assume there exists a regularly varying normalizing function* $\sigma(\nu) > 0$ *with index* $\beta \geqslant 0$ *such that* $\sigma(\nu) \to \infty$ *as* $\nu \to \infty$, *a centring function* $b(\nu)$ *and an* α*-stable process* ξ *on* $[0, \infty)$ *such that, for every choice of indices* $t_1 < \ldots < t_k$,

$$[S(\nu t_1) - b(\nu t_1), \ldots, S(\nu t_k) - b(\nu t_k)]/\sigma(\nu) \Rightarrow (\xi(t_1), \ldots, \xi(t_k)). \tag{2.15}$$

Then the relations (i)–(iii) *of Theorem 2.2 hold for any choice of indices* $\mathbf{s}$*:*

(i)

$$\nu \int_0^1 P(\mathbf{X}_{k,s}(\nu, \nu z)/\sigma(\nu) \in A(r, S))\, \mathrm{d}z \to \mu_{\mathbf{s}}(A(r, S)) \tag{2.16}$$

for all continuity sets $A(r, S)$ *of a measure* $\mu_{\mathbf{s}}$ *satisfying the homogeneity condition*

$$\mu_{\mathbf{s}}(A(r, S)) = r^{-\alpha}\mu_{\mathbf{s}}(A(1, S)), \qquad \text{for } r > 0.$$

(ii) $\boldsymbol{\gamma}_{\mathbf{s}}(\nu) - \mathbf{b}_{\mathbf{s}}(\nu)/\sigma(\nu) \to \boldsymbol{\gamma}_{\mathbf{s}}$, *where* $\mathbf{b}_{\mathbf{s}}(\nu)$ *is defined in* (2.10) *with* $\delta = 1$ *and* $\mu(\nu, \cdot)$ *replaced by* $\mu_{\mathbf{s}}(\nu, \cdot)$.

(iii)

$$\lim_{\epsilon \downarrow 0} \limsup_{\nu \to \infty} \int_{|\mathbf{x}| < \epsilon} (\boldsymbol{\theta}, \mathbf{x})^2\, \mathrm{d}\mu_{\mathbf{s}}(\nu, \mathbf{x}) = 0 \tag{2.17}$$

for all $\boldsymbol{\theta} \in \mathbb{R}^{k+1}$.

(b) *Under the assumptions of* (a), *the limiting vector in* (2.15) *can be written as*

$$(\xi(t_1), \xi(t_2), \xi(t_3), \ldots, \xi(t_k))^{\mathrm{T}} \stackrel{d}{=}$$

$$\begin{pmatrix} \Delta_{1,0}^{\beta}\xi^{(1)}(0) \\ \Delta_{1,0}^{\beta}\xi^{(1)}(\Delta_{2,1}/\Delta_{1,0}) + \Delta_{2,1}^{\beta}\xi^{(2)}(0) \\ \Delta_{1,0}^{\beta}\xi^{(1)}(\Delta_{3,1}/\Delta_{1,0}) + \Delta_{2,1}^{\beta}\xi^{(2)}(\Delta_{3,2}/\Delta_{2,1}) + \Delta_{3,2}^{\beta}\xi^{(3)}(0) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \Delta_{1,0}^{\beta}\xi^{(1)}(\Delta_{k,1}/\Delta_{1,0}) + \Delta_{2,1}^{\beta}\xi^{(2)}(\Delta_{k,2}/\Delta_{2,1}) + \Delta_{3,2}^{\beta}\xi^{(3)}(\Delta_{k,3}/\Delta_{3,2}) + \ldots + \Delta_{k,k-1}^{\beta}\xi^{(k)}(0) \end{pmatrix},$$

where $\Delta_{i,j} = t_j - t_i$ *and* $t_0 = 0$, *and the processes* $\xi^{(i)}$ *on* $[0, \infty)$ *are i.i.d.* α*-stable with finite-dimensional distributions determined by the pairs* $(\mu_{\mathbf{s}}, \boldsymbol{\gamma}_{\mathbf{s}})$ *in* (i) *and* (ii) *of part* (a).

(c) *If* (i)–(iii) *of* (a) *hold then* (2.15) *is valid for appropriate normalizing and centring constants and an* α*-stable limit vector. Moreover, if* $\sigma(\nu)$ *is regularly varying with a positive index, then the structure of the limit process is given by part* (b).

Proof. The proofs of parts (a) and (c) follow from Theorem 2.2, taking the remarks before the corollary into account. The structure of the limiting process (part (b)) is a consequence of the dependence structure of the Poisson shot noise as explained in Section 2.1 and the regular variation of $\sigma(\nu)$. $\square$

Remark 2.8. If relation (2.15) holds for all choices of index sets $(t_1, \ldots, t_n)$, the normalizing function $\sigma(\nu)$ is necessarily regularly varying. Indeed, we then have, for $t, s \geqslant 0$,

$$\frac{S(ts\nu) - b(ts\nu)}{\sigma(\nu)} \Rightarrow \xi(ts) \quad \text{and} \quad \frac{S(ts\nu) - b(ts\nu)}{\sigma(t\nu)} \Rightarrow \xi(s),$$

and the convergence-to-types theorem (see, for example, Embrechts *et al.*, 1997, p. 554) implies that the limit $\lim_{\nu\to\infty} \sigma(t\nu)/\sigma(\nu)$ exists and is positive for every $t > 0$, that is, $\sigma(\nu)$ is regularly varying. In contrast to the degenerate case when $\mathbf{X}_1$ is a regularly varying vector with index $\alpha \in (0, 2)$ and $\sigma(\nu)$ is necessarily regularly varying with index $1/\alpha$, in the case of shot noise such a relationship is in general not true; see the examples considered below. In particular, $\sigma(\nu)$ can be a slowly varying function; see Section 3.7.

Remark 2.9. Under the conditions of part (c) with $\beta > 0$, the limiting process in (b) satisfies the scaling property

$$(\xi(st))_{t\geqslant 0} \stackrel{d}{=} s^{\beta}(\xi(t))_{t\geqslant 0} \qquad \text{for any } s > 0,$$

where $\stackrel{d}{=}$ stands for identity of the finite-dimensional distributions. This means that the limiting process is a β-*self-similar* α-stable process.

Remark 2.10. The structure of the limiting process given in part (b) might lead one to the conclusion that ξ has independent increments. This is not correct since the processes $\xi^{(i)}$ generally have dependent increments. An exception is the compound Poisson process; see Section 3.1 below.

3. Applications

We consider various examples in order to illustrate different stable limiting behaviours of Poisson shot noise. We focus on the verification of condition (2.16) which characterizes the finite-dimensional distributions of the α-stable limit process up to centring. Only in one example (Section 3.2) do we show explicitly that (2.17) is satisfied. The other cases are similar and boil down to standard calculations.

3.1. Degenerate noise

We start with the simplest example, where the noise processes are given by

$$X_n(t) = Y_n I_{[0,\infty)}(t), \qquad t \in \mathbb{R},$$

in which (Y_n) is an i.i.d. sequence. This means that $S(t)$ is a compound Poisson process. We assume that the distribution of Y_1 is in the domain of attraction of an α-stable law for some $\alpha \in (0, 2)$. This means, in particular, that Y_1 is *regularly varying with index* α, that is, there exist constants $p, q \geqslant 0$ and a slowly varying function L such that

$$P(Y_1 > x) \sim p\frac{L(x)}{x^\alpha} \quad \text{and} \quad P(Y_1 \leqslant -x) \sim q\frac{L(x)}{x^\alpha}, \qquad x \to \infty. \tag{3.1}$$

In this case it is well known (see, for example, Gut 1988; Jacod and Shiryaev 1987) that

$$(S(\nu\cdot) - b(\nu\cdot))/\sigma(\nu) \Rightarrow \xi, \tag{3.2}$$

where $P(|Y_1| > \sigma(\nu)) \sim \nu^{-1}$, b is an appropriate centring function and ξ is an α-stable Lévy motion. The convergence in (3.2) is understood as convergence of the underlying finite-dimensional distributions and can be strengthened to distributional convergence in the Skorokhod space $\mathbb{D}[0, \infty)$ equipped with the J_1-topology.

For illustrational purposes we investigate condition (2.16) which characterizes the finite-dimensional distributions of ξ. In this case, the integrand does not depend on z and the condition becomes

$$\nu P(\sqrt{k+1}|Y_1| > r\sigma(\nu); (\sqrt{k+1})^{-1}(\text{sign}(Y_1), \ldots, \text{sign}(Y_1)) \in S) \to r^{-\alpha}\mu_{\mathbf{s}}(A(1, S)).$$

It is not difficult to see that the latter condition is equivalent to the regular variation condition (3.1). The measure $\mu_{\mathbf{s}}$ is concentrated at the atoms $(\sqrt{k+1})^{-1}(1, \ldots, 1)$ and $(\sqrt{k+1})^{-1}(-1, \ldots, -1)$ with corresponding probabilities p and q. This kind of measure characterizes an α-stable random vector whose components are identical, and from part (b) of Corollary 2.7 one may conclude that the limiting process ξ has independent and stationary increments.

3.2. Lévy motion as noise

Assume that the noise processes are given by a strictly α-stable Lévy motion with index $\alpha < 2$, skewness parameter $\beta \in [-1, 1]$ and scale parameter $c > 0$, that is, the log-characteristic function of $X_1(1)$ has form

$$-f(x) = \begin{cases} -c|x|^\alpha\left(1 - \mathrm{i}\beta\,\text{sign}(x)\tan\left(\dfrac{\pi\alpha}{2}\right)\right), & \text{if } \alpha \neq 1, \\ -c|x|\left(1 + \beta\log(|x|)\dfrac{2}{\pi}\,\text{sign}(x)\right), & \text{if } \alpha = 1. \end{cases} \tag{3.3}$$

For $\alpha = 1$ strict stability implies that $\beta = 0$, that is, $X_1(1)$ is symmetric in this case. Choose

$$\sigma(\nu) = \nu^{2/\alpha}.$$

By strict stability the left-hand side of (2.16) turns into

$$\begin{aligned} &\nu\int_0^1 P(|(X_1(\nu z), X_1(\nu(z+s_1)), \ldots, X_1(\nu(z+s_k)))| > \nu^{2/\alpha} r; \\ &\qquad (X_1(\nu z), X_1(\nu(z+s_1)), \ldots, X_1(\nu(z+s_k)))^\sim \in S)\,\mathrm{d}z \\ &= \nu\int_0^1 P(|(X_1(z), X_1(z+s_1), \ldots, X_1(z+s_k))| > \nu^{1/\alpha} r; \end{aligned}$$

$$(X_1(z),\, X_1(z+s_1),\, \ldots,\, X_1(z+s_k))^{\sim} \in S)\,\mathrm{d}z. \tag{3.4}$$

For fixed $z \in [0, 1]$, the limit of

$$\nu P(|(X_1(z),\, X_1(z+s_1),\, \ldots,\, X_1(z+s_k))| > \nu^{1/\alpha} r;\, (X_1(z),\, X_1(z+s_1),\, \ldots,\, X_1(z+s_k))^{\sim} \in S)$$

is, up to a multiple $cr^{-\alpha}$, the spectral measure of the stable process X_1. The latter can be read off from the characteristic function of the Lévy motion. This follows, for example, by an application of Rvačeva's results for sums of i.i.d. random vectors in the domain of attraction of a stable distribution; see Rvačeva (1962). Using the independent stationary increments of this stable process, we see that

$$(\boldsymbol{\theta},\, (X_1(z),\, X_1(z+s_1),\, \ldots,\, X_1(z+s_k))^{\mathrm{T}})$$

$$\stackrel{d}{=} X_1(z)(\theta_1 + \ldots + \theta_{k+1}) + X_2(s_1)(\theta_2 + \ldots + \theta_{k+1}) + \ldots + X_{k+1}(s_k - s_{k-1})\theta_{k+1}$$

$$= z^{1/\alpha} X_1(1)(\theta_1 + \ldots + \theta_{k+1}) + s_1^{1/\alpha} X_2(1)(\theta_2 + \ldots + \theta_{k+1}) + \ldots + (s_k - s_{k-1})^{1/\alpha} X_{k+1}(1)\theta_{k+1}.$$

Switching to characteristic functions (see (3.3)), we see that

$$\begin{aligned}
&\mathrm{E}\exp\{\mathrm{i}(\boldsymbol{\theta},\, (X_1(z),\, X_1(z+s_1),\, \ldots,\, X_1(z+s_k))^{\mathrm{T}})\}\\
&\quad = \exp\{-[zf(\theta_1 + \ldots + \theta_{k+1}) + s_1 f(\theta_2 + \ldots + \theta_{k+1}) + \ldots + (s_k - s_{k-1}) f(\theta_{k+1})]\}\\
&\quad = \exp\left\{-\int_{\mathbb{R}^{k+1}} f((\boldsymbol{\theta},\, \mathbf{x}))(z\varepsilon_{(1,\ldots,1)} + s_1\varepsilon_{(0,1,\ldots,1)} + \ldots + (s_k - s_{k-1})\varepsilon_{(0,\ldots,0,1)})(\mathrm{d}\mathbf{x})\right\}\\
&\quad = \exp\left\{-\int_{\mathbb{S}^k} f_1((\boldsymbol{\theta},\, \tilde{\mathbf{x}}))\Gamma_{z,s_1,\ldots,s_k}(\mathrm{d}\tilde{\mathbf{x}})\right\}
\end{aligned}$$

where $\varepsilon_{\mathbf{x}}$ denotes Dirac measure at $\mathbf{x}$, f_1 is the characteristic function (3.3) with $\beta = 1$ for $\alpha \neq 1$ and $\beta = 0$ for $\alpha = 1$, and the spectral measure $\Gamma_{z,s_1,\ldots,s_k}$ on $\mathbb{S}^k$ is the superposition of the two measures

$$\frac{1+\beta}{2}\left[z\left(\sqrt{k+1}\right)^{\alpha}\varepsilon_{(1,\ldots,1)/\sqrt{k+1}} + s_1\left(\sqrt{k}\right)^{\alpha}\varepsilon_{(0,1,\ldots,1)/\sqrt{k}} + \ldots + (s_k - s_{k-1})\varepsilon_{(0,\ldots,0,1)}\right]$$

and

$$\frac{1-\beta}{2}\left[z\left(\sqrt{k+1}\right)^{\alpha}\varepsilon_{-(1,\ldots,1)/\sqrt{k+1}} + s_1\left(\sqrt{k}\right)^{\alpha}\varepsilon_{-(0,1,\ldots,1)/\sqrt{k}} + \ldots + (s_k - s_{k-1})\varepsilon_{-(0,\ldots,0,1)}\right].$$

Notice that for every $T > 0$,

$$\limsup_{\nu\to\infty} \nu P\left(\sup_{0\leqslant t\leqslant T} |X_1(t)| > \nu^{1/\alpha}\right) < \infty.$$

This follows from strict stability and a maximal inequality of Lévy–Skorohod–Ottaviani type; see, for example, Petrov (1995, Theorem 2.3). Hence a domination argument – Pratt's (1960) lemma; cf. Resnick (1987), p. 289) – yields that the limiting measure in (3.4), up to a constant multiple, is given by

$$
\begin{aligned}
&r^{-\alpha}\int_0^1 \Gamma_{z,s_1,\dots,s_k}(S)\,\mathrm{d}z\\
&= r^{-\alpha}\frac{1+\beta}{2}(0.5(\sqrt{k+1})^{\alpha}\varepsilon_{(1,\dots,1)/\sqrt{k+1}} + s_1(\sqrt{k})^{\alpha}\varepsilon_{(0,1,\dots,1)/\sqrt{k}} + \dots + (s_k - s_{k-1})\varepsilon_{(0,\dots,0,1)})\ (S)\\
&\quad + r^{-\alpha}\frac{1-\beta}{2}(0.5(\sqrt{k+1})^{\alpha}\varepsilon_{-(1,\dots,1)/\sqrt{k+1}} + s_1(\sqrt{k})^{\alpha}\varepsilon_{-(0,1,\dots,1)/\sqrt{k}} + \dots\\
&\quad + (s_k - s_{k-1})\varepsilon_{-(0,\dots,0,1)})\ (S),
\end{aligned}
$$

where $A(r, S)$ is any continuity set of the limiting measure. Thus the finite-dimensional distributions of the Poisson shot noise process are α-stable.

It remains to check condition (2.17). We indicate this in the case $k = 1$ and write $\mathbf{s} = (0, s)$ for some $s = s_1 > 0$. Write

$$\mathbf{Y}(z, s) = (X_1(\nu s), X_1(\nu(z+s))).$$

We have for large ν,

$$
\begin{aligned}
\int_{|x|\leqslant\epsilon} (\boldsymbol{\theta}, \mathbf{x})^2 \mu_{\mathbf{s}}(\nu, \mathrm{d}\mathbf{x}) &= \nu\int_{|\mathbf{x}|\leqslant\epsilon} (\boldsymbol{\theta}, \mathbf{x})^2 \int_0^1 P(\mathbf{Y}(z, s)/\sigma(\nu) \in \mathrm{d}\mathbf{x})\,\mathrm{d}z\\
&\leqslant c\nu\int_{|\mathbf{x}|\leqslant\epsilon} |\mathbf{x}|^2 \int_0^1 P(\mathbf{Y}(z, s)/\sigma(\nu) \in \mathrm{d}\mathbf{x})\,\mathrm{d}z\\
&\leqslant c\nu\int_0^1 \sum_{k\leqslant\epsilon\sigma(\nu)+1} k^2 P(k-1 < |\mathbf{Y}(z, s)| \leqslant k)\,\mathrm{d}z\\
&\leqslant c\nu\int_0^1 \sum_{k\leqslant\epsilon\sigma(\nu)} kP(|\mathbf{Y}(z, s)| > k)\,\mathrm{d}z\\
&\leqslant c\int_{x\leqslant\epsilon} x\left[\nu\int_0^1 P(|\mathbf{Y}(z, s)| > x\sigma(\nu))\right]\mathrm{d}z\,\mathrm{d}x.
\end{aligned}
$$

Now one can proceed in a similar way to the first part of the proof, using Pratt's lemma, to conclude that the right-hand side converges as $\nu \to \infty$ to

$$\int_{x\leqslant\epsilon} x\mu_{\mathbf{s}}(A(x, \mathbb{S}^1))\,\mathrm{d}x = \mu_{\mathbf{s}}(A(1, \mathbb{S}^1))\int_{x\leqslant\epsilon} x^{1-\alpha}\,\mathrm{d}x = c\epsilon^{2-\alpha} \to 0 \qquad \text{as } \epsilon \to 0.$$

This concludes the proof.

3.3. Multiplicative noise

Another simple example is given by the noise processes

$$X_n(t) = Y_n f(t),$$

where (Y_n) is an i.i.d. sequence of almost surely positive random variables, regularly varying with index $\alpha \in (0, 2)$ (see (3.1)), and f is a deterministic function on $\mathbb{R}$ with $f(t) = 0$ for $t < 0$. We also assume that f is bounded on compact intervals, positive for $t > 0$ and regularly varying at infinity with index $\beta > 0$. Finally, we assume one of the following conditions: Y_n has only positive or negative values; or Y_n is symmetric. In both cases we know that $\operatorname{sign}(Y_n)$ and $|Y_n|$ are independent.

We choose

$$\sigma(\nu) = a_\nu f(\nu),$$

where $P(|Y_1| > a_\nu) \sim \nu^{-1}$. Then a_ν is regularly varying with index $1/\alpha$ and $\sigma(\nu)$ with index $\beta + 1/\alpha$.

The left-hand side of (2.16) becomes

$$\nu \int_0^1 P(|Y_1|\,|(f(\nu z), f(\nu(z+s_1)), \ldots, f(\nu(z+s_k)))| > a_\nu f(\nu) r)$$
$$\times P(\operatorname{sign}(Y_1)(f(\nu z), f(\nu(z+s_1)), \ldots, f(\nu(z+s_k)))^\sim \in S)\,\mathrm{d}z. \tag{3.5}$$

Since f is regularly varying with positive index, the uniform convergence theorem (Bingham *et al.* 1987, Theorem 1.5.2) yields

$$\frac{|(f(\nu z), f(\nu(z+s_1)), \ldots, f(\nu(z+s_k)))|}{f(\nu)} \to |(|z|^\beta, |z+s_1|^\beta, \ldots, |z+s_k|^\beta)|, \tag{3.6}$$

$$(f(\nu z), f(\nu(z+s_1)), \ldots, f(\nu(z+s_k)))^\sim \to (|z|^\beta, |z+s_1|^\beta, \ldots, |z+s_k|^\beta)^\sim \tag{3.7}$$

uniformly for $z \in (0, 1)$. By the same theorem and (3.6), (3.5) is asymptotically equivalent to

$$\sim r^{-\alpha} \int_0^1 |(|z|^\beta, |z+s_1|^\beta, \ldots, |z+s_k|^\beta)|^\alpha$$
$$\times P(\operatorname{sign}(Y_1)(f(\nu z), f(\nu(z+s_1)), \ldots, f(\nu(z+s_k)))^\sim \in S)\,\mathrm{d}z$$
$$\sim r^{-\alpha} \int_0^1 |(|z|^\beta, |z+s_1|^\beta, \ldots, |z+s_k|^\beta)|^\alpha$$
$$\times P(\operatorname{sign}(Y_1)(|z|^\beta, |z+s_1|^\beta, \ldots, |z+s_k|^\beta)^\sim \in S)\,\mathrm{d}z.$$

In the last step we use (3.7) and Pratt's (1960) lemma. The convergence holds for every $S \subset \mathbb{S}^k$. Thus the limit of the shot noise process is again an α-stable process. The verification of condition (2.17) is analogous to the previous example and therefore omitted.

Remark 3.1. In the above calculations the uniform convergence in (3.6) and (3.7) for $z \in (0, 1]$ was crucial. For slowly varying f, that is, $\beta = 0$, uniform convergence can be achieved only on compact sets bounded away from zero. For uniformity of (3.6) and (3.7) one would have to assume a slow variation condition with remainder term. Then the above calculations go through with $\beta = 0$. Notice that the spherical part of the limiting measure is concentrated at the two points $(1, \ldots, 1)/\sqrt{k+1}$ and $-(1, \ldots, 1)/\sqrt{k+1}$. This means that the limiting α-stable vector has identical components. This is analogous to the compound Poisson case considered in Section 3.1.

3.4. Sub-Gaussian processes

In this subsection we assume that the i.i.d. noise processes are given by

$$X_n(t) = Y_n B_H(t), \qquad t \geqslant 0, \tag{3.8}$$

where B_H is standard H-fractional Brownian motion for some $H \in (0, 1)$ and (Y_n) is an i.i.d. sequence, independent of B_H. We also assume that Y_1 is regularly varying with index $\alpha \in (0, 2)$.

Recall, for example from Samorodnitsky and Taqqu (1994, Chapter 7), that standard H-fractional Brownian motion is a process with almost surely continuous sample paths, stationary mean-zero increments and covariance structure

$$\mathrm{cov}(B_H(t), B_H(s)) = 0.5\ [|t|^{2H} + |s|^{2H} - |t-s|^{2H}].$$

For $H = 0.5$, B_H is standard Brownian motion. In contrast to the case $H \leqslant 0.5$, for fractional Brownian motion with $H \in (0.5, 1)$ the stationary noise sequence $(B_H(n) - B_H(n-1))$ has a non-summable autocovariance function. The latter fact is referred to as *long-range dependence*. Moreover, B_H is H-self-similar, that is $(B_H(ct)) \stackrel{d}{=} c^H(B_H(t))$, where $\stackrel{d}{=}$ refers to identity of the finite-dimensional distributions.

In order to ensure the 'heavy-tailedness' of the noise, we also assume that Y_1 is regularly varying with index $\alpha \in (0, 2)$; see Section 3.3.

Samorodnitsky and Taqqu (1994, Section 3.7) call a process X sub-Gaussian if it can be written in the form $X(t) = A^{1/2}G(t)$, where G is a Gaussian process and A is a positive $(\gamma/2)$-stable random variable for some $\gamma < 2$, independent of G. The resulting process X is then γ-stable. On the one hand, the noise process (3.8) is more general since the multipliers Y_n do not necessarily have the structure mentioned above. On the other hand, we do not allow for general Gaussian processes G. Nevertheless, we call the noise process (3.8) *sub-Gaussian*.

Write

$$\sigma(\nu) = \nu^H a_\nu,$$

where $P(|Y_1| > a_\nu) \sim \nu^{-1}$. We only check condition (2.16) in order to obtain a description of the dependence in the limiting α-stable process. Write

$$\mathbf{Z_s} = (B_H(z),\ B_H(z+s_1),\ \ldots,\ B_H(z+s_k)).$$

Using the self-similarity of B_H, the left-hand side of (2.16) becomes

$$\nu\int_0^1 P(|Y_1|\,|(B_H(\nu z),\ B_H(\nu(z+s_1)),\ \ldots,\ B_H(\nu(z+s_k)))| > \nu^H a_\nu r;$$

$$\operatorname{sign}(Y_1)(B_H(\nu z),\ B_H(\nu(z+s_1)),\ \ldots,\ B_H(\nu(z+s_k)))^{\sim} \in S)\,\mathrm{d}z$$

$$= \nu\int_0^1 P(|Y_1|\,|\mathbf{Z_s}| > a_\nu r;\ \operatorname{sign}(Y_1)\tilde{\mathbf{Z}}_{\mathbf{s}} \in S)\,\mathrm{d}z$$

$$= \nu\int_0^1 P(|Y_1|\,|\mathbf{Z_s}| > a_\nu r;\ \tilde{\mathbf{Z}}_{\mathbf{s}} \in S)\,\mathrm{d}z). \tag{3.9}$$

In the last step we use the fact that B_H is a symmetric random element with values in the space of continuous functions. Hence $(|Y_1|, \operatorname{sign}(Y_1)B_H)$ and $(|Y_1|, B_H)$ have the same distribution. A result of Breiman (1965) and the independence of $|Y_1|$ and B_H ensure that

$$\nu P(|Y_1|\,|\mathbf{Z_s}| > a_\nu r;\ \tilde{\mathbf{Z}}_{\mathbf{s}} \in S) \sim P(|Y_1| > a_\nu r)\mathrm{E}|\mathbf{Z_s}|^\alpha I_S(\tilde{\mathbf{Z}}_{\mathbf{s}})$$

$$\sim r^{-\alpha}\mathrm{E}|\mathbf{Z_s}|^\alpha I_S(\tilde{\mathbf{Z}}_{\mathbf{s}}).$$

In the last step we use the definition of a_ν. The right-hand side determines the radial and spherical parts of the Lévy measure of an α-stable distribution. Moreover, one can interchange the integral and the limit in (3.9), yielding the desired Lévy measure of the limit of the shot noise process. Indeed, this interchange is again justified by an application of Pratt's lemma which is based on the relation.

$$\limsup_{\nu\to\infty} \nu P\Big(|Y_1| \sup_{0\leq t\leq T} |B_H(t)| > a_\nu\Big) < \infty,$$

for every $T > 0$. The latter fact follows by another application of Breiman's result.

3.5. Compound Poisson noise

In this subsection we assume that the noise processes have a compound Poisson structure, that is, the i.i.d. noise processes are of the form

$$X_n(t) = \sum_{i=1}^{N_n(t)} Y_{ni},$$

where N_n are i.i.d. homogeneous Poisson processes on $(0, \infty)$ with (without loss of generality) unit rate and Y_{ni}, i, $n = 1, 2, \ldots,$ are i.i.d. random variables. For convenience we write $Y_i = Y_{1i}$. We also assume that the Y_i are strictly α-stable, that is, for every $k \geq 1$ and non-negative c_i,

$$c_1 Y_1 + \ldots + c_k Y_k \stackrel{d}{=} \left(\sum_{i=1}^k |c_i|^\alpha\right)^{1/\alpha} Y_1.$$

We only give the verification of (2.16) in order to characterize the limiting stable process, and for ease of presentation we focus on the case $k = 1$, the general case being analogous. Observe that

$$(X_1(\nu z),\, X_1(\nu(z+s))) \stackrel{d}{=} ([N(\nu z)]^{1/\alpha} Y_1,\, [N(\nu z)]^{1/\alpha} Y_1 + [N(\nu s)]^{1/\alpha} Y_2). \tag{3.10}$$

Conditionally on N, this vector is α-stable. Then condition (2.16) with $k = 1$ (we set $s_1 = s$) becomes

$$\nu \int_0^1 P(|([N(\nu z)]^{1/\alpha} Y_1,\, [N(\nu z)]^{1/\alpha} Y_1 + [N(\nu s)]^{1/\alpha} Y_2)| > \sigma(\nu) r;$$
$$([N(\nu z)]^{1/\alpha} Y_1,\, [N(\nu z)]^{1/\alpha} Y_1 + [N(\nu s)]^{1/\alpha} Y_2)^{\sim} \in S)\, \mathrm{d}z \to \mu_{\mathbf{s}}(A(r, S)).$$

By the law of large numbers, $N(\nu)/\nu \xrightarrow{\text{a.s.}} 1$. Therefore the left-hand expression becomes

$$\nu \int_0^1 P(|(z^{1/\alpha} Y_1,\, z^{1/\alpha} Y_1 + s^{1/\alpha} Y_2)| > [1 + o(1)]\nu^{-1/\alpha} \sigma(\nu) r;$$
$$(z^{1/\alpha} Y_1,\, z^{1/\alpha} Y_1 + s^{1/\alpha} Y_2)^{\sim}[1 + o(1)] \in S)\, \mathrm{d}z.$$

Choose $\sigma(\nu) = \nu^{2/\alpha}$. Conditionally on N, as $\nu \to \infty$ with probability 1,

$$\begin{aligned} &\nu P(|(z^{1/\alpha} Y_1,\, z^{1/\alpha} Y_1 + s^{1/\alpha} Y_2)| > [1 + o(1)]\nu^{1/\alpha} r; \\ &(z^{1/\alpha} Y_1,\, z^{1/\alpha} Y_1 + s^{1/\alpha} Y_2)^{\sim}[1 + o(1)] \in S \mid N)\, \mathrm{d}z \\ &\to \mu_{\mathbf{s}}(A(r, S)), \end{aligned} \tag{3.11}$$

where the limit is the same as in Section 3.2, that is, for noise processes which are α-stable Lévy motions with $X_1(1) \stackrel{d}{=} Y_1$. Hence the limiting measure in (2.16) is exactly the same as in Section 3.2 provided we can show that the interchange of limit and integration is justified. For an application of Pratt's lemma it suffices to show that the terms in (3.11) are dominated by some functions which are integrable and whose integrals converge to a finite number. Indeed, for this purpose we may choose

$$\nu P(2[N(\nu)]^{1/\alpha} |Y_1| + [N(\nu s)]^{1/\alpha} |Y_2| > \sigma(\nu) r \mid N). \tag{3.12}$$

Since this expression is independent of z, it suffices to show that the expectation with respect to N can be dominated by a function which converges as $\nu \to \infty$. This can be seen as follows. First intersect the event in (3.12) with

$$A = \{|N(\nu) - \nu| \leqslant \nu,\, |N(\nu s) - \nu s| \leqslant \nu\},$$

and use the fact that $P(|Y_1| > x) \sim c x^{-\alpha}$. Then

$$\nu P(\{2[N(\nu)]^{1/\alpha} |Y_1| + [N(\nu s)]^{1/\alpha} |Y_2| > \sigma(\nu) r\} \cap A)$$
$$\leqslant \nu P(\text{const. } [|Y_1| + |Y_2|] > \nu^{1/\alpha} r) \to c$$

for some positive constant. Moreover,

$$\nu P(2[N(\nu)]^{1/\alpha}|Y_1| + [N(\nu s)]^{1/\alpha}|Y_2| > \sigma(\nu)r \cap A^c) \leqslant \nu P(A^c),$$

but $P(A^c)$ decays exponentially fast in ν. This proves that Pratt's lemma is applicable and finishes the proof.

Remark 3.2. Although desirable, it is more difficult to replace the Y_{ni} by random variables in the domain of attraction of an α-stable distribution. In this case, exact scaling as in (3.10) is not valid, and so one would depend on a large-deviation argument in higher dimensions which does not seem to be available at the moment.

3.6. A teletraffic model

We now consider a model introduced by Konstantopoulos and Lin (1998) for heavy-tailed teletraffic. The T_i are interpreted as the times when a new on period of an individual source in a computer network starts. The i.i.d. lengths (X_i) of the on periods are independent of the Poisson points (T_i), and X_1 is a positive regularly varying random variable of index $\alpha \in (1, 2)$. During an on period the source sends a signal at unit rate. At time t the number of active computers in the network is given by the shot noise process

$$Q(t) = \sum_{i=1}^{\infty} I_{(T_i T_i + X_i]}(t). \tag{3.13}$$

The corresponding workload process in $[0, t]$ is then given as the integrated Q-process

$$S(t) = \int_0^t Q(s)\,\mathrm{d}s = \sum_{i=1}^{N(t)} \min(X_i,\ t - T_i) I_{(0,\infty)}(t - T_i). \tag{3.14}$$

Thus the workload process is a shot noise process. If $T_i + X_i \leqslant t$, then the full period X_i contributes to the workload. Otherwise, only the length of the unfinished on period $t - T_i$ is taken into account. Konstantopoulos and Lin (1998) also allowed for more general noise than (3.14), assuming some kind of a regular variation condition of the noise. First we consider the simple model (3.14), but more general ones can be considered as well, including reward processes, where the indicators in (3.13) are multiplied by random variables Y_i, being independent of (T_i) and (X_i); see the discussion below. Models of this type, but in the slightly different context of on/off models, were considered in Levy and Taqqu (2000), Pipiras and Taqqu (2000) and Pipiras *et al.* (2000). All the papers mentioned showed convergence of $S(t)$ to infinite-variance stable limits under various additional assumptions. Since the limit results are known, we do not intend to give a complete proof but rather wish to show that Corollary 2.7 gives the convergence of the finite-dimensional distributions without too much effort. We again restrict ourselves to check relation (2.16) which will characterize the α-stable limit. From the verification of (2.16) it will become transparent why $\alpha \in (1, 2)$ is a necessary requirement.

We choose $\sigma(\nu)$ as $P(X_1 > \sigma(\nu)) \sim \nu^{-1}$. For the shot noise process (3.14) the left-hand side of the relation (2.16) with $k = 1$ and $s = s_1$ (the general case $k \geqslant 1$ is analogous) reads as follows:

$$\nu\int_0^1 P(|(\min(X_1,\,\nu z),\,\min(X_1,\,\nu(z+s)))|>r\sigma(\nu);\,(\min(X_1,\,\nu z),\,\min(X_1,\,\nu(z+s)))^{\sim}\in S)\,\mathrm{d}z$$

$$\begin{aligned}&=\nu\int_0^1 P(|(z,\,z+s)|>r\sigma(\nu)/\nu;\,(z,\,z+s)^{\sim}\in S,\,X_1>\nu(z+s))\,\mathrm{d}z\\&\quad+\nu\int_0^1 P(|(\nu z,\,X_1)|>r\sigma(\nu);\,(\nu z,\,X_1)^{\sim}\in S,\,\nu z<X_1\leqslant\nu(z+s))\,\mathrm{d}z\\&\quad+\nu\int_0^1 P(|(X_1,\,X_1)|>r\sigma(\nu);\,(X_1,\,X_1)^{\sim}\in S,\,X_1\leqslant\nu z)\,\mathrm{d}z\\&=I_1+I_2+I_3.\end{aligned}$$

We will show that I_1 and I_2 do not contribute to the limit, and the term I_3 yields the same limiting measure as for a compound Poisson process described in Section 3.1. Hence α-stable Lévy motion is the limit of the shot noise process. A comparison with Section 3.1 shows that it remains to remove the event $\{X_1\leqslant\nu z\}$. However, by definition of $\sigma(\nu)$,

$$\nu\int_0^1 P(|(X_1,\,X_1)|>r\sigma(\nu);\,(X_1,\,X_1)^{\sim}\in S,\,X_1>\nu z)\,\mathrm{d}z\leqslant\nu P(|(X_1,\,X_1)|>r\sigma(\nu))\to\text{const.}\tag{3.15}$$

On the other hand, $\nu P(X_1>\nu z)\to 0$ for every z. An application of Pratt's lemma shows that (3.15) converges to zero. It is easily seen that

$$I_1\leqslant\nu P(X_1>\nu s)\to 0,$$

$$I_2\leqslant\nu\int_0^1 P(|(X_1,\,X_1)|>r\sigma(\nu);\,(\nu z,\,X_1)^{\sim}\in S,\,X_1>\nu z)\,\mathrm{d}z\to 0,$$

where the latter convergence follows in the same way as for (3.15). Analogous arguments show the convergence for the general case in (2.16); the limiting measure characterizes the limit as an α-stable Lévy motion. Notice that the condition $\alpha\in(1,\,2)$ was crucial since we needed that $\sigma(\nu)/\nu\to 0$. This is clearly satisfied since $(\sigma(\nu))$ is regularly varying with index $1/\alpha$.

We now consider a reward process in the spirit of Levy and Taqqu (2000), Pipiras and Taqqu (2000) and Pipiras *et al.* (2000). Consider the analogue to (3.13):

$$\tilde{Q}(t)=\sum_{i=1}^{\infty}Y_i I_{(T_i,T_i+X_i]}(t),$$

where (Y_i) is an i.i.d. process of rewards, independent of (X_i). The reward process is then the integrated version of $\tilde{Q}$:

$$S(t)=\int_0^t\tilde{Q}(s)\,\mathrm{d}s=\sum_{i=1}^{N(t)}Y_i\min(X_i,\,t-T_i)I_{(0,\infty)}(t-T_i).$$

The left-hand side of condition (2.16) becomes

$$\nu\int_0^1 P(|Y_1|\,|(\min(X_1,\,\nu z),\,\min(X_1,\,\nu(z+s)))|>r\sigma(\nu);$$

$$\operatorname{sign}(Y_1)\,(\min(X_1,\,\nu z),\,\min(X_1,\,\nu(z+s)))^{\sim}\in S)\,\mathrm{d}z$$

$$=\nu\int_0^1 P(|Y_1|\,|(z,\,z+s)|>r\sigma(\nu)/\nu;\ \operatorname{sign}(Y_1)\,(z,\,z+s)^{\sim}\in S,\ X_1>\nu(z+s))\,\mathrm{d}z$$

$$+\nu\int_0^1 P(|Y_1|\,|(\nu z,\,X_1)|>r\sigma(\nu);\ \operatorname{sign}(Y_1)\,(\nu z,\,X_1)^{\sim}\in S,\ \nu z<X_1\leqslant\nu(z+s))\,\mathrm{d}z$$

$$+\nu\int_0^1 P(|Y_1|\,|(X_1,\,X_1)|>r\sigma(\nu);\ \operatorname{sign}(Y_1)\,(X_1,\,X_1)^{\sim}\in S,\ X_1\leqslant\nu z)\,\mathrm{d}z$$

$$=J_1+J_2+J_3.$$

Assume that Y_1 is such that $\mathrm{E}|y_1|^{\alpha+\epsilon}<\infty$ for some $\epsilon>0$. Since X_1 is positive and regularly varying with index α, it follows from Breiman's (1965) result that

$$P(Y_1X_1>x)\sim\mathrm{E}[Y_1^+]^{\alpha}P(X_1>x)\quad\text{and}\quad P(Y_1X_1\leqslant -x)\sim\mathrm{E}[Y_1^-]^{\alpha}P(X_1>x),$$

that is, Y_1X_1 is regularly varying with index α. Choose $\sigma(\nu)$ such that

$$\nu P(|X_1Y_1|>\sigma(\nu))\sim 1. \tag{3.16}$$

Now one can follow the lines of the proof above to conclude that J_1 and J_2 are asymptotically negligible, whereas

$$J_3\sim\nu P(|(Y_1X_1,\,Y_1X_1)|>r\sigma(\nu);\ (\operatorname{sign}(Y_1X_1),\,\operatorname{sign}(Y_1X_1))^{\sim}\in S). \tag{3.17}$$

Hence the limiting finite-dimensional distributions are those of an α-stable Lévy motion. The picture changes if Y_1 is a positive regularly varying random variable of index β and $\beta<\alpha$. Choosing $(\sigma(\nu))$ as for (3.16), the same arguments as above show that J_1 and J_2 are asymptotically negligible, but the limits J_3 now characterize a β-stable Lévy motion. Indeed, Y_1X_1 is regularly varying with index β, as follows again from an application of Breiman's result:

$$P(Y_1X_1>x)\sim\mathrm{E}X_1^{\beta}P(Y_1>x)\quad\text{and}\quad P(Y_1X_1\leqslant -x)\sim\mathrm{E}X_1^{\beta}P(Y_1\leqslant -x).$$

One can proceed along the same lines as above to conclude that (3.17) remains valid. Hence the limiting finite-dimensional distributions are those of a β-stable Lévy motion. The case $\alpha=\beta$ can be treated as well, but requires more information about the slowly varying functions in the tails of X_1 and Y_1.

In Mikosch *et al.* (2002) the model (3.14) was considered under the assumption that the intensity $\lambda(\nu)$ of the Poisson process N_ν is a function of ν and increases to infinity. The latter assumption ensures that, in any finite interval of time, there is an increase of the number of sources feeding the network at unit rate. Mikosch *et al.* proved that the normalized and centred workload process $S(t)$ in (3.14) converges to an α-stable Lévy motion provided the *slow-growth condition*

$$F^{\leftarrow}(1 - [\nu\lambda(\nu)]^{-1})/\nu \to 0 \tag{3.18}$$

holds, where $F^{\leftarrow}(t) = \inf\{x : F(x) \geqslant t\},\ t \in (0, 1)$, denotes the generalized inverse of the distribution function F of X_1. In contrast to the latter, it turns out that weak limits of $S(t)$ are fractional Brownian motions if the *fast-growth condition*

$$F^{\leftarrow}(1 - [\nu\lambda(\nu)]^{-1})/\nu \to \infty$$

holds. Now assume that the slow-growth condition (3.18) holds and $\lambda(\nu) \to \infty$. Then the same calculations that led to (2.16) give the corresponding condition for convergence to an α-stable process:

$$\nu\lambda(\nu)\int_0^1 P(\mathbf{X}_{k,s}(z) \in A(r, S))\,\mathrm{d}z \to r^{-\alpha}\mu(A(1, S)). \tag{3.19}$$

Now one can follow the lines of the proof for constant λ. Choose $\sigma(\nu)$ such that

$$\nu\lambda(\nu)P(X_1 > \sigma(\nu)) \to 1.$$

This means that $\sigma(\nu)$ can be chosen as

$$\sigma(\nu) = F^{\leftarrow}(1 - [\nu\lambda(\nu)]^{-1}),$$

and the slow-growth condition then turns into $\sigma(\nu)/\nu \to 0$. The left-hand side of (3.19) for $k = 1$ then reads as follows:

$$\begin{aligned}
&\nu\lambda(\nu)\int_0^1 P(|(z, z+s)| > r\sigma(\nu)/\nu;\ (z, z+s)^{\sim} \in S,\ X_1 > \nu(z+s))\,\mathrm{d}z \\
&\quad + \nu\lambda(\nu)\int_0^1 P(|(\nu z, X_1)| > r\sigma(\nu);\ (\nu z, X_1)^{\sim} \in S,\ \nu z < X_1 \leqslant \nu(z+s))\,\mathrm{d}z \\
&\quad + \nu\lambda(\nu)\int_0^1 P(|(X_1, X_1)| > r\sigma(\nu);\ (X_1, X_1)^{\sim} \in S,\ X_1 \leqslant \nu z)\,\mathrm{d}z \\
&= K_1 + K_2 + K_3.
\end{aligned}$$

Then, since $\sigma(\nu)/\nu \to 0$ and by our choice of $\sigma(\nu)$,

$$K_1 \leqslant \nu\lambda(\nu)P(X_1 > \nu s) = [\nu\lambda(\nu)P(X_1 > \sigma(\nu))]\frac{P(X_1 > \nu s)}{P(X_1 > \sigma(\nu))} = o(1).$$

A similar argument, together with an application of Pratt's lemma, shows that $K_2 \to 0$ and that

$$K_3 \sim \nu\lambda(\nu)P(|(X_1, X_1)| > r\sigma(\nu);\ (X_1, X_1)^{\sim} \in S).$$

Similar calculations in the general case $k \geqslant 1$ show that the limit of (3.19) characterizes

α-stable Lévy motion. Moreover, similar calculations are possible for the corresponding reward processes with changing intensity.

The rationale for the validity of this limit result is the slow-growth condition $\sigma(\nu)/\nu \to 0$ and the fact that $\alpha \in (1, 2)$. These conditions ensure that there is enough 'space' for noise processes $\min(X_i, t - T_i)$ with values in the interval $[\nu, \sigma(\nu)]$. However, if the intensity $\lambda(\nu)$ grows too fast there is no space between ν and $\sigma(\nu)$ and therefore the left-hand side of (3.19) converges to zero. The limiting Gaussian process then exhibits an extremely strong kind of dependence. From an extreme value theory point of view, this behaviour is described in Stegeman (2002).

We finally mention that Maulik *et al.* (2000) and Maulik and Resnick (2001) showed weak convergence to α-stable Lévy motion for the accumulated workload in the framework of the infinite-source Poisson mode. Their model is again a Poisson shot noise process and the convergence of the finite-dimensional distributions could be derived by using the approach advocated in this paper.

3.7. An example where the normalizing function can be slowly varying

In this subsection we wish to illustrate that the normalizing function $\sigma(\nu)$ for the weak convergence to an infinite-variance stable limit can be slowly varying. This is very much in contrast to classical limit theory for i.i.d. vectors and compound Poisson processes.

We consider the Poisson shot noise process

$$S(t) = \sum_{i=1}^{\infty} Y_i I_{[0,X_i)}(t - T_i) = \sum_{i=1}^{\infty} Y_i I_{[T_i T_i + X_i)}(t), \qquad t \geqslant 0,$$

where (T_i), (X_i) and (Y_i) are independent, X_i are i.i.d. positive random variables and Y_i are i.i.d. positive random variables. This model looks similar to the shot noise process of Section 3.6 but, in contrast to the latter, the indicator functions $I_{[T_i,T_i+X_i)}(t)$ are not integrated. In order to achieve weak convergence to an α-stable limit for some $\alpha < 2$ and to identify the limiting Lévy measure, we need to verify that the limits

$$\begin{aligned} I = \int_0^{\nu} & P(Y_1 |(I_{[0,X_1)}(z),\, I_{[0,X_1)}(z + \nu s_1),\, \ldots,\, I_{[0,X_1)}(z + \nu s_k))| > r\sigma(\nu); \\ & (I_{[0,X_1)}(z),\, I_{[0,X_1)}(z + \nu s_1),\, \ldots,\, I_{[0,X_1)}(z + \nu s_k))^{\sim} \in S)\, \mathrm{d}z \end{aligned} \tag{3.20}$$

for $0 = s_0 < \ldots < s_k$ exist. In the event that the vector $(I_{[0,X_1)}(z),\ I_{[0,X_1)}(z + \nu s_1),\ \ldots,\ I_{[0,X_1)}(z + s_k))$ contains only zero components, we interpret the corresponding probabilities as zeroes. Obviously,

$$I = P(Y_1\sqrt{k+1} > r\sigma(\nu))\int_0^{\nu} P(z+\nu s_k < X_1)\,\mathrm{d}z I_{(1,1,\dots,1)/\sqrt{k+1}}(S)$$
$$+ P(Y_1\sqrt{k} > r\sigma(\nu))\int_0^{\nu} P(z+\nu s_{k-1} < X_1 < z+\nu s_k)\,\mathrm{d}z I_{(0,1,\dots,1)/\sqrt{k}}(S) + \dots$$
$$+ P(Y_1 > r\sigma(\nu))\int_0^{\nu} P(z < X_1 < z+\nu s_1)\,\mathrm{d}z I_{(0,\dots,0,1)}(S).$$

Assume that $P(Y_1 > x) \sim cx^{-\alpha}$ for some $\alpha \in (0, 2)$. Then, if $\sigma(\nu) \to \infty$,

$$I \sim cr^{-\alpha}\sigma(\nu)^{-\alpha}\Bigg[(k+1)^{-\alpha/2} I_{(1,1,\dots,1)/\sqrt{k+1}}(S)\int_{\nu s_k}^{\nu(1+s_k)} P(X_1 > z)\,\mathrm{d}z$$
$$+ k^{-\alpha/2} I_{(0,1,\dots,1)/\sqrt{k}}(S)\left[\int_{\nu s_{k-1}}^{\nu(1+s_{k-1})} P(X_1 > z)\,\mathrm{d}z - \int_{\nu s_k}^{\nu(1+s_k)} P(X_1 > z)\,\mathrm{d}z\right]$$
$$+ \dots + I_{(0,\dots,0,1)}(S)\left[\int_0^{\nu} P(X_1 > z)\,\mathrm{d}z - \int_{\nu s_1}^{\nu(1+s_1)} P(X_1 > z)\,\mathrm{d}z\right]\Bigg].$$

Regular variation of $\sigma(\nu)$ with some non-negative index and $\sigma(\nu) \to \infty$ are necessary conditions for weak convergence of the shot noise process to a stable limit. Thus, in order to ensure that $I = I_\nu$ has a limit as $\nu \to \infty$, one needs to assume that $c\sigma(\nu)^{-\alpha}\int_0^{y\nu} P(X_1 > z)\,\mathrm{d}z$ has a limit for every $y > 0$ and that $\sigma(\nu) \to \infty$. This means we have to assume that $\int_0^{\nu} P(X_1 > z)\,\mathrm{d}z$ is regularly varying with some index $\beta \geqslant 0$ and, for $\beta = 0$, it is not equivalent to a constant. The condition $\sigma(\nu) \to \infty$ is then only possible if $\mathrm{E}X_1 = \infty$. The monotone density theorem for regularly varying functions (cf. Embrechts *et al.* 1997, p. 586) implies that $P(X_1 > x)$ is regularly varying with index $\beta - 1$. Hence $\beta \leqslant 1$ is a necessary condition. We conclude that we can choose

$$c\sigma(\nu)^{-\alpha}\int_0^{\nu} P(X_1 > z)\,\mathrm{d}z \sim 1,$$

that is, $\sigma(\nu)$ is regularly varying with index β/α. Then we have

$$I \sim r^{-\alpha}\Big[(k+1)^{-\alpha/2} I_{(1,1,\dots,1)/\sqrt{k+1}}(S)[(s_k+1)^{\beta} - s_k^{\beta}]$$
$$+ k^{-\alpha/2} I_{(0,1,\dots,1)/\sqrt{k}}(S)[(s_{k-1}+1)^{\beta} - s_{k-1}^{\beta} - (s_k+1)^{\beta} + s_k^{\beta}] + \dots$$
$$+ I_{(0,0,\dots,1)}(S)[1 - (s_1+1)^{\beta} + s_1^{\beta}]\Big].$$

We omit the verification of the other assumptions of Corollary 2.7.

We consider some special cases. Assume first that $\beta = 0$, that is, $\int_0^{\nu} P(X_1 > z)\,\mathrm{d}z$ is a slowly varying function, or equivalently $P(X_1 > x)$ is regularly varying with index -1 and $\mathrm{E}X_1 = \infty$. Then $\sigma(\nu)$ is a slowly varying function and $I \sim r^{-\alpha} I_{(0,0,\dots,1)}(S)$. The latter corresponds to a stable degenerate vector where all components are zero with the exception of the last one.

Another special case corresponds to $\beta = 1$, that is, $P(X_1 > x)$ is slowly varying. Then $\sigma(\nu)$ is regularly varying with index $1/\alpha$ and $I \sim r^{-\alpha}(k+1)^{-\alpha/2} I_{(1,1,\ldots,1)/\sqrt{k+1}}(S)$. The latter corresponds to the case of a stable vector whose components are identical. The same limit occurs for the compound Poisson process; cf. Section 3.1.

3.8. Regular variation and convergence of point proceses

It is well known from classical extreme value theory that regular variation with index $-\alpha < 0$ of the right tail of the distribution of the i.i.d. random variables X_i is equivalent to the weak convergence of the point processes

$$\sum_{i=1}^{n} \varepsilon_{X_i/\sigma(n)} \Rightarrow \mathrm{PRM}(\mu), \tag{3.21}$$

where σ_n is the $(1 - n^{-1})$ quantile of the distribution of X_1 and the limiting process is a Poisson random measure with mean measure μ of the interval $(a, b]$ given by $a^{-\alpha} - b^{-\alpha}$. Here $\Rightarrow$ denotes weak convergence in the space of point measures on $(0, \infty)$ equipped with the vague topology; see Kallenberg (1983) or Resnick (1987). The convergence in (3.21) is equivalent to the weak convergence of the maxima $M_n = \max(X_1, \ldots, X_n)$, that is,

$$M_n/\sigma_n \Rightarrow Y,$$

where Y has the Fréchet distribution $P(Y \leqslant x) = \exp\{-x^{-\alpha}\} = \Phi_\alpha(x)$, $x > 0$.

Analogous results hold for the extremes and point processes constructed from the Poisson shot noise. To be precise, introduce the point processes

$$R_\nu = \sum_{n=1}^{N(\nu)} \varepsilon_{X_n(\nu - T_n)/\sigma(\nu)},$$

where $\sigma(\nu)$ is supposed to satisfy the relation

$$\nu \int_0^1 P(\sigma(\nu)x < X_1(\nu u))\,\mathrm{d}u \sim x^{-\alpha} \qquad \text{for every } x > 0. \tag{3.22}$$

Using the order statistics property of the Poisson process, it is not difficult to see that

$$P\Big(\max_{i=1,\ldots,N(\nu)} X_i(\nu - T_i) \leqslant x\Big) = P\Big(\max_{i=1,\ldots,N(\nu)} X_i(\nu U_i) \leqslant x\Big),$$

where N, the sequence (U_i) of random variables i.i.d. uniform on $(0, 1)$, and (X_i) are independent. A conditioning argument gives, for $x > 0$,

$$P\left([\sigma(\nu)]^{-1} \max_{i=1,\ldots,N(\nu)} X_i(\nu - T_i) \leqslant x\right) = \exp\left\{-\nu \int_0^1 P(X_1(\nu u) > x\sigma(\nu))\,\mathrm{d}u\right\}. \tag{3.23}$$

Hence the right-hand side converges to $\Phi_\alpha(x)$ if and only if the regular variation condition (3.22) holds.

Proposition 3.3. *The relation* $R_\nu \Rightarrow \mathrm{PRM}(\mu)$ *with mean measure* $\mu(a, b] = a^{-\alpha} - b^{-\alpha}$, $0 < a < b$, *holds if and only if* (3.22) *is satisfied.*

Proof. We commence by assuming that (3.22) holds. According to Kallenberg's theorem (see Resnick (1987), p. 157), one has to show that, for any $0 < a < b$,

$$\mathrm{E}R_\nu((a, b]) \to \mu(a, b] \tag{3.24}$$

and for $B = (c_1, d_1] \cup \cdots \cup (c_k, d_k]$, $0 < c_1 < d_1 < \ldots < c_k < d_k$,

$$P(R_\nu(B) = 0) \to \mathrm{e}^{-\mu(B)}. \tag{3.25}$$

We have, by the order statistics property of the Poison process,

$$\begin{aligned}\mathrm{E}R_\nu((a, b]) &= \mathrm{E}\left(\sum_{n=1}^{N(\nu)} I_{(a,b]}(X_n(\nu - T_n)/\sigma(\nu))\right)\\ &= \mathrm{E}\left(\sum_{n=1}^{N(\nu)} I_{(a,b]}(X_n(\nu U_n)/\sigma(\nu))\right),\end{aligned}$$

where (T_n), (X_n) and (U_n) are independent. Hence, using assumption (3.22),

$$\begin{aligned}\mathrm{E}R_\nu((a, b]) &= \nu P(\sigma(\nu)a < X_1(\nu U_1) \leqslant \sigma(\nu)b)\\ &= \int_0^\nu P(\sigma(\nu)a < X_1(u) \leqslant \sigma(\nu)b)\,\mathrm{d}u\\ &\sim a^{-\alpha} - b^{-\alpha} = \mu(a, b].\end{aligned}$$

This proves (3.24). Now we turn to the proof of (3.25). Notice that

$$P(R_\nu(B) = 0) = P(Q_\nu(B) = 0) = \mathrm{E}[P(Q_\nu(B) = 0|N)],$$

where the random variable

$$Q_\nu(B) = \sum_{n=1}^{N(\nu)} I_B(\nu U_n)/\sigma(\nu))$$

is conditionally $Bin(N(\nu), P(X_1(\nu U_1)/\sigma(\nu) \in B))$ distributed. By the law of large numbers, $N(\nu)/\nu \stackrel{\text{a.s.}}{\to} 1$. Therefore, and by virtue of (3.22), it follows that, with probability 1 as $\nu \to \infty$,

$$N(\nu)P(X_1(\nu U_1)/\sigma(\nu) \in B) \to \mu(B).$$

This and Poisson's limit theorem imply that $Q_\nu(B) \Rightarrow Poi(\mu(B))$, conditionally on N. This, together with a dominated convergence argument, yields

$$\mathrm{E}[P(Q_\nu = 0 \,|\, N)] \to \mathrm{e}^{-\mu(B)}.$$

This proves the sufficiency part.

Now assume that $R_\nu \Rightarrow \mathrm{PRM}(\mu)$. Then (3.23) necessarily has a Fréchet limit since, for $x > 0$,

$$P\left([\sigma(\nu)]^{-1} \max_{i=1,\ldots,n(\nu)} X_i(\nu - T_i) \leqslant x\right) = P(R_\nu((x, \infty)) = 0) \to \mathrm{e}^{-\mu(x,\infty)} = \Phi_\alpha(x),$$

and the argument before the proposition then yields that (3.22) holds. This concludes the proof. □

Remark 3.4. The extremal behaviour of the shot noise process, not the noise processes themselves, has been intensively investigated in the case where a stationary version of S exists. We refer to Doney and O'Brien (1991), Hsing and Teugels (1989), McCormick (1997) and the references therein.

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