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Asymptotics of random processes with immigration I: Scaling limits

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Let $(X_1, \xi_1), (X_2, \xi_2), \dots$ be i.i.d. copies of a pair (X, ξ) where X is a random process with paths in the Skorokhod space $D[0,\infty)$ and ξ is a positive random variable. Define $S_k := \xi_1 + \cdots + \xi_k$, $k \in \mathbb{N}_0$ and $Y(t) := \sum_{k\geq 0} X_{k+1}(t - S_k)\mathbb{1}_{\{S_k \leq t\}}$, $t \geq 0$. We call the process $(Y(t))_{t\geq 0}$ random process with immigration at the epochs of a renewal process. We investigate weak convergence of the finite-dimensional distributions of $(Y(ut))_{u>0}$ as $t \to \infty$. Under the assumptions that the covariance function of X is regularly varying in $(0,\infty) \times (0,\infty)$ in a uniform way, the class of limiting processes is rather rich and includes Gaussian processes with explicitly given covariance functions, fractionally integrated stable Lévy motions and their sums when the law of ξ belongs to the domain of attraction of a stable law with finite mean, and conditionally Gaussian processes with explicitly given (conditional) covariance functions, fractionally integrated inverse stable subordinators and their sums when the law of ξ belongs to the domain of attraction of a stable law with infinite mean.

Keywords: random process with immigration; renewal theory; shot noise processes; weak convergence of finite-dimensional distributions

1. Introduction

1.1. Random processes with immigration at the epochs of a renewal process

Denote by $D[0,\infty)$ and $D(0,\infty)$ the Skorokhod spaces of right-continuous real-valued functions which are defined on $[0,\infty)$ and $(0,\infty)$, respectively, and have finite limits from the left at each point of the domain. Throughout the paper, we abbreviate $D[0,\infty)$ by D. Let $X := (X(t))_{t\in\mathbb{R}}$ be a random process with paths in D satisfying $X(t) = 0$ for all $t < 0$, and let ξ be a positive random variable. Arbitrary dependence between X and ξ is allowed. It is worth stating explicitly that we do not exclude the possibility $X = h$ a.s. for a deterministic function h.

Further, let $(X_1, \xi_1), (X_2, \xi_2), \dots$ be i.i.d. copies of the pair (X, ξ) and denote by $(S_k)_{k\in\mathbb{N}_0}$ the zero-delayed random walk with increments ξ_j, that is,

$$S_0 := 0, \qquad S_k := \xi_1 + \cdots + \xi_k, \qquad k \in \mathbb{N}.$$

We write $(\nu(t))_{t\in\mathbb{R}}$ for the corresponding first-passage time process, that is,

$$\nu(t) := \inf\{k \in \mathbb{N}_0 : S_k > t\} = \#\{k \in \mathbb{N}_0 : S_k \le t\}, \qquad t \in \mathbb{R},$$

where the last equality holds a.s. The process $Y := (Y(t))_{t\in\mathbb{R}}$ defined by

$$Y(t) := \sum_{k\ge 0} X_{k+1}(t - S_k) = \sum_{k=0}^{\nu(t)-1} X_{k+1}(t - S_k), \qquad t \in \mathbb{R} \tag{1}$$

will be called *random process with immigration at the epochs of a renewal process* or, for short, *random process with immigration*. The interpretation is as follows: at time $S_0 = 0$ the immigrant 1 starts running a random process X_1, for $k \in \mathbb{N}$, at time S_k the immigrant $k + 1$ starts running a random process X_{k+1}, $Y(t)$ being the sum of all processes run by the immigrants up to and including time t. We advocate using this term for two reasons. First, we believe that it is more informative than the more familiar term *renewal shot noise process with random response functions* X_k; in particular, the random process Y defined by (1) has little in common with the originally defined shot noise processes [45] intended to model the real shot noise in vacuum tubes which were based on Poisson inputs and deterministic response functions. Second, the new term was inspired by the fact that if X is a continuous-time branching process, then Y is known in the literature as a *branching process with immigration*.

Random processes with immigration have been used to model various phenomena. An incomplete list of possible areas of applications includes anomalous diffusion in physics [37], earthquakes occurrences in geology [47], rainfall modeling in meteorology [43,49], network traffic in computer sciences [33,38,41,42] as well as insurance [30,31] and finance [29,44]. Further references concerning mainly renewal shot noise processes can be found in [2,17,22,48]. Although we do not have any particular application in mind, our results are potentially useful for either of the aforementioned fields.

1.2. Weak convergence of random processes with immigration

The paper at hand is part of a series of papers that further contains the references [21–23] in which we investigate the asymptotic distribution of Y. When $\mu := \mathbb{E}\xi < \infty$ and $X(t)$ tends to 0 quickly as $t \to \infty$ (more precisely, if $t \mapsto \mathbb{E}[|X(t)| \wedge 1]$ is a directly Riemann integrable function), then, under mild technical assumptions, $(Y(u + t))_{u\ge 0}$ converges to a stationary version of the process. This convergence is investigated in [23]. In the present paper, we focus on the case where the law of ξ is in the domain of attraction of a stable law of index $\alpha \neq 1$ and, if $\mu < \infty$ (equivalently, $\alpha > 1$), either $\mathbb{E}[X(t)]$ or $\mathrm{Var}[X(t)]$ is too large for convergence to stationarity. In this situation, we investigate the weak convergence of the finite-dimensional distributions of $Y_t(u) := a(t)^{-1}(Y(ut) - b(ut))$ with suitable norming constants $a(t) > 0$ and shifts $b(t) \in \mathbb{R}$. This convergence is mainly regulated by two factors: the tail behavior of ξ and the asymptotics of the finite-dimensional distributions of $X(t)$ as $t \to \infty$. The various combinations of these give rise to a broad spectrum of possible limit results. In this paper, assuming that $h(t) := \mathbb{E}[X(t)]$ is finite for all $t \geq 0$, we start with the decomposition

$$Y(t) - b(t) = \Big(Y(t) - \sum_{k \geq 0} h(t - S_k)\mathbb{1}_{\{S_k \leq t\}}\Big) + \Big(\sum_{k \geq 0} h(t - S_k)\mathbb{1}_{\{S_k \leq t\}} - b(t)\Big) \tag{2}$$

and observe that $Y_t(u)$ may converge if at least one summand in (2), properly normalized, converges weakly.

The asymptotic behavior of the second summand, properly normalized, is driven by the functional limit theorems for the first-passage time process $(\nu(t))_{t \geq 0}$ as well as the behavior of the function h at infinity.

The asymptotics of the first summand, properly normalized, is accessible via martingale central limit theory or convergence results for triangular arrays. When $\mathbb{E}\xi$ is finite, the normalizing constants and limiting processes for the first summand are completely determined by properties of X, the influence of the law of ξ is small. This phenomenon can easily be understood: the randomness induced by the ξ_k's is governed by the law of large numbers for $\nu(t)$ and is thus degenerate in the limit. When $\mathbb{E}\xi$ is infinite and $\mathbb{P}\{\xi > t\}$ is regularly varying with index larger than -1, $\nu(t)$, properly normalized, weakly converges to a non-degenerate law. Hence, unlike the finite-mean case, the randomness induced by ξ persists in the limit.

It turns out that there are situations in which one of the summands in (2) dominates (cases $p = 0$ and $p = 1$ of Theorem 2.4; cases $q = 0$ and $q = 1$ of Theorem 2.5; the case where $h \equiv 0$), and those in which the contributions of the summands are comparable (case $p \in (0, 1)$ of Theorem 2.4 and case $q \in (0, 1)$ of Theorem 2.5). A nice feature of the former situation is that possible dependence of X and ξ gets neutralized by normalization (provided $\lim_{t \to \infty} a(t) = +\infty$) so that the limit results are only governed by individual contributions of X and ξ. Suppose, for the time being, that the latter situation prevails, that is, the two summands in (2) are of the same order, and that X and ξ are independent. From the discussion above it should be clear that whenever $\mathbb{E}\xi$ is finite, the two limit random processes corresponding to the summands in (2) are independent, whereas this is not the case, otherwise. Still, we are able to show that the summands in (2) converge jointly.

When X and ξ are dependent, proving such a joint convergence remains an open problem. In the particular case where $X(t) = \mathbb{1}_{\{|\log(1-W)| > t\}}$ and $\xi = |\log W|$ for some random variable $W \in (0, 1)$ a.s., this problem, already reported in Section 1 of [24], turned out to be the major obstacle on the way towards obtaining the description of *all* possible modes of weak convergence of the number of empty boxes in the Bernoulli sieve.

Adequacy of the aforementioned approach was realized by the authors some time ago, and as a preparation for its implementation the articles [21,22] were written. In the first of these papers, functional limit theorems for the second summand have been established in the case where h is eventually increasing,[1] while in the second convergence of the finite-dimensional distributions of the second summand has been proved in the case where h is eventually decreasing.[2]

[1] We call a function h increasing (decreasing) if $s < t$ implies $h(s) \leq h(t)$ (resp., $h(s) \geq h(t)$) and strictly increasing (decreasing) if $s < t$ implies $h(s) < h(t)$ (resp., $h(s) > h(t)$).

[2] The present paper does not offer new results about weak convergence of the second summand in (2) alone. However, the joint convergence of the summands in (2) is treated here for the first time.

1.3. Bibliographic comments and known results

In the case where ξ has an exponential law, the process Y (or its stationary version) is a *Poisson shot noise process*. Weak convergence of Poisson shot noise processes has received considerable attention. In some papers of more applied nature weak convergence of $Y_t(u)$ for X having a specific form is investigated. In the list to be given next η denotes a random variable independent of ξ and f a deterministic function which satisfies certain restrictions which are specified in the cited papers:

- $X(t) = \mathbb{1}_{\{\eta>t\}}$ and $X(t) = t \wedge \eta$, functional convergence, see [41];
- $X(t) = \eta f(t)$, stationary version of Y, functional convergence, see [29];
- $X(t) = f(t \wedge \eta)$, convergence of finite-dimensional distributions, see [33]; functional convergence, see [42];
- $X(t) = \eta_1/\eta_2 f(t\eta_2)$, stationary version, convergence of finite-dimensional distributions, see [11,12].

The articles [16,27,30,32,35] are of more theoretical nature, and study weak convergence of $Y_t(u)$ for general (not explicitly specified) X. The work [27] contains further pointers to relevant literature which could have extended our list of particular cases given above.

In the case where the law of ξ is exponential, the variables $Y_t(u)$ have infinitely divisible laws with characteristic functions of a rather simple form. Furthermore, the convergence, as $t \to \infty$, of these characteristic functions to a characteristic function of a limiting infinitely divisible law follows from the general theory. Also, in this context Poisson random measures arise naturally and working with them considerably simplifies the analysis. In the cases where the law of ξ is not exponential, the aforementioned approaches are not applicable. We are aware of several papers in which weak convergence of processes Y, properly normalized, centered and rescaled, is investigated in the case where ξ has distribution other than exponential. Iglehart [18] has proved weak convergence of

$$\frac{1}{\sqrt{n}}\biggl(\sum_{k\geq 0} X_{k+1}\bigl(u - n^{-1}S_k\bigr)\mathbb{1}_{\{S_k \leq nu\}} - \frac{n}{\mathbb{E}\xi}\int_0^u \mathbb{E}\bigl[X(y)\bigr]\,\mathrm{d}y\biggr)$$

in $D[0,1]$ to a Gaussian process, as $n \to \infty$, under rather restrictive assumptions (in particular, concerning the existence of moments of order four). See also Theorem 1 on page 103 of [8] for a similar result with $X(t) = \mathbb{1}_{\{\eta>t\}}$ in a more general setting. For $X(t) = \int_0^t f(s,\eta)\,\mathrm{d}s$, weak convergence of $(Y_t(u))_{0\leq u\leq 1}$ on $D[0,1]$ as $t \to \infty$ was established in [19] under the assumptions that ξ and η are independent, that $\int_0^\infty |f(s,x)|\,\mathrm{d}s < \infty$ for every $x \in \mathbb{R}$ and some other conditions. For $X(t) = \mathbb{1}_{\{\eta>t\}}$, weak convergence of finite-dimensional distributions of $(Y_t(u))_u$, as $t \to \infty$, has been settled in [38] under the assumption that ξ and η are independent and some moment-type conditions.

Last but not least, weak convergence of $Y_t(1)$ has been much investigated, especially in the case where X is a branching process (see, for instance, [3,26,40]).

1.4. Additional definitions

Throughout the paper, we assume that $h(t) := \mathbb{E}[X(t)]$ is finite for all $t \geq 0$ and that the covariance

$$f(s,t) := \mathrm{Cov}\big[X(s), X(t)\big] = \mathbb{E}\big[X(s)X(t)\big] - \mathbb{E}\big[X(s)\big]\mathbb{E}\big[X(t)\big]$$

is finite for all $s, t \geq 0$. The variance of X will be denoted by v, that is, $v(t) := f(t,t) = \mathrm{Var}[X(t)]$. In what follows we assume that $h, v \in D$. By Lebesgue's dominated convergence theorem, local uniform integrability of X^2 is sufficient for this to be true since the paths of X belong to D. $h, v \in D$ implies that h and v are a.e. continuous and locally bounded. Consequently, $\int_0^t h(y)\,\mathrm{d}y$ and $\int_0^t v(y)\,\mathrm{d}y$ are well-defined as Riemann integrals.

Regular variation in $\mathbb{R}^2$

Recall that a positive measurable function ℓ, defined on some neighborhood of ∞, is called *slowly varying* at ∞ if $\lim_{t\to\infty} \frac{\ell(ut)}{\ell(t)} = 1$ for all $u > 0$, see [7], page 6.

Definition 1.1. *A function* $r : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *is regularly varying*[3] *in* $\mathbb{R}_+^2 := (0,\infty) \times (0,\infty)$ *if there exists a function* $C : \mathbb{R}_+^2 \to (0,\infty)$, *called limit function*, *such that*

$$\lim_{t\to\infty} \frac{r(ut, wt)}{r(t,t)} = C(u,w), \qquad u, w > 0.$$

The definition implies that $r(t,t)$ is regularly varying at ∞, that is, $r(t,t) \sim t^{\beta}\ell(t)$ as $t \to \infty$ for some ℓ slowly varying at ∞ and some $\beta \in \mathbb{R}$ which is called the *index of regular variation*. In particular, $C(a,a) = a^{\beta}$ for all $a > 0$ and further

$$C(au, aw) = C(a,a)C(u,w) = a^{\beta}C(u,w)$$

for all $a, u, w > 0$.

Definition 1.2. *A function* $r : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *will be called fictitious regularly varying of index* β *in* $\mathbb{R}_+^2$ *if*

$$\lim_{t\to\infty} \frac{r(ut, wt)}{r(t,t)} = C(u,w), \qquad u, w > 0,$$

where $C(u,u) := u^{\beta}$ *for* $u > 0$ *and* $C(u,w) := 0$ *for* $u, w > 0$, $u \neq w$. *A function* r *will be called wide-sense regularly varying of index* β *in* $\mathbb{R}_+^2$ *if it is either regularly varying or fictitious regularly varying of index* β *in* $\mathbb{R}_+^2$.

The function C corresponding to a fictitious regularly varying function will also be called *limit function*.

[3]The canonical definition of the regular variation in $\mathbb{R}_+^2$ (see, for instance, [9]) requires nonnegativity of r.

Definition 1.3. *A function $r : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is uniformly regularly varying of index β in strips in $\mathbb{R}_+^2$ if it is regularly varying of index β in $\mathbb{R}_+^2$ and*

$$\lim_{t\to\infty} \sup_{a\le u\le b} \left| \frac{r(ut,(u+w)t)}{r(t,t)} - C(u,u+w) \right| = 0 \tag{3}$$

for every $w > 0$ and all $0 < a < b < \infty$.

Limit processes for $Y_t(u)$

The processes introduced in Definition 1.4 arise as weak limits of the first summand in (2) in the case $\mathbb{E}\xi < \infty$. We shall check that they are well-defined at the beginning of Section 4.1.

Definition 1.4. *Let C be the limit function for a wide-sense regularly varying function (see Definition 1.2) in $\mathbb{R}_+^2$ of index β for some $\beta \in (-1,\infty)$. We shall denote by $V_\beta := (V_\beta(u))_{u>0}$ a centered Gaussian process with covariance function*

$$\mathbb{E}\big[V_\beta(u)V_\beta(w)\big] = \int_0^u C(u-y, w-y)\,\mathrm{d}y, \qquad 0 < u \le w,$$

when $C(s,t) \ne 0$ for some $s,t > 0$, $s \ne t$, and a centered Gaussian process with independent values and variance $\mathbb{E}[V_\beta^2(u)] = (1+\beta)^{-1}u^{1+\beta}$, otherwise.

Definition 1.5 reminds the notion of an inverse subordinator.

Definition 1.5. *For $\alpha \in (0,1)$, let $W_\alpha := (W_\alpha(t))_{t\ge 0}$ be an α-stable subordinator (nondecreasing Lévy process) with Laplace exponent $-\log \mathbb{E}[\exp(-zW_\alpha(t))] = \Gamma(1-\alpha)tz^\alpha$, $z \ge 0$, where $\Gamma(\cdot)$ is the gamma function. The inverse α-stable subordinator $W_\alpha^\leftarrow := (W_\alpha^\leftarrow(s))_{s\ge 0}$ is defined by*

$$W_\alpha^\leftarrow(s) := \inf\big\{t \ge 0 : W_\alpha(t) > s\big\}, \qquad s \ge 0.$$

The processes introduced in Definition 1.6 arise as weak limits of the first summand in (2) in the case $\mathbb{E}\xi = \infty$. We shall check that these are well-defined in Lemma 5.7.

Definition 1.6. *Let $W_\alpha^\leftarrow$ be an inverse α-stable subordinator and C the limit function for a wide-sense regularly varying function (see Definition 1.2) in $\mathbb{R}_+^2$ of index β for some $\beta \in [-\alpha,\infty)$. We shall denote by $Z_{\alpha,\beta} := (Z_{\alpha,\beta}(u))_{u>0}$ a process which, given $W_\alpha^\leftarrow$, is centered Gaussian with (conditional) covariance*

$$\mathbb{E}\big[Z_{\alpha,\beta}(u)Z_{\alpha,\beta}(w)|W_\alpha^\leftarrow\big] = \int_{[0,u]} C(u-y,w-y)\,\mathrm{d}W_\alpha^\leftarrow(y), \qquad 0 < u \le w,$$

when $C(s,t) \ne 0$ for some $s,t > 0$, $s \ne t$, and a process which, given W_α, is centered Gaussian with independent values and (conditional) variance $\mathbb{E}[Z_{\alpha,\beta}(u)^2|W_\alpha^\leftarrow] = \int_{[0,u]}(u-y)^\beta\,\mathrm{d}W_\alpha^\leftarrow(y)$, otherwise.

Throughout the paper, we use $\stackrel{\mathrm{d}}{\to}$, $\stackrel{\mathbb{P}}{\to}$ and $\Rightarrow$ to denote weak convergence of one-dimensional distributions, convergence in probability and convergence in distribution in a function space, respectively. Additionally, we write $Z_t(u) \stackrel{\text{f.d.}}{\Rightarrow} Z(u)$, $t \to \infty$ to denote weak convergence of finite-dimensional distributions, that is, for any $n \in \mathbb{N}$ and any $0 < u_1 < u_2 < \cdots < u_n < \infty$,

$$\big(Z_t(u_1), \dots, Z_t(u_n)\big) \stackrel{\mathrm{d}}{\to} \big(Z(u_1), \dots, Z(u_n)\big), \qquad t \to \infty.$$

We stipulate hereafter that ℓ, $\widehat{\ell}$ and ℓ^* denote functions slowly varying at ∞ and that all unspecified limit relations hold as $t \to \infty$.

2. Main results

2.1. Asymptotic distribution of the first summand in (2)

Proposition 2.1 (case $\mathbb{E}\xi < \infty$) and Proposition 2.2 (case $\mathbb{E}\xi = \infty$) deal with the asymptotics of the first summand in (2).

Proposition 2.1. *Assume that*:

- $\mu := \mathbb{E}\xi \in (0, \infty)$;
- $f(u, w) = \mathrm{Cov}[X(u), X(w)]$ *is either uniformly regularly varying in strips in* $\mathbb{R}^2_+$ *or fictitious regularly varying in* $\mathbb{R}^2_+$, *in either of the cases*, *of index* β *for some* $\beta \in (-1, \infty)$ *and with limit function* C; *when* $\beta = 0$, *there exists a positive monotone function* u *satisfying* $v(t) = \mathrm{Var}[X(t)] \sim u(t)$ *as* $t \to \infty$;
- *for all* $y > 0$

$$v_y(t) := \mathbb{E}\big[\big(X(t) - h(t)\big)^2 \mathbb{1}_{\{|X(t)-h(t)|>y\sqrt{tv(t)}\}}\big] = o\big(v(t)\big), \qquad t \to \infty. \tag{4}$$

Then

$$\frac{Y(ut) - \sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k \leq ut\}}}{\sqrt{\mu^{-1} t v(t)}} \stackrel{f.d.}{\Rightarrow} V_\beta(u), \qquad t \to \infty, \tag{5}$$

where V_β *is a centered Gaussian process as introduced in Definition* 1.4.

Proposition 2.2. *Assume that*:

- X *is independent of* ξ;
- *for some* $\alpha \in (0, 1)$ *and some* ℓ^*

$$\mathbb{P}\{\xi > t\} \sim t^{-\alpha} \ell^*(t), \qquad t \to \infty; \tag{6}$$

- $f(u, w) = \mathrm{Cov}[X(u), X(w)]$ *is either uniformly regularly varying in strips in* $\mathbb{R}^2_+$ *or fictitious regularly varying in* $\mathbb{R}^2_+$, *in either of cases*, *of index* β *for some* $\beta \in [-\alpha, \infty)$ *and with limit function* C; *when* $\beta = -\alpha$, *there exists a positive increasing function* u *with* $\lim_{t\to\infty} \frac{v(t)}{\mathbb{P}\{\xi > t\} u(t)} = 1$;

- *for all* $y > 0$

$$v_y(t) := \mathbb{E}\big[\big(X(t) - h(t)\big)^2 \mathbb{1}_{\{|X(t)-h(t)|>y\sqrt{v(t)/\mathbb{P}\{\xi>t\}}\}}\big] = o\big(v(t)\big), \qquad t \to \infty. \tag{7}$$

Then

$$\sqrt{\frac{\mathbb{P}\{\xi > t\}}{v(t)}}\Big(Y(ut) - \sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k \leq ut\}}\Big) \stackrel{f.d.}{\Rightarrow} Z_{\alpha,\beta}(u), \qquad t \to \infty,$$

where $Z_{\alpha,\beta}$ *is a conditionally Gaussian process as introduced in Definition 1.6.*

Remark 2.3. There is an interesting special case of Proposition 2.2 in which the finite-dimensional distributions of Y converge weakly, that is, without normalization and centering. Namely, if $h(t) \equiv 0$, $\lim_{t\to\infty} v(t)/\mathbb{P}\{\xi > t\} = c$ for some $c > 0$, and the assumptions of Proposition 2.2 hold (note that $\beta = -\alpha$ and one may take $u(t) \equiv c$), then

$$Y(ut) \stackrel{\text{f.d.}}{\Rightarrow} \sqrt{c}Z_{\alpha,-\alpha}(u).$$

When $h(t) = \mathbb{E}[X(t)]$ is not identically zero, the centerings used in Propositions 2.1 and 2.2 are random which is undesirable. Theorem 2.4 (case $\mathbb{E}\xi < \infty$) and Theorem 2.5 (case $\mathbb{E}\xi = \infty$) stated below give limit results with non-random centerings. These are obtained by combining the results concerning weak convergence of the second summand in (2) with Proposition 2.1 and Proposition 2.2, respectively.

2.2. Domains of attraction

To fix notation for our main results, we recall here that the law of ξ belongs to the domain of attraction of a 2-stable (normal) law if, and only if, either $\sigma^2 := \operatorname{Var}\xi < \infty$, or $\operatorname{Var}\xi = \infty$ and

$$\mathbb{E}\big[\xi^2 \mathbb{1}_{\{\xi\leq t\}}\big] \sim \ell^*(t) \tag{8}$$

for some ℓ^*. Further, the law of ξ belongs to the domain of attraction of an α-stable law, $\alpha \in (0, 2)$ if, and only if,

$$\mathbb{P}\{\xi > t\} \sim t^{-\alpha}\ell^*(t) \tag{9}$$

for some ℓ^*. In the present paper, we do not treat the case $\alpha = 1$, for it is technically more complicated than the others and does not shed any new light on weak convergence of random processes with immigration.

If $\mu = \mathbb{E}\xi = \infty$, then necessarily $\alpha \in (0, 1)$ (because we excluded the case $\alpha = 1$) and according to Corollary 3.4 in [36] we have

$$\mathbb{P}\{\xi > t\}\nu(ut) \Rightarrow W_\alpha^{\leftarrow}(u) \tag{10}$$

in the J_1-topology on D.

If $\mu < \infty$, then necessarily $\alpha \in (1, 2]$ (where $\alpha = 2$ corresponds to the case of attraction to a normal law) and according to Theorem 5.3.1 and Theorem 5.3.2 in [14] or Section 7.3.1 in [50] we have

$$\frac{\nu(ut) - \mu^{-1}ut}{\mu^{-1-1/\alpha}c(t)} \Rightarrow \mathcal{S}_\alpha(u), \tag{11}$$

where:

- if $\sigma^2 < \infty$, then $\mathcal{S}_2 := (\mathcal{S}_2(u))_{u\geq 0}$ is a Brownian motion; $c(t) = \sigma\sqrt{t}$ and the convergence takes place in the J_1-topology on D;
- if $\sigma^2 = \infty$ and (8) holds, then $c(t)$ is some positive continuous function such that

$$\lim_{t\to\infty} t\ell^*\big(c(t)\big)/c(t)^2 = 1,$$

  and the convergence takes place in the J_1-topology on D;
- if in (9) $\alpha \in (1, 2)$, then $\mathcal{S}_\alpha := (\mathcal{S}_\alpha(u))_{u\geq 0}$ is a spectrally negative α-stable Lévy process such that $\mathcal{S}_\alpha(1)$ has the characteristic function

$$\mathbb{E}\big[\exp\big(iz\mathcal{S}_\alpha(1)\big)\big] = \exp\big\{-|z|^\alpha \Gamma(1-\alpha)\big(\cos(\pi\alpha/2) + \mathrm{i}\,\mathrm{sign}(z)\sin(\pi\alpha/2)\big)\big\}, \qquad z \in \mathbb{R},$$

  where $\Gamma(\cdot)$ denotes Euler's gamma function; $c(t)$ is some positive continuous function satisfying

$$\lim_{t\to\infty} t\ell^*\big(c(t)\big)/c(t)^\alpha = 1,$$

  and the convergence takes place in the M_1-topology on D.

In any case, $c(t)$ is regularly varying at ∞ of index $1/\alpha$, see Lemma 5.3. We refer to [50] for extensive information concerning both the J_1- and M_1-convergence on D.

2.3. Scaling limits of random processes with immigration

Theorem 2.4. *Assume that the law of ξ belongs to the domain of attraction of an α-stable law, $\alpha \in (1, 2]$, that c is as in (11), that h is eventually monotone and not identically zero, and that the following limit*

$$p := \lim_{t\to\infty} \frac{c(t)^2 h(t)^2}{\int_0^t v(y)\,\mathrm{d}y + c(t)^2 h(t)^2} \in [0, 1],$$

exists. Assume further that:

- *if $p < 1$, then the assumptions of Proposition 2.1 hold;*
- *if $p > 0$, then $h(t) \sim t^\rho \widehat{\ell}(t)$ as $t \to \infty$ for some $\rho > -1/\alpha$ and some $\widehat{\ell}$;*
- *if $p = 1$, then $\lim_{t\to\infty} \int_0^t v(y)\,\mathrm{d}y = \infty$ and there exists a positive monotone function u such that $v(t) \sim u(t)$, $t \to \infty$, or v is directly Riemann integrable on $[0, \infty)$;*
- *if $p \in (0, 1)$, then X is independent of ξ.*

Then, as $t \to \infty$,

$$\frac{Y(ut) - (1/\mu)\int_0^{ut} h(y)\,\mathrm{d}y}{\sqrt{\int_0^t v(y)\,\mathrm{d}y + c(t)^2 h(t)^2}} \overset{f.d.}{\Rightarrow} \sqrt{\frac{(1-p)(1+\beta)}{\mu}} V_\beta(u) + \sqrt{p}\mu^{-(\alpha+1)/\alpha} \int_0^u (u-y)^\rho \,\mathrm{d}\mathcal{S}_\alpha(y), \tag{12}$$

where V_β is as in Definition 1.4, and $\mathcal{S}_\alpha$ is assumed independent of V_β.

Theorem 2.5. *Suppose that (9) holds for $\alpha \in (0,1)$ and that h is not identically zero. Assume further that the following limit*

$$q := \lim_{t\to\infty} \frac{h(t)^2}{v(t)\mathbb{P}\{\xi > t\} + h(t)^2} \in [0,1]$$

exists and that:

- *if $q < 1$, then the assumptions of Proposition 2.2 hold (with the same α as above);*
- *if $q = 1$, then $h(t) \sim t^\rho \widehat{\ell}(t)$, $t \to \infty$ for some $\rho \ge -\alpha$ and some $\widehat{\ell}$; if $\rho = -\alpha$, then there exists a positive increasing function w such that $\lim_{t\to\infty} w(t) = \infty$ and $\lim_{t\to\infty} \frac{h(t)}{\mathbb{P}\{\xi>t\}w(t)} = 1$.*

Then, setting $\rho := (\beta - \alpha)/2$ when $q \in (0,1)$,

$$\frac{\mathbb{P}\{\xi > t\} Y(ut)}{\sqrt{v(t)\mathbb{P}\{\xi > t\} + h(t)^2}} \overset{f.d.}{\Rightarrow} \sqrt{1-q}\, Z_{\alpha,\beta}(u) + \sqrt{q} \int_{[0,u]} (u-y)^\rho \,\mathrm{d}W_\alpha^{\leftarrow}(y), \qquad t \to \infty,$$

where $Z_{\alpha,\beta}$ is as in Definition 1.6, and $W_\alpha^{\leftarrow}$ under the integral sign is the same as in the definition of $Z_{\alpha,\beta}$. In particular, the summands defining the limit process are dependent.

There is a simple situation where the weak convergence of finite-dimensional distributions obtained in Theorem 2.5 implies the J_1-convergence on D. Of course, the case where the limit process in Proposition 2.2 is a conditional white noise (equivalently, $C(u,w) = 0$ for $u \ne w$) must be eliminated as no version of such a process belongs to D.

Corollary 2.6. *Let $X(t)$ be almost surely increasing with $\lim_{t\to\infty} X(t) \in (0,\infty]$ almost surely. Assume that the assumptions of Theorem 2.5 are in force with the exception that in the case $q < 1$ the conditions on the function $f(u,w)$ are replaced by the condition that the function $(u,w) \mapsto \mathbb{E}[X(u)X(w)]$ is regularly varying in $\mathbb{R}_+^2$ of index β with limit function C. Then the limit relations of Theorem 2.5 hold in the sense of weak convergence in the J_1-topology on D, where $Z_{\alpha,\beta}(0) = 0$ is defined as the limit in probability of $Z_{\alpha,\beta}(u)$ as $u \downarrow 0$.*

We close the section with a negative result which implies that weak convergence of the finite-dimensional distributions in Theorem 2.5 cannot be strengthened to weak convergence on $D(0,\infty)$ whenever $Z_{\alpha,-\alpha}$ arises in the limit.

Proposition 2.7. *Any version of the process $Z_{\alpha,-\alpha}$ has paths in the Skorokhod space $D(0,\infty)$ with probability strictly less than* 1. *If further $C(u,w)=0$ for all $u\neq w$, $u,w>0$, then any version has paths in $D(0,\infty)$ with probability* 0.

3. Applications

Unless the contrary is stated, the random variable η appearing in this section may be arbitrarily dependent on ξ, and (ξ_k,η_k), $k\in\mathbb{N}$ denote i.i.d. copies of (ξ,η).

Example 3.1. Let $X(t)=\mathbb{1}_{\{\eta>t\}}$, $\sigma^2<\infty$ and suppose that $\mathbb{P}\{\eta>t\}\sim t^\beta\ell(t)$ for some $\beta\in(-1,0]$. Since $h(t)=\mathbb{E}[X(t)]=\mathbb{P}\{\eta>t\}$ and $v(t)=\mathbb{P}\{\eta>t\}\mathbb{P}\{\eta\le t\}$ we infer $\lim_{t\to\infty}v(t)/h(t)^2=\infty$. Further

$$\frac{f(ut,wt)}{v(t)}=\frac{\mathbb{P}\{\eta>(u\vee w)t\}\mathbb{P}\{\eta\le(u\wedge w)t\}}{\mathbb{P}\{\eta>t\}\mathbb{P}\{\eta\le t\}}\to(u\vee w)^\beta,\qquad u,w>0,$$

and this convergence is locally uniform in $\mathbb{R}_+^2$ as it is the case for $\lim_{t\to\infty}\mathbb{P}\{\eta>(u\vee w)t\}/\mathbb{P}\{\eta>t\}=(u\vee w)^\beta$ by Lemma 5.2(a). In particular, condition (3) holds with $C(u,w)=(u\vee w)^\beta$. Finally, condition (4) holds because $|\mathbb{1}_{\{\eta>t\}}-\mathbb{P}\{\eta>t\}|\le 1$ a.s. Now we conclude that, according to the case $p=0$ of Theorem 2.4,

$$\frac{\sum_{k\ge0}\mathbb{1}_{\{S_k\le ut<S_k+\eta_{k+1}\}}-(1/\mu)\int_0^{ut}\mathbb{P}\{\eta>y\}\,\mathrm{d}y}{\sqrt{\mu^{-1}t\mathbb{P}\{\eta>t\}}}\overset{\text{f.d.}}{\Rightarrow}V_\beta(u),$$

where V_β is a centered Gaussian process with covariance

$$\mathbb{E}\big[V_\beta(u)V_\beta(w)\big]=(1+\beta)^{-1}\big(w^{1+\beta}-(w-u)^{1+\beta}\big),\qquad 0\le u\le w.$$

Assuming that ξ and η are independent, a counterpart of this result with a random centering (i.e. a result that follows from Proposition 2.1) was obtained in Proposition 3.2 of [38].

Example 3.2. Let $X(t)=\mathbb{1}_{\{\eta\le t\}}$. Since $h(t)=\mathbb{P}\{\eta\le t\}$ and $v(t)=\mathbb{P}\{\eta\le t\}\mathbb{P}\{\eta>t\}\sim\mathbb{P}\{\eta>t\}$, we infer $\lim_{t\to\infty}th(t)^2/\int_0^t v(y)\,\mathrm{d}y=\infty$. Further, if $\mathbb{E}\eta<\infty$, then v is dRi on $[0,\infty)$ because it is nonnegative, bounded, a.e. continuous and dominated by the decreasing and integrable function $\mathbb{P}\{\eta>t\}$. If $\mathbb{E}\eta=\infty$, that is, $\lim_{t\to\infty}\int_0^t v(y)\,\mathrm{d}y=\infty$, v is equivalent to the monotone function $u(t)=\mathbb{P}\{\eta>t\}$. If $\sigma^2<\infty$ then, according to the case $p=1$ of Theorem 2.4,

$$\frac{\sum_{k\ge0}\mathbb{1}_{\{S_k+\eta_{k+1}\le ut\}}-(1/\mu)\int_0^{ut}\mathbb{P}\{\eta\le y\}\,\mathrm{d}y}{\sqrt{\sigma^2\mu^{-3}t}}\overset{\text{f.d.}}{\Rightarrow}\mathcal{S}_2(u),$$

where $\mathcal{S}_2$ is a Brownian motion, because h is regularly varying at ∞ of index $\rho=0$. If $\mathbb{P}\{\xi>t\}$ is regularly varying at ∞ of index $-\alpha$, $\alpha\in(0,1)$, then, by Corollary 2.6,

$$\mathbb{P}\{\xi>t\}\sum_{k\ge0}\mathbb{1}_{\{S_k+\eta_{k+1}\le ut\}}\Rightarrow W_\alpha^{\leftarrow}(u)$$

in the J_1-topology on D.

Example 3.3. Let $X(t) = \eta g(t)$ with $\operatorname{Var} \eta < \infty$ and let $g : \mathbb{R}^+ \to \mathbb{R}$ be regularly varying at ∞ of index $\beta/2$ for some $\beta > -1$. Then $h(t) = g(t)\mathbb{E}\eta$ and $v(t) = g(t)^2 \operatorname{Var} \eta$. While $f(u, w) = g(u)g(w) \operatorname{Var} \eta$ is clearly regularly varying in $\mathbb{R}^2_+$ of index β with limit function $C(u, w) = (uw)^{\beta/2}$, (3) holds by virtue of Lemma 5.2(a). Further observe that $\lim_{t\to\infty} \sqrt{tv(t)}/|g(t)| = \infty$ implies

$$\mathbb{E}\big[\big(X(t) - h(t)\big)^2 \mathbb{1}_{\{|X(t)-h(t)|>y\sqrt{tv(t)}\}}\big]$$
$$= g(t)^2 \mathbb{E}\big[(\eta - \mathbb{E}\eta)^2 \mathbb{1}_{\{|\eta-\mathbb{E}\eta|>y\sqrt{tv(t)}/|g(t)|\}}\big] = o\big(v(t)\big)$$

and thereupon (4). Also, as a consequence of $\lim_{t\to\infty} \sqrt{v(t)/\mathbb{P}\{\xi > t\}}/|g(t)| = \infty$, which holds whatever the law of ξ is, we have

$$\mathbb{E}\big[\big(X(t) - h(t)\big)^2 \mathbb{1}_{\{|X(t)-h(t)|>y\sqrt{v(t)/\mathbb{P}\{\xi>t\}}\}}\big]$$
$$= g(t)^2 \mathbb{E}\big[(\eta - \mathbb{E}\eta)^2 \mathbb{1}_{\{|\eta-\mathbb{E}\eta|>y\sqrt{v(t)/\mathbb{P}\{\xi>t\}}/|g(t)|\}}\big] = o\big(v(t)\big)$$

which means that condition (7) holds.

If $\mathbb{E}\eta = 0$ and $\mu \in (0, \infty)$, then, according to Proposition 2.1,

$$\frac{\sum_{k\ge0} \eta_{k+1} g(ut - S_k) \mathbb{1}_{\{S_k\le ut\}}}{\sqrt{\mu^{-1} t \mathbb{E}[\eta^2]}\, g(t)} \overset{\text{f.d.}}{\Rightarrow} V_\beta(u),$$

where V_β is a centered Gaussian process with covariance

$$\mathbb{E}\big[V_\beta(u)V_\beta(w)\big] = \int_0^u (u - y)^{\beta/2} (w - y)^{\beta/2}\, \mathrm{d}y, \qquad 0 < u \le w.$$

Furthermore, the limit process can be represented as a stochastic integral

$$V_\beta(u) = \int_{[0,u]} (u - y)^{\beta/2}\, \mathrm{d}\mathcal{S}_2(y), \qquad u > 0.$$

Throughout the rest of this example, we assume that η is independent of ξ.

If $\mathbb{E}\eta = 0$ and $\mathbb{P}\{\xi > t\}$ is regularly varying at ∞ of index $-\alpha$, $\alpha \in (0, 1)$ and $\beta > -\alpha$ then, according to Proposition 2.2,

$$\frac{\sqrt{\mathbb{P}\{\xi > t\}}}{g(t)} \sum_{k\ge0} \eta_{k+1} g(ut - S_k) \mathbb{1}_{\{S_k\le ut\}} \overset{\text{f.d.}}{\Rightarrow} \sqrt{\mathbb{E}\big[\eta^2\big]}\, Z_{\alpha,\beta}(u).$$

Furthermore, the limit process can be represented as a stochastic integral

$$Z_{\alpha,\beta}(u) = \int_{[0,u]} (u - y)^{\beta/2}\, \mathrm{d}\mathcal{S}_2\big(W_\alpha^{\leftarrow}(y)\big), \qquad u > 0,$$

where $\mathcal{S}_2$ is a Brownian motion independent of $W_\alpha^{\leftarrow}$, which can be seen by calculating the conditional covariance of the last integral.

If $\mathbb{E}\eta \neq 0$, $\sigma^2 < \infty$ and g is eventually monotone, then, according to Theorem 2.4,

$$\frac{\sum_{k\geq 0}\eta_{k+1}g(ut-S_k)\mathbb{1}_{\{S_k\leq ut\}} - \mu^{-1}\mathbb{E}\eta\int_0^{ut}g(y)\,\mathrm{d}y}{\mathbb{E}\eta\sqrt{t}g(t)} \overset{\text{f.d.}}{\Rightarrow} \left(\frac{\sigma^2}{\mu^3}\right)^2\int_{[0,u]}(u-y)^{\beta/2}\,\mathrm{d}\mathcal{S}_2(u) + \left(\frac{\operatorname{Var}\eta}{(\mathbb{E}\eta)^2\mu}\right)^{1/2}V_\beta(u).$$

If $\mathbb{E}\eta \neq 0$, $\mathbb{P}\{\xi > t\}$ is regularly varying at ∞ of index $-\alpha$, $\alpha \in (0,1)$, and $\beta > -2\alpha$, then, since $\lim_{t\to\infty} v(t)\mathbb{P}\{\xi > t\}/h(t)^2 = 0$, an application of Theorem 2.5 with $q = 1$ gives

$$\frac{\mathbb{P}\{\xi > t\}}{g(t)}\sum_{k\geq 0}\eta_{k+1}g(ut-S_k)\mathbb{1}_{\{S_k\leq ut\}} \overset{\text{f.d.}}{\Rightarrow} \mathbb{E}\eta\int_{[0,u]}(u-y)^{\beta/2}\,\mathrm{d}W_\alpha^{\leftarrow}(y).$$

If further $\eta \geq 0$ a.s. and g is increasing (which implies $\beta \geq 0$), then, according to Corollary 2.6, the limit relation takes place in the J_1-topology on D.

Example 3.4. Let $Z := (Z(t))_{t\geq 0}$ be a stationary Ornstein–Uhlenbeck process defined by

$$Z(t) = e^{-t}\theta + \int_{[0,t]} e^{-(t-y)}\,\mathrm{d}\mathcal{S}_2(y), \qquad t \geq 0,$$

where θ is a normal random variable with mean zero and variance $1/2$ independent of a Brownian motion $\mathcal{S}_2$. Z and ξ may be arbitrarily dependent. Put $X(t) = (t+1)^{\beta/2}Z(t)$ for $\beta \in (-1, 0)$. Then $\mathbb{E}[X(t)] = 0$ and $f(u,w) = \mathbb{E}[X(u)X(w)] = 2^{-1}(u+1)^{\beta/2}(w+1)^{\beta/2}e^{-|u-w|}$ from which we conclude that f is fictitious regularly varying in $\mathbb{R}_+^2$ of index β. By stationarity, for each $t > 0$, $Z(t)$ has the same law as θ. Hence

$$\mathbb{E}\big[X(t)^2\mathbb{1}_{\{|X(t)|>y\}}\big] = (t+1)^\beta\mathbb{E}\big[\theta^2\mathbb{1}_{\{|\theta|>y(t+1)^{-\beta/2}\}}\big] = o\big(t^\beta\big),$$

that is, condition (4) holds. If $\mu < \infty$ an application of Proposition 2.1 yields

$$\frac{\sum_{k\geq 0}X_{k+1}(ut-S_k)\mathbb{1}_{\{S_k\leq ut\}}}{\sqrt{(2\mu)^{-1}t^{\beta+1}}} \overset{\text{f.d.}}{\Rightarrow} V_\beta(u),$$

the limiting process being a centered Gaussian process with independent values (white noise).

Example 3.5. Let $X(t) = \mathcal{S}_2((t+1)^{-\alpha})$, $\mathbb{P}\{\xi > t\} \sim t^{-\alpha}$ and assume that X and ξ are independent. Then $f(u,w) = \mathbb{E}[X(u)X(w)]$ is uniformly regularly varying of index $-\alpha$ in strips in $\mathbb{R}_+^2$ with limit function $C(u,w) = (u \vee w)^{-\alpha}$. (7) follows from

$$\mathbb{E}\big[X(t)^2\mathbb{1}_{\{|X(t)|>y\}}\big] = (t+1)^{-\alpha}\mathbb{E}\big[\mathcal{S}_2(1)^2\mathbb{1}_{\{|\mathcal{S}_2(1)|>y(t+1)^{\alpha/2}\}}\big] = o\big(t^{-\alpha}\big)$$

for all $y > 0$. Thus, Proposition 2.2 (in which we take $u(t) \equiv 1$) applies and yields $\sum_{k\geq 0}X_{k+1}(ut-S_k)\mathbb{1}_{\{S_k\leq ut\}} \overset{\text{f.d.}}{\Rightarrow} Z_{\alpha,-\alpha}(u)$.

4. Proofs of main results

4.1. Proofs of Propositions 2.1 and 2.2

For a σ-algebra $\mathcal{G}$ we shall write $\mathbb{E}_{\mathcal{G}}[\cdot]$ for $\mathbb{E}[\cdot|\mathcal{G}]$. Recalling that $\nu(t) = \inf\{k \in \mathbb{N}_0 : S_k > t\}$, $t \geq 0$, we define the renewal function $U(t) := \mathbb{E}[\nu(t)] = \sum_{k\geq 0} \mathbb{P}\{S_k \leq t\}$, $t \geq 0$.

Proof of Proposition 2.1. We only investigate the case where $C(u,w) > 0$ for some $u, w > 0$, $u \neq w$. Modifications needed in the case where $C(u,w) = 0$ for all $u, w > 0$, $u \neq w$ should be clear from the subsequent presentation.

Note that relation (3) ensures continuity of the function $u \mapsto C(u, u+w)$ on $(0,\infty)$ for each $w > 0$ (an accurate proof of a similar fact is given in [51], pages 2–3). From the Cauchy–Schwarz inequality, we deduce that

$$\bigl|f(u,w)\bigr| \leq 2^{-1}\bigl(v(u) + v(w)\bigr), \qquad u, w \geq 0, \tag{13}$$

and hence

$$C(u-y, w-y) \leq 2^{-1}\bigl((u-y)^{\beta} + (w-y)^{\beta}\bigr). \tag{14}$$

Consequently, as $\beta > -1$,

$$\int_0^u C(u-y, w-y)\,\mathrm{d}y < \infty, \qquad 0 < u \leq w.$$

Since $(u,w) \mapsto C(u,w)$ is positive semidefinite, so is $(u,w) \mapsto \int_0^u C(u-y,w-y)\,\mathrm{d}y$, $0 < u \leq w$. Hence the process V_β does exist.

Without loss of generality we can and do assume that X is centered, for it is the case for $X(t) - h(t)$. According to the Cramér–Wold device (see Theorem 29.4 in [5]) it suffices to prove that

$$\frac{\sum_{j=1}^m \alpha_j \sum_{k\geq 0} X_{k+1}(u_j t - S_k)\mathbb{1}_{\{S_k \leq u_j t\}}}{\sqrt{\mu^{-1} t v(t)}} \xrightarrow{\mathrm{d}} \sum_{j=1}^m \alpha_j V_\beta(u_j) \tag{15}$$

for all $m \in \mathbb{N}$, all $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ and all $0 < u_1 < \cdots < u_m < \infty$. Note that the random variable $\sum_{j=1}^m \alpha_j V_\beta(u_j)$ has a normal law with mean 0 and variance

$$(1+\beta)^{-1} \sum_{j=1}^m \alpha_j^2 u_j^{1+\beta} + 2 \sum_{1\leq i<j\leq m} \alpha_i \alpha_j \int_0^{u_i} C(u_i - y, u_j - y)\,\mathrm{d}y =: D(u_1, \ldots, u_m). \tag{16}$$

Define the σ-algebras $\mathcal{F}_0 := \{\varnothing, \Omega\}$ and $\mathcal{F}_k := \sigma((X_1,\xi_1), \ldots, (X_k,\xi_k))$, $k \in \mathbb{N}$ and observe that

$$\mathbb{E}_{\mathcal{F}_k}\Biggl[\sum_{j=1}^m \alpha_j \mathbb{1}_{\{S_k \leq u_j t\}} X_{k+1}(u_j t - S_k)\Biggr] = 0.$$

Thus, in order to prove (15), one may use the martingale central limit theorem (Corollary 3.1 in [15]), whence it suffices to verify

$$\sum_{k\geq 0} \mathbb{E}_{\mathcal{F}_k}\big[Z_{k+1,t}^2\big] \xrightarrow{\mathbb{P}} D(u_1,\dots,u_m), \tag{17}$$

and

$$\sum_{k\geq 0} \mathbb{E}_{\mathcal{F}_k}\big[Z_{k+1,t}^2 \mathbb{1}_{\{|Z_{k+1,t}|>y\}}\big] \xrightarrow{\mathbb{P}} 0 \tag{18}$$

for all $y > 0$, where

$$Z_{k+1,t} := \frac{\sum_{j=1}^m \alpha_j \mathbb{1}_{\{S_k\leq u_j t\}} X_{k+1}(u_j t - S_k)}{\sqrt{\mu^{-1} t v(t)}}, \qquad k \in \mathbb{N}_0, t > 0.$$

Proof of (18). In view of the inequality

$$\begin{aligned}(a_1+\dots+a_m)^2\mathbb{1}_{\{|a_1+\dots+a_m|>y\}} &\leq \big(|a_1|+\dots+|a_m|\big)^2\mathbb{1}_{\{|a_1|+\dots+|a_m|>y\}}\\ &\leq m^2\big(|a_1|\vee\dots\vee|a_m|\big)^2\mathbb{1}_{\{m(|a_1|\vee\dots\vee|a_m|)>y\}}\\ &\leq m^2\big(a_1^2\mathbb{1}_{\{|a_1|>y/m\}}+\dots+a_m^2\mathbb{1}_{\{|a_m|>y/m\}}\big)\end{aligned} \tag{19}$$

which holds for $a_1,\dots,a_m \in \mathbb{R}$, it is sufficient to show that

$$\sum_{k\geq 0} \mathbb{1}_{\{S_k\leq t\}} \mathbb{E}_{\mathcal{F}_k}\bigg[\frac{X_{k+1}(t-S_k)^2}{\mu^{-1} t v(t)} \mathbb{1}_{\{|X_{k+1}(t-S_k)|>y\sqrt{\mu^{-1}tv(t)}\}}\bigg] \xrightarrow{\mathbb{P}} 0 \tag{20}$$

for all $y > 0$. We can take t instead of $u_j t$ here because v is regularly varying and $y > 0$ is arbitrary.

Without loss of generality we assume that the function $t \mapsto t v(t)$ is increasing, for we could otherwise work with $(\beta+1)\int_0^t v(y)\,\mathrm{d}y$ (see Lemma 5.2(c)). By Markov's inequality and the aforementioned monotonicity relation (20), follows if we can prove that

$$\lim_{t\to\infty} \frac{1}{t v(t)} \int_{[0,t]} v_y(t-x)\,\mathrm{d}U(x) = 0 \tag{21}$$

for all $y > 0$, where the definition of v_y is given in (4). Recalling that $\mu < \infty$ and that v is locally bounded, measurable and regularly varying at infinity of index $\beta \in (-1,\infty)$ an application of Lemma 5.11 with $r_1 = 0$ and $r_2 = 1$ yields

$$\int_{[0,t]} v(t-x)\,\mathrm{d}U(x) \sim \mathrm{const}\, t v(t).$$

Since, according to (4), $v_y(t) = o(v(t))$, (21) follows from Lemma 5.10(b).

Proof of (17). It can be checked that

$$\sum_{k\geq 0}\mathbb{E}_{\mathcal{F}_k}\big[Z_{k+1,t}^2\big]=\frac{\sum_{j=1}^m \alpha_j^2\sum_{k\geq 0}\mathbb{1}_{\{S_k\leq u_j t\}}v(u_j t-S_k)}{\mu^{-1}tv(t)}$$
$$+\frac{2\sum_{1\leq i<j\leq m}\alpha_i\alpha_j\sum_{k\geq 0}\mathbb{1}_{\{S_k\leq u_i t\}}f(u_i t-S_k,u_j t-S_k)}{\mu^{-1}tv(t)}.$$

We shall prove that

$$\frac{\sum_{k\geq 0}\mathbb{1}_{\{S_k\leq u_i t\}}v(u_i t-S_k)}{\mu^{-1}tv(t)}=\frac{\int_{[0,u_i]}v((u_i-y)t)\,\mathrm{d}\nu(ty)}{\mu^{-1}tv(t)}\overset{\mathbb{P}}{\to}\frac{u_i^{1+\beta}}{1+\beta} \tag{22}$$

and

$$\begin{aligned}\frac{\sum_{k\geq 0}\mathbb{1}_{\{S_k\leq u_i t\}}f(u_i t-S_k,u_j t-S_k)}{\mu^{-1}tv(t)}&=\frac{\int_{[0,u_i]}f((u_i-y)t,(u_j-y)t)\,\mathrm{d}\nu(ty)}{\mu^{-1}tv(t)}\\&\overset{\mathbb{P}}{\to}\int_0^{u_i}C(u_i-y,u_j-y)\,\mathrm{d}y\end{aligned} \tag{23}$$

for all $1\leq i<j\leq m$.

Fix any $u_i<u_j$ and pick $\varepsilon\in(0,u_i)$. By the functional strong law of large numbers (Theorem 4 in [13])

$$\lim_{t\to\infty}\sup_{y\in[0,u_i]}\left|\frac{\nu(ty)}{\mu^{-1}t}-y\right|=0\qquad\text{a.s.}$$

Also,

$$\lim_{t\to\infty}\frac{v((u_i-y)t)}{v(t)}=(u_i-y)^\beta$$

uniformly in $y\in[0,u_i-\varepsilon]$ by Lemma 5.2(a), and

$$\lim_{t\to\infty}\frac{f((u_i-y)t,(u_j-y)t)}{v(t)}=C(u_i-y,u_j-y)$$

uniformly in $y\in[0,u_i-\varepsilon]$, by virtue of (3). Two applications of Lemma 5.4(a) (with $X_t(y)=\nu(ty)/(\mu^{-1}t)$) yield

$$\int_{[0,u_i-\varepsilon]}\frac{v((u_i-y)t)}{v(t)}\,\mathrm{d}\frac{\nu(ty)}{\mu^{-1}t}\overset{\mathbb{P}}{\to}\int_0^{u_i-\varepsilon}(u_i-y)^\beta\,\mathrm{d}y=\frac{u_i^{1+\beta}-\varepsilon^{1+\beta}}{1+\beta}$$

and

$$\int_{[0,u_i-\varepsilon]}\frac{f((u_i-y)t,(u_j-y)t)}{v(t)}\,\mathrm{d}\frac{\nu(ty)}{\mu^{-1}t}\overset{\mathbb{P}}{\to}\int_0^{u_i-\varepsilon}C(u_i-y,u_j-y)\,\mathrm{d}y.$$

Observe that since $\nu(y)$ is a.s. increasing, so is $X_t(y)$.

As $\varepsilon \downarrow 0$, the right-hand sides of the last two equalities converge to $(1+\beta)^{-1} u_i^{1+\beta}$ and $\int_0^{u_1} C(u_i - y, u_j - y)\,\mathrm{d}y$, respectively. Therefore, for (22) and (23) to hold it is sufficient (see Lemma 5.1) that

$$\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty} \mathbb{P}\left\{\frac{\int_{(u_i-\varepsilon,u_i]} v(t(u_i-y))\,\mathrm{d}\nu(ty)}{tv(t)} > \delta\right\} = 0$$

and

$$\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty} \mathbb{P}\left\{\frac{|\int_{(u_i-\varepsilon,u_i]} f(t(u_i-y),t(u_j-y))\,\mathrm{d}\nu(ty)|}{tv(t)} > \delta\right\} = 0$$

for all $\delta > 0$. By Markov's inequality, it thus suffices to check that

$$\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty} \frac{\int_{(u_i-\varepsilon,u_i]} v((u_i-y)t)\,\mathrm{d}U(ty)}{tv(t)} = 0 \tag{24}$$

and

$$\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty} \frac{\int_{(u_i-\varepsilon,u_i]} |f((u_i-y)t,(u_j-y)t)|\,\mathrm{d}U(ty)}{tv(t)} = 0, \tag{25}$$

respectively. Changing the variable $s = u_i t$ and recalling that v is regularly varying of index $\beta \in (-1,\infty)$, we apply Lemma 5.11 with $r_1 = 1 - \varepsilon u_i^{-1}$ and $r_2 = 1$ to infer

$$\begin{aligned}\int_{((u_i-\varepsilon)t,u_it]} v(u_it-y)\,\mathrm{d}U(y) &= \int_{((1-\varepsilon u_i^{-1})s,s]} v(s-y)\,\mathrm{d}U(y)\\ &\sim \left(\frac{\varepsilon}{u_i}\right)^{1+\beta}\frac{sv(s)}{(1+\beta)\mu} \sim \frac{\varepsilon^{1+\beta}tv(t)}{(1+\beta)\mu}.\end{aligned}$$

Using (13), we further obtain

$$\begin{aligned}&\int_{((u_i-\varepsilon)t,u_it]} \big|f(u_it-y,u_jt-y)\big|\,\mathrm{d}U(y)\\ &\quad\le \frac{1}{2}\int_{((u_i-\varepsilon)t,u_it]} v(u_it-y)\,\mathrm{d}U(y) + \frac{1}{2}\int_{((u_i-\varepsilon)t,u_it]} v(u_jt-y)\,\mathrm{d}U(y)\\ &\quad\sim \frac{1}{2\mu(1+\beta)}\big(\varepsilon^{1+\beta} + (u_j-u_i+\varepsilon)^{1+\beta} - (u_j-u_i)^{1+\beta}\big)tv(t),\end{aligned}$$

where for the second integral we have changed the variable $s = u_jt$, invoked Lemma 5.11 with $r_1 = (u_i-\varepsilon)u_j^{-1}$ and $r_2 = u_iu_j^{-1}$ and then got back to the original variable t. These relations entail both, (24) and (25). The proof of Proposition 2.1 is complete. □

In what follows, $\mathcal{F}$ denotes the σ-algebra generated by $(S_n)_{n\in\mathbb{N}_0}$.

Proof of Proposition 2.2. As in the previous proof, we can and do assume that X is centered. Put $r(t) := v(t)/\mathbb{P}\{\xi > t\}$. The process $Z_{\alpha,\beta}$ is well-defined by Lemma 5.7. In view of the Cramér–Wold device, it suffices to check that

$$\frac{1}{\sqrt{r(t)}} \sum_{j=1}^{m} \gamma_j Y(u_j t) \stackrel{\mathrm{d}}{\to} \sum_{j=1}^{m} \gamma_j Z_{\alpha,\beta}(u_j) \tag{26}$$

for all $\gamma_1, \dots, \gamma_m \in \mathbb{R}$. Since $C(y, y) = y^{\beta}$, then, given $W_{\alpha}^{\leftarrow}$, the random variable $\sum_{j=1}^{m} \gamma_j Z_{\alpha,\beta}(u_j)$ is centered normal with variance

$$\begin{aligned} D_{\alpha,\beta}(u_1, \dots, u_m) := & \sum_{j=1}^{m} \gamma_j^2 \int_{[0,u_j]} (u_j - y)^{\beta}\, \mathrm{d}W_{\alpha}^{\leftarrow}(y) \\ & + 2 \sum_{1 \le i < j \le m} \gamma_i \gamma_j \int_{[0,u_i]} C(u_i - y, u_j - y)\, \mathrm{d}W_{\alpha}^{\leftarrow}(y). \end{aligned} \tag{27}$$

Equivalently,

$$\mathbb{E}\Bigg[\exp\Bigg(\mathrm{i}z \sum_{j=1}^{m} \gamma_j Z_{\alpha,\beta}(u_j)\Bigg)\Bigg] = \mathbb{E}\big[\exp\big(-D_{\alpha,\beta}(u_1, \dots, u_m) z^2/2\big)\big], \qquad z \in \mathbb{R},$$

where here and throughout the paper, i denotes the imaginary unit. Hence, according to Lemma 5.6, (26) is a consequence of

$$\sum_{k \ge 0} \mathbb{E}_{\mathcal{F}}\big[Z_{k+1,t}^2\big] \stackrel{\mathrm{d}}{\to} D_{\alpha,\beta}(u_1, \dots, u_m), \tag{28}$$

where $Z_{k+1,t} := (r(t))^{-1/2} \sum_{j=1}^{m} \gamma_j X_{k+1}(u_j t - S_k) \mathbb{1}_{\{S_k \le u_j t\}}$, and

$$\sum_{k \ge 0} \mathbb{E}_{\mathcal{F}}\big[Z_{k+1,t}^2 \mathbb{1}_{\{|Z_{k+1,t}| > y\}}\big] \stackrel{\mathbb{P}}{\to} 0 \tag{29}$$

for all $y > 0$. Since $r(t)$ is regularly varying at ∞ of index $\beta + \alpha$ we have

$$\begin{aligned} & \limsup_{t \to \infty} \frac{1}{r(t)} \int_{(\rho z, z]} v\big(t(z - y)\big)\, \mathrm{d}U(ty) \\ & \le \lim_{t \to \infty} \frac{r(tz)}{r(t)} \limsup_{t \to \infty} \frac{1}{r(tz)} \int_{(\rho tz, tz]} v(tz - y)\, \mathrm{d}U(y) \\ & = z^{\beta+\alpha} \limsup_{t \to \infty} \frac{1}{r(t)} \int_{(\rho t, t]} v(t - y)\, \mathrm{d}U(y) \end{aligned}$$

for all $z > 0$. Hence, the relation

$$\lim_{\rho\uparrow 1}\limsup_{t\to\infty}\frac{1}{r(t)}\int_{(\rho z,z]} v\big(t(z-y)\big)\,\mathrm{d}U(ty)=0 \tag{30}$$

for all $z > 0$ is an immediate consequence of Lemma 5.12(a). Using the representation

$$\sum_{k\ge 0}\mathbb{E}_{\mathcal{F}}\big[Z_{k+1,t}^2\big]=\frac{1}{r(t)}\int_{[0,u_m]}\Bigg(\sum_{j=1}^{m}\gamma_j^2 v\big((u_j-y)t\big)\mathbb{1}_{[0,u_j]}(y)$$
$$+2\sum_{1\le i<j\le m}\gamma_i\gamma_j f\big((u_i-y)t,(u_j-y)t\big)\mathbb{1}_{[0,u_i]}(y)\Bigg)\,\mathrm{d}\nu(ty)$$

we further conclude that (28) follows from Lemma 5.8 with $\lambda_1=0$ (observe that conditions (67) and (68) are then not needed and (65) coincides with (30)). In view of (19), (29) is a consequence of

$$\frac{1}{r(t)}\sum_{k\ge 0}\mathbb{1}_{\{S_k\le t\}}\mathbb{E}_{\mathcal{F}}\Big[\big(X_{k+1}(t-S_k)\big)^2\mathbb{1}_{\{|X_{k+1}(t-S_k)|>y\sqrt{r(t)}\}}\Big]\overset{\mathbb{P}}{\to}0 \tag{31}$$

for all $y > 0$. To prove (31) we assume, without loss of generality, that the function r is increasing, for in the case $\beta=-\alpha$ it is asymptotically equivalent to an increasing function $u(t)$ by assumption, while in the case $\beta>-\alpha$ the existence of such a function is guaranteed by Lemma 5.2(b) because r is then regularly varying of positive index. Using this monotonicity and recalling that we are assuming that $h\equiv 0$, whence $v_y(t)=\mathbb{E}[(X(t))^2\mathbb{1}_{\{|X(t)|>y\sqrt{r(t)}\}}]$, we conclude that it is sufficient to check that

$$\mathbb{E}\Bigg[\sum_{k\ge 0}\mathbb{1}_{\{S_k\le t\}}\mathbb{E}_{\mathcal{F}}\Big[\big(X_{k+1}(t-S_k)\big)^2\mathbb{1}_{\{|X_{k+1}(t-S_k)|>y\sqrt{r(t-S_k)}\}}\Big]\Bigg]$$
$$=\int_{[0,t]} v_y(t-x)\,\mathrm{d}U(x)=o\big(r(t)\big)$$

for all $y > 0$, by Markov's inequality. In view of (7) the latter is an immediate consequence of Lemma 5.12(b) with $\phi_1(t)=v_y(t)$, $\phi(t)=v(t)$, $q(t)=u(t)$ and $\gamma=\beta$. The proof of Proposition 2.2 is complete. □

4.2. Proofs of Theorems 2.4 and 2.5

For the proof of Theorem 2.4, we need two auxiliary results, Lemma 4.1 and Lemma 4.2. Replacing the denominator in (5) by a function which grows faster leads to weak convergence of finite-dimensional distributions to zero. However, this result holds without the regular variation assumptions of Proposition 2.1.

Lemma 4.1. *Assume that*:

- $\mu=\mathbb{E}\xi<\infty$;

- *either*

$$\lim_{t\to\infty}\int_0^t v(y)\,\mathrm{d}y=\infty \quad \textit{and} \quad \lim_{t\to\infty}\frac{v(t)}{\int_0^t v(y)\,\mathrm{d}y}=0$$

  and there exists a monotone function u *such that* $v(t)\sim u(t)$ *as* $t\to\infty$, *or* v *is directly Riemann integrable* (*dRi*) *on* $[0,\infty)$.

Then

$$\frac{Y(ut)-\sum_{k\geq 0}h(ut-S_k)\mathbb{1}_{\{S_k\leq ut\}}}{s(t)}\overset{f.d.}{\Rightarrow}0, \qquad t\to\infty \tag{32}$$

for any positive function $s(t)$ *regularly varying at* ∞ *which satisfies*

$$\lim_{t\to\infty}s(t)^2\Big/\int_0^t v(y)\,\mathrm{d}y=\infty.$$

Proof. By Chebyshev's inequality and the Cramér–Wold device, it suffices to prove that

$$s(t)^{-2}\mathbb{E}\biggl[\biggl(Y(t)-\sum_{k\geq 0}h(t-S_k)\mathbb{1}_{\{S_k\leq t\}}\biggr)^2\biggr]\to 0.$$

The expectation above equals $\int_{[0,t]}v(t-y)\,\mathrm{d}U(y)$. If v is dRi, the latter integral is bounded (this is clear from the key renewal theorem when the law of ξ is nonlattice while in the lattice case, it follows from Lemma 8.2 in [24]). If v is non-integrable and u is a monotone function such that $v(t)\sim u(t)$, Lemma 5.10(a) with $r_1=0$ and $r_2=1$ yields

$$\int_{[0,t]}v(t-y)\,\mathrm{d}U(y)\sim\int_{[0,t]}u(t-y)\,\mathrm{d}U(y).$$

Modifying u if needed in the right vicinity of zero we can assume that u is monotone and locally integrable. Since $u\sim v$, we have $\lim_{t\to\infty}(u(t)/\int_0^t u(y)\,\mathrm{d}y)=0$ as the corresponding relation holds for v, and an application of Lemma 5.9 applied to $\phi=u$ with $r_1=0$ and $r_2=1$ gives

$$\int_{[0,t]}u(t-y)\,\mathrm{d}U(y)\sim\frac{1}{\mu}\int_0^t u(y)\,\mathrm{d}y$$

and again using $u\sim v$ we obtain

$$\int_0^t u(y)\,\mathrm{d}y\sim\int_0^t v(y)\,\mathrm{d}y=o\bigl(s(t)^2\bigr),$$

where the last equality follows from the assumption on s. The proof of Lemma 4.1 is complete. □

Lemma 4.2. *Assume that* h *is eventually monotone and eventually nonnegative and that the law of* ξ *belongs to the domain of attraction of an* α*-stable law,* $\alpha\in(1,2]$ (*i.e.*, *relation* (11) *holds*).

Then

$$\frac{\sum_{k\geq 0} h(ut-S_k)\mathbb{1}_{\{S_k\leq ut\}} - (1/\mu)\int_0^{ut} h(y)\,\mathrm{d}y}{r(t)} \overset{f.d.}{\Rightarrow} 0, \qquad t\to\infty$$

for any positive function $r(t)$ regularly varying at ∞ of positive index satisfying

$$\lim_{t\to\infty} \frac{r(t)}{c(t)h(t)} = \infty,$$

where c is the same as in (11).

Proof. Using the Cramér–Wold device and taking into account the regular variation of r, it suffices to prove that

$$\frac{\int_{[0,t]} h(t-y)\,\mathrm{d}(\nu(y)-y/\mu)}{r(t)} = \frac{\sum_{k\geq 0} h(t-S_k)\mathbb{1}_{\{S_k\leq t\}} - (1/\mu)\int_0^t h(y)\,\mathrm{d}y}{r(t)} \overset{\mathbb{P}}{\to} 0. \tag{33}$$

By assumption, there exists a $t_0>0$ such that h is monotone and nonnegative on $[t_0,\infty)$. Let $h^*\in D$ be an arbitrary function which coincides with h on $[t_0,\infty)$. Then, for $t>t_0$,

$$\begin{aligned}\left|\int_{[0,t]} \big(h(t-y)-h^*(t-y)\big)\,\mathrm{d}\nu(y)\right| &= \left|\int_{(t-t_0,t]} \big(h(t-y)-h^*(t-y)\big)\,\mathrm{d}\nu(y)\right| \\ &\leq \sup_{0\leq y\leq t_0} \big|h(y)-h^*(y)\big|\big(\nu(t)-\nu(t-t_0)\big) \\ &\overset{\mathrm{d}}{\leq} \sup_{0\leq y\leq t_0} \big|h(y)-h^*(y)\big|\nu(t_0),\end{aligned}$$

where $Z_1 \overset{\mathrm{d}}{\leq} Z_2$ means that $\mathbb{P}\{Z_1>x\}\leq \mathbb{P}\{Z_2>x\}$ for all $x\in\mathbb{R}$, and the last inequality in the displayed formula follows from the distributional subadditivity of ν. Analogously,

$$\left|\int_{[0,t]} \big(h(t-y)-h^*(t-y)\big)\,\mathrm{d}y\right| \leq \sup_{0\leq y\leq t_0} \big|h(y)-h^*(y)\big|t_0.$$

Hence while proving (33), we can replace h with h^*. Choosing t_0 large enough we make h^* monotone and nonnegative on $[0,\infty)$. Furthermore, if h^* is increasing on $[t_0,\infty)$, we set $h^*(t)=0$ for $t\in[0,t_0)$ thereby ensuring that $h^*(0)=0$.

Case where h^* is increasing. Integration by parts reveals that it is enough to prove

$$\frac{1}{r(t)}\int_{[0,1]} \big(\nu(t)-\nu\big(t(1-y)-\big)-\mu^{-1}ty\big)\,\mathrm{d}\big(-h^*\big(t(1-y)\big)\big) \overset{\mathbb{P}}{\to} 0. \tag{34}$$

By monotonicity, $h^*(t(1-y))/h^*(t)\leq 1$ for all $y\in[0,1]$. Hence, $\lim_{t\to\infty}\frac{h^*(t(1-y))}{r(t)/c(t)}=0$. For sufficiently large t, define finite measures ρ_t on $[0,1]$ by

$$\rho_t\big([0,a]\big) = \frac{r(t)/c(t)-h^*(t(1-a))}{r(t)/c(t)}, \qquad a\in[0,1].$$

Then the ρ_t converge weakly to δ_0 as $t \to \infty$. Applying the continuous mapping $\mathcal{V}: D \to D[0,1]$ with $\mathcal{V}(f(\cdot)) = f(1) - f((1-\cdot)-)$ to (11) we obtain

$$\frac{\nu(t) - \nu(t(1-y)-) - \mu^{-1}ty}{\mu^{-1-1/\alpha}c(t)} \Rightarrow \mathcal{S}_\alpha(1) - \mathcal{S}_\alpha\big((1-y)-\big)$$

in the J_1- or M_1-topology on $D[0,1]$. Invoking Lemma 5.4(b) yields (34), since $(\mathcal{S}_\alpha(1) - \mathcal{S}_\alpha((1-y)-))_{y\in[0,1]}$ is a.s. continuous at zero and $\mathcal{S}_\alpha(1) - \mathcal{S}_\alpha(1-) = 0$ a.s.

Case where h^* is decreasing. Integration by parts reveals that we have to prove

$$\begin{aligned} \frac{\nu(t) - \mu^{-1}t}{r(t)}h^*(t) &\overset{\mathbb{P}}{\to} 0 \quad \text{and} \\ \frac{1}{r(t)}\int_{[0,t]}\left(\nu(t) - \nu\big((t-y)-\big) - \frac{y}{\mu}\right)\mathrm{d}\big(-h^*(y)\big) &\overset{\mathbb{P}}{\to} 0. \end{aligned} \tag{35}$$

The first of these is a consequence of the assumption $\lim_{t\to\infty} r(t)/(c(t)h(t)) = \infty$ and (11). Arguing as in the proof of Theorem 2.7 on pages 2160–2161 in [22] (note that $g(t)$ in [22] corresponds to $\mu^{-1-1/\alpha}c(t)$ in this paper), we observe that the second relation in (35) follows once we can check that

$$\lim_{t\to\infty} \frac{\int_{[t_0,t]} y^{1/\alpha-\delta}\,\mathrm{d}(-h^*(y))}{t^{1/\alpha-\delta}r(t)/c(t)} = 0$$

for some $\delta \in (0, 1/\alpha)$ and $t_0 = t_0(\delta) > 0$ specified in Lemma 3.2 of [22] (recall that $\alpha = 2$ corresponds to the case where the limit process in (11) is a Brownian motion). Pick δ to further satisfy $\delta < \gamma$, where γ is the index of regular variation of r. By Lemma 5.3, $c(t)$ is regularly varying at ∞ of index $1/\alpha$. Hence, the function $t \mapsto t^{1/\alpha-\delta}r(t)/c(t)$ is regularly varying at ∞ of the positive index $\gamma - \delta$ which particularly implies $\lim_{t\to\infty} t^{1/\alpha-\delta}r(t)/c(t) = \infty$. Integration by parts yields

$$\begin{aligned} \frac{\int_{[t_0,t]} y^{1/\alpha-\delta}\,\mathrm{d}(-h^*(y))}{t^{1/\alpha-\delta}r(t)/c(t)} &= \frac{-t^{1/\alpha-\delta}h^*(t)}{t^{1/\alpha-\delta}r(t)/c(t)} \\ &\quad + \frac{t_0^{1/\alpha-\delta}h^*(t_0)}{t^{1/\alpha-\delta}r(t)/c(t)} + \left(\frac{1}{\alpha} - \delta\right)\frac{\int_{t_0}^t y^{1/\alpha-\delta-1}h^*(y)\,\mathrm{d}y}{t^{1/\alpha-\delta}r(t)/c(t)}. \end{aligned}$$

As $t \to \infty$, the first two terms converge to zero. As for the third, observe that for any $d > 0$ there exists $t(d)$ such that $h^*(t) \le d^{-1}r(t)/c(t) = d^{-1}t^{\gamma-1/\alpha}\ell(t)$ for all $t \ge t(d)$. With this at hand, we infer

$$\begin{aligned} \frac{\int_{t_0}^t y^{1/\alpha-\delta-1}h^*(y)\,\mathrm{d}y}{t^{1/\alpha-\delta}r(t)/c(t)} &= \frac{\int_{t_0}^{t(d)} y^{1/\alpha-\delta-1}h^*(y)\,\mathrm{d}y}{t^{1/\alpha-\delta}r(t)/c(t)} + \frac{\int_{t(d)}^t y^{1/\alpha-\delta-1}h^*(y)\,\mathrm{d}y}{t^{1/\alpha-\delta}r(t)/c(t)} \\ &\le o(1) + d^{-1}\int_{t(d)}^t y^{\gamma-\delta-1}\ell(y)\,\mathrm{d}y/t^{\gamma-\delta}\ell(t) \to d^{-1}(\gamma-\delta+1)^{-1} \end{aligned}$$

by Lemma 5.2(c). Letting $d \to \infty$ completes the proof of Lemma 4.2. □

Proof of Theorem 2.4.

Case $p = 0$: According to Proposition 2.1, (5) holds which is equivalent to

$$\frac{Y(ut) - \sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k \leq ut\}}}{\sqrt{\int_0^t v(y)\,\mathrm{d}y}} \overset{\text{f.d.}}{\Rightarrow} \sqrt{\frac{1+\beta}{\mu}} V_\beta(u) \tag{36}$$

because v is regularly varying at ∞ of index $\beta \in (-1, \infty)$.

Since $(\int_0^t v(y)\,\mathrm{d}y)^{1/2}$ is regularly varying at ∞ of positive index $\frac{1}{2}(1+\beta)$ and

$$\lim_{t\to\infty} \frac{\sqrt{\int_0^t v(y)\,\mathrm{d}y}}{c(t)|h(t)|} = +\infty,$$

Lemma 4.2[4] (with $r(t) = \sqrt{\int_0^t v(y)\,\mathrm{d}y}$) applies and yields

$$\frac{\sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k \leq ut\}} - \mu^{-1}\int_0^{ut} h(y)\,\mathrm{d}y}{\sqrt{\int_0^t v(y)\,\mathrm{d}y}} \overset{\text{f.d.}}{\Rightarrow} 0.$$

Summing the last relation and (36) finishes the proof for this case because

$$\int_0^t v(y)\,\mathrm{d}y \sim \int_0^t v(y)\,\mathrm{d}y + c(t)^2 h(t)^2.$$

Case $p > 0$: Using Theorem 1.1 in [21] when $h(t)$ is eventually nondecreasing and Theorem 2.7 in [22] when $h(t)$ is eventually nonincreasing, we infer

$$\frac{\sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k \leq ut\}} - \mu^{-1}\int_0^{ut} h(y)\,\mathrm{d}y}{c(t)h(t)} \overset{\text{f.d.}}{\Rightarrow} \mu^{-(\alpha+1)/\alpha} \int_{[0,u]} (u-y)^\rho \,\mathrm{d}\mathcal{S}_\alpha(y). \tag{37}$$

Subcase $p = 1$: By Lemma 5.3, $c(t)$ is regularly varying at ∞ of index $1/\alpha$. Hence, $c(t)h(t)$ is regularly varying of positive index. If v is dRi, an application of Lemma 4.1 (with $s(t) = c(t)h(t)$) yields

$$\frac{Y(ut) - \sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k \leq ut\}}}{c(t)h(t)} \overset{\text{f.d.}}{\Rightarrow} 0. \tag{38}$$

If $\lim_{t\to\infty} \int_0^t v(y)\,\mathrm{d}y = \infty$, then the assumption $\lim_{t\to\infty}(c(t)^2 h(t)^2 / \int_0^t v(y)\,\mathrm{d}y) = \infty$ implies that $\lim_{t\to\infty}(v(t)/\int_0^t v(y)\,\mathrm{d}y) = 0$. To see this, we can assume without loss of generality that v is

[4] Lemma 4.2 requires that h be eventually monotone and eventually nonnegative. If h is eventually nonpositive we simply replace it with $-h$.

monotone. If v is decreasing, then the claimed convergence follows immediately. Hence, consider the case where v is increasing. Since $c(t)^2h(t)^2$ is regularly varying and $\int_0^t v(y)\,\mathrm{d}y \geq v(t/2)t/2$, we conclude that there exists an $a > 0$ such that $\lim_{t\to\infty} t^a/v(t) = \infty$. Let a_* denote the infimum of these a. Then, there exists $\varepsilon > 0$ such that $t^{a_*+\varepsilon}/v(t) \to \infty$ whereas $t^{a_*+\varepsilon-1}/v(t) \to 0$. Consequently,

$$\frac{v(t)}{\int_0^t v(y)\,\mathrm{d}y} \leq \frac{v(t)}{\int_{t/2}^t v(y)\,\mathrm{d}y} \leq \frac{2v(t)}{tv(t/2)} = 2^{a_*+\varepsilon}\frac{v(t)}{t^{a_*+\varepsilon}}\frac{(t/2)^{a_*+\varepsilon-1}}{v(t/2)} \to 0$$

because both factors tend to zero by our choice of ε. Invoking Lemma 4.1 again allows us to conclude that (38) holds in this case, too. Summing (37) and (38) finishes the proof for this subcase because

$$c(t)^2h(t)^2 \sim \int_0^t v(y)\,\mathrm{d}y + c(t)^2h(t)^2.$$

Subcase $p \in (0,1)$: We only give a proof in the case $\sigma^2 < \infty$, the other cases being similar. Relation (12) then reads

$$\frac{Y(ut) - \mu^{-1}\int_0^{ut} h(y)\,\mathrm{d}y}{\sigma\sqrt{t}h(t)} \overset{\text{f.d.}}{\Rightarrow} c_1 V_\beta(u) + c_2\int_0^u (u-y)^\rho\,\mathrm{d}\mathcal{S}_\alpha(y), \tag{39}$$

where $c_1 := \sqrt{\frac{(1-p)(1+\beta)}{p\mu}}$ and $c_2 := \mu^{-(\alpha+1)/\alpha}$. Write

$$\begin{aligned}\frac{Y(ut) - \mu^{-1}\int_0^{ut} h(y)\,\mathrm{d}y}{\sigma\sqrt{t}h(t)} &= \frac{Y(ut) - \sum_{k\geq 0} h(ut-S_k)\mathbb{1}_{\{S_k\leq ut\}}}{\sigma\sqrt{t}h(t)} \\ &\quad + \frac{\sum_{k\geq 0} h(ut-S_k)\mathbb{1}_{\{S_k\leq ut\}} - \mu^{-1}\int_0^{ut} h(y)\,\mathrm{d}y}{\sigma\sqrt{t}h(t)} \\ &=: A_t(u) + B_t(u).\end{aligned}$$

According to Proposition 2.1, (36) holds which is equivalent to

$$A_t(u) \overset{\text{f.d.}}{\Rightarrow} c_1 V_\beta(u).$$

From (37), we already know that

$$B_t(u) \overset{\text{f.d.}}{\Rightarrow} c_2\int_{[0,u]} (u-y)^\rho\,\mathrm{d}\mathcal{S}_2(y). \tag{40}$$

By the Cramér–Wold device and Lévy's continuity theorem, in order to prove (39) it suffices to check that, for any $m \in \mathbb{N}$, any real numbers $\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_m$, any $0 < u_1 < \dots < u_m < \infty$

and any $w, z \in \mathbb{R}$,

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[\exp\left(\mathrm{i}w\sum_{j=1}^{m}\alpha_j A_t(u_j)+\mathrm{i}z\sum_{r=1}^{m}\beta_r B_t(u_r)\right)\right] \\
&= \mathbb{E}\left[\exp\left(\mathrm{i}wc_1\sum_{j=1}^{m}\alpha_j V_\beta(u_j)\right)\right]\mathbb{E}\left[\exp\left(\mathrm{i}zc_2\sum_{r=1}^{m}\beta_r\int_{[0,u_r]}(u_r-y)^\rho\,\mathrm{d}\mathcal{S}_2(y)\right)\right] \qquad (41)\\
&= \exp\bigl(-D(u_1,\dots,u_m)c_1^2w^2/2\bigr)\mathbb{E}\left[\exp\left(\mathrm{i}zc_2\sum_{r=1}^{m}\beta_r\int_{[0,u_r]}(u_r-y)^\rho\,\mathrm{d}\mathcal{S}_2(y)\right)\right]
\end{aligned}$$

with $D(u_1,\dots,u_m)$ defined in (16).

The idea behind the subsequent proof is that while the B_t is $\mathcal{F}$-measurable, the finite-dimensional distributions of the A_t converge weakly conditionally on $\mathcal{F}$. To make this precise, we write

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}}\left[\exp\left(\mathrm{i}w\sum_{j=1}^{m}\alpha_j A_t(u_j)+\mathrm{i}z\sum_{r=1}^{m}\beta_r B_t(u_r)\right)\right]\\
&\quad=\exp\left(\mathrm{i}z\sum_{r=1}^{m}\beta_r B_t(u_r)\right)\mathbb{E}_{\mathcal{F}}\left[\exp\left(\mathrm{i}w\sum_{j=1}^{m}\alpha_j A_t(u_j)\right)\right].
\end{aligned}$$

In view of (40)

$$\exp\left(\mathrm{i}z\sum_{r=1}^{m}\beta_r B_t(u_r)\right)\xrightarrow{\mathrm{d}}\exp\left(\mathrm{i}zc_2\sum_{r=1}^{m}\beta_r\int_{[0,u_r]}(u_r-y)^\rho\,\mathrm{d}\mathcal{S}_2(y)\right).$$

Since X and ξ are assumed independent, relations (17) and (18) read

$$\sum_{k\ge0}\mathbb{E}_{\mathcal{F}}\bigl[Z_{k+1,t}^2\bigr]\xrightarrow{\mathbb{P}}D(u_1,\dots,u_m)$$

and

$$\sum_{k\ge0}\mathbb{E}_{\mathcal{F}}\bigl[Z_{k+1,t}^2\mathbb{1}_{\{|Z_{k+1,t}|>y\}}\bigr]\xrightarrow{\mathbb{P}}0$$

for all $y>0$, respectively. With these at hand and noting that

$$y(t):=\frac{\sqrt{\mu^{-1}tv(t)}}{\sigma\sqrt{t}h(t)}\to c_1,$$

we infer

$$\mathbb{E}_{\mathcal{F}}\left[\exp\left(\mathrm{i}w\sum_{j=1}^{m}\alpha_j A_t(u_j)\right)\right] = \mathbb{E}_{\mathcal{F}}\left[\exp\left(\mathrm{i}wy(t)\sum_{k\geq 0}Z_{k+1,t}\right)\right]$$
$$\stackrel{\mathrm{d}}{\to} \exp\bigl(-D(u_1,\dots,u_m)c_1^2w^2/2\bigr)$$

by formula (55) of Lemma 5.6. Since the right-hand side of the last expression is non-random, Slutsky's lemma implies

$$\exp\left(\mathrm{i}z\sum_{r=1}^{m}\beta_r B_t(u_r)\right)\mathbb{E}_{\mathcal{F}}\left[\exp\left(\mathrm{i}w\sum_{j=1}^{m}\alpha_j A_t(u_j)\right)\right]$$
$$\stackrel{\mathrm{d}}{\to} \exp\left(\mathrm{i}zc_2\sum_{r=1}^{m}\beta_r\int_{[0,u_r]}(u_r-y)^{\rho}\,\mathrm{d}\mathcal{S}_2(y)\right)\exp\bigl(-D(u_1,\dots,u_m)c_1^2w^2/2\bigr).$$

Invoking the Lebesgue dominated convergence theorem completes the proof of (41). □

Proof of Theorem 2.5.

Case $q=0$: According to Proposition 2.2

$$\sqrt{\frac{\mathbb{P}\{\xi>t\}}{v(t)}}\left(Y(ut)-\sum_{k\geq 0}h(ut-S_k)\mathbb{1}_{\{S_k\leq ut\}}\right)\stackrel{\text{f.d.}}{\Rightarrow} Z_{\alpha,\beta}(u). \tag{42}$$

It remains to show that

$$\sqrt{\frac{\mathbb{P}\{\xi>t\}}{v(t)}}\sum_{k\geq 0}h(ut-S_k)\mathbb{1}_{\{S_k\leq ut\}}\stackrel{\text{f.d.}}{\Rightarrow} 0.$$

Invoking the Cramér–Wold device, Markov's inequality and the regular variation of the normalization factor, we conclude that it is enough to prove that

$$\sqrt{\frac{\mathbb{P}\{\xi>t\}}{v(t)}}\mathbb{E}\left[\sum_{k\geq 0}\bigl|h(t-S_k)\bigr|\mathbb{1}_{\{S_k\leq t\}}\right]=\sqrt{\frac{\mathbb{P}\{\xi>t\}}{v(t)}}\int_{[0,t]}\bigl|h(t-x)\bigr|\,\mathrm{d}U(x)\to 0. \tag{43}$$

This follows immediately from Lemma 5.12(b) with $\phi_1(t)=|h(t)|$, $\phi(t)=\sqrt{v(t)\mathbb{P}\{\xi>t\}}$, $\gamma=(\beta-\alpha)/2$ and $q(t)=\sqrt{u(t)}$ for $u(t)$ defined in Proposition 2.2. Note that $\phi_1=o(\phi)$ in view of the assumption $q=0$. The proof for this case is complete because

$$\frac{\mathbb{P}\{\xi>t\}}{\sqrt{v(t)\mathbb{P}\{\xi>t\}+h(t)^2}}\sim\sqrt{\frac{\mathbb{P}\{\xi>t\}}{v(t)}}.$$

Case $q = 1$: Using Theorem 1.1 in [21] when $\rho > 0$[5] and Theorem 2.9 in [22] when $\rho \in [-\alpha, 0]$, we infer

$$\frac{\mathbb{P}\{\xi > t\}}{h(t)} \sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k \leq ut\}} \overset{\text{f.d.}}{\Rightarrow} \int_{[0,u]} (u-y)^\rho \, \mathrm{d}W_\alpha^{\leftarrow}(y).$$

It remains to show that

$$\frac{\mathbb{P}\{\xi > t\}}{h(t)} \Big(Y(ut) - \sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k \leq ut\}} \Big) \overset{\text{f.d.}}{\Rightarrow} 0.$$

Appealing to Markov's inequality and the Cramér–Wold device we conclude that it suffices to prove

$$\Big(\frac{\mathbb{P}\{\xi > t\}}{h(t)}\Big)^2 \mathbb{E}\Big[\Big(Y(ut) - \sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k \leq ut\}}\Big)^2\Big]$$
$$= \Big(\frac{\mathbb{P}\{\xi > t\}}{h(t)}\Big)^2 \int_{[0,t]} v(t-y)\,\mathrm{d}U(y) \to 0.$$

This immediately follows from Lemma 5.12(b) with $\phi_1(t) = v(t)$, $\phi(t) = h(t)^2/\mathbb{P}\{\xi > t\}$, $\gamma = 2\rho + \alpha$ and $q(t) = w(t)^2$. Note that $\phi_1 = o(\phi)$ in view of the assumption $q = 1$. The proof for this case is complete because (trivially)

$$\frac{\mathbb{P}\{\xi > t\}}{\sqrt{v(t)\mathbb{P}\{\xi > t\} + h(t)^2}} \sim \frac{\mathbb{P}\{\xi > t\}}{h(t)}.$$

Case $q \in (0, 1)$: Put

$$\bar{A}_t(u) := \sqrt{\frac{\mathbb{P}\{\xi > t\}}{v(t)}} \sum_{k\geq 0} \big(X_{k+1}(ut - S_k) - h(ut - S_k)\big)\mathbb{1}_{\{S_k \leq ut\}},$$
$$\bar{B}_t(u) := \sqrt{\frac{\mathbb{P}\{\xi > t\}}{v(t)}} \sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k \leq ut\}}$$

and

$$A_{\alpha,\beta}(u) := q^{1/2}(1-q)^{-1/2} \int_{[0,u]} (u-y)^{(\beta-\alpha)/2} \, \mathrm{d}W_\alpha^{\leftarrow}(y).$$

[5] In Theorem 1.1 of [21] functional limit theorems were proved under the assumption that h is eventually nondecreasing. The latter assumption is not needed for weak convergence of finite-dimensional distributions which can be seen by mimicking the proof of Theorem 2.9 in [22].

We shall prove that

$$\sum_{j=1}^{m}\gamma_j\big(\bar{A}_t(u_j)+\bar{B}_t(u_j)\big)\overset{\mathrm{d}}{\to}\sum_{j=1}^{m}\gamma_j\big(Z_{\alpha,\beta}(u_j)+A_{\alpha,\beta}(u_j)\big)$$

for any $m\in\mathbb{N}$, any $\gamma_1,\dots,\gamma_m\in\mathbb{R}$ and any $0<u_1<\cdots<u_m<\infty$.

Set

$$\bar{Z}_{k+1,t}:=\sqrt{\mathbb{P}\{\xi>t\}/v(t)}\sum_{j=1}^{m}\gamma_j\big(X_{k+1}(u_jt-S_k)-h(u_jt-S_k)\big)\mathbb{1}_{\{S_k\le u_jt\}},\qquad k\in\mathbb{N}_0,t>0.$$

Then $\sum_{j=1}^{m}\gamma_j\bar{A}_t(u_j)=\sum_{k\ge 0}\bar{Z}_{k+1,t}$ and

$$\sum_{k\ge 0}\mathbb{E}_{\mathcal{F}}\big[\bar{Z}_{k+1,t}^2\big]=\frac{\mathbb{P}\{\xi>t\}}{v(t)}\int_{[0,u_m]}\Bigg(\sum_{j=1}^{m}\gamma_j^2v\big(t(u_j-y)\big)\mathbb{1}_{[0,u_j]}(y)$$
$$+2\sum_{1\le r<l\le m}\gamma_r\gamma_l f\big(t(u_r-y),t(u_l-y)\big)\mathbb{1}_{[0,u_r]}(y)\Bigg)\,\mathrm{d}\nu(ty).$$

With this at hand, we write

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}}\Bigg[\exp\Bigg(\mathrm{i}z\sum_{j=1}^{m}\gamma_j\big(\bar{A}_t(u_j)+\bar{B}_t(u_j)\big)\Bigg)\Bigg]\\
&=\exp\Bigg(\mathrm{i}z\sum_{j=1}^{m}\gamma_j\bar{B}_t(u_j)\Bigg)\mathbb{E}_{\mathcal{F}}\Bigg[\exp\Bigg(\mathrm{i}z\sum_{k\ge 0}\bar{Z}_{k+1,t}\Bigg)\Bigg]\\
&=\exp\Bigg(\mathrm{i}z\sum_{j=1}^{m}\gamma_j\bar{B}_t(u_j)\Bigg)\Bigg(\mathbb{E}_{\mathcal{F}}\Bigg[\exp\Bigg(\mathrm{i}z\sum_{k\ge 0}\bar{Z}_{k+1,t}\Bigg)\Bigg]-\exp\Bigg(-\sum_{k\ge 0}\mathbb{E}_{\mathcal{F}}\big[\bar{Z}_{k+1,t}^2\big]z^2/2\Bigg)\Bigg)\\
&\quad+\exp\Bigg(\mathrm{i}z\sum_{j=1}^{m}\gamma_j\bar{B}_t(u_j)-\sum_{k\ge 0}\mathbb{E}_{\mathcal{F}}\big[\bar{Z}_{k+1,t}^2\big]z^2/2\Bigg)
\end{aligned}\tag{44}$$

for $z\in\mathbb{R}$.

By formula (66) of Lemma 5.8 (with $b=q^{-1}(1-q)$)

$$\lambda_1\sum_{j=1}^{m}\gamma_j\bar{B}_t(u_j)+\lambda_2\sum_{k\ge 0}\mathbb{E}_{\mathcal{F}}\big[\bar{Z}_{k+1,t}^2\big]$$
$$=\lambda_1\sqrt{\frac{\mathbb{P}\{\xi>t\}}{v(t)}}\int_{[0,u_m]}\sum_{j=1}^{m}\gamma_jh\big(t(u_j-y)\big)\mathbb{1}_{[0,u_j]}(y)\,\mathrm{d}\nu(ty)$$

$$+\lambda_2 \frac{\mathbb{P}\{\xi > t\}}{v(t)} \int_{[0,u_m]} \Bigg(\sum_{j=1}^{m} \gamma_j^2 v\big(t(u_j - y)\big) \mathbb{1}_{[0,u_j]}(y) \tag{45}$$

$$+ 2 \sum_{1 \le r < l \le m} \gamma_r \gamma_l f\big(t(u_r - y), t(u_l - y)\big) \mathbb{1}_{[0,u_r]}(y) \Bigg) \mathrm{d}\nu(ty)$$

$$\stackrel{\mathrm{d}}{\to} \lambda_1 \sum_{j=1}^{m} \gamma_j A_{\alpha,\beta}(u_j) + \lambda_2 D_{\alpha,\beta}(u_1, \ldots, u_m)$$

for any real λ_1 and λ_2 with $D_{\alpha,\beta}(u_1, \ldots, u_m)$ defined in (27). Hence,

$$\exp\Bigg(\mathrm{i}z \sum_{j=1}^{m} \gamma_j \bar{B}_t(u_j) - \sum_{k \ge 0} \mathbb{E}_{\mathcal{F}}[\bar{Z}_{k+1,t}^2 | z^2/2 \Bigg)$$

$$\stackrel{\mathrm{d}}{\to} \exp\Bigg(\mathrm{i}z \sum_{j=1}^{m} \gamma_j A_{\alpha,\beta}(u_j) - D_{\alpha,\beta}(u_1, \ldots, u_m) z^2/2 \Bigg)$$

for each $z \in \mathbb{R}$, and thereupon

$$\lim_{t \to \infty} \mathbb{E}\Bigg[\exp\Bigg(\mathrm{i}z \sum_{j=1}^{m} \gamma_j \bar{B}_t(u_j) - \sum_{k \ge 0} \mathbb{E}_{\mathcal{F}}\big[\bar{Z}_{k+1,t}^2\big] z^2/2 \Bigg) \Bigg]$$

$$= \mathbb{E}\Bigg[\exp\Bigg(\mathrm{i}z \sum_{j=1}^{m} \gamma_j A_{\alpha,\beta}(u_j) - D_{\alpha,\beta}(u_1, \ldots, u_m) z^2/2 \Bigg) \Bigg]$$

$$= \mathbb{E}\Bigg[\exp\Bigg(\mathrm{i}z \sum_{j=1}^{m} \gamma_j \big(A_{\alpha,\beta}(u_j) + Z_{\alpha,\beta}(u_j) \big) \Bigg) \Bigg]$$

by Lebesgue's dominated convergence theorem, the second equality following from the fact that $\sum_{j=1}^{m} \gamma_j Z_{\alpha,\beta}(u_j)$ is centered normal with variance $D_{\alpha,\beta}(u_1, \ldots, u_m)$.

According to Formula (57) of Lemma 5.6

$$\mathbb{E}_{\mathcal{F}}\Bigg[\exp\Bigg(\mathrm{i}z \sum_{k \ge 0} \bar{Z}_{k+1,t} \Bigg) \Bigg] - \exp\Bigg(- \sum_{k \ge 0} \mathbb{E}_{\mathcal{F}}\big[\bar{Z}_{k+1,t}^2\big] z^2/2 \Bigg) \stackrel{\mathbb{P}}{\to} 0.$$

Hence the first summand on the right-hand side of (44) tends to zero in probability if we verify that

$$\sum_{k \ge 0} \mathbb{E}_{\mathcal{F}}\big[\bar{Z}_{k+1,t}^2\big] \stackrel{\mathrm{d}}{\to} D_{\alpha,\beta}(u_1, \ldots, u_m) \tag{46}$$

and

$$\sum_{k\geq 0}\mathbb{E}_{\mathcal{F}}\big[\bar{Z}_{k+1,t}^2\mathbb{1}_{\{|\bar{Z}_{k+1,t}|>y\}}\big]\overset{\mathbb{P}}{\to}0 \tag{47}$$

for all $y>0$. Relation (46) follows from (45) with $\lambda_1=0$ and $\lambda_2=1$. In view of the inequality (19) relation (47) is implied by (31) which has already been checked. This finishes the proof for this case because $\lim_{t\to\infty}\frac{\mathbb{P}\{\xi>t\}v(t)}{\mathbb{P}\{\xi>t\}v(t)+h(t)^2}=1-q$ ensures that

$$\sqrt{1-q}\sqrt{\frac{\mathbb{P}\{\xi>t\}}{v(t)}}\sim\frac{\mathbb{P}\{\xi>t\}}{\sqrt{\mathbb{P}\{\xi>t\}v(t)+h(t)^2}}.$$

The proof of Theorem 2.5 is complete. □

4.3. Proofs of Corollary 2.6 and Proposition 2.7

Proof of Corollary 2.6. We first show that the function $f(u,w)=\mathbb{E}[X(u)X(w)]-\mathbb{E}[X(u)]\times\mathbb{E}[X(w)]$ is uniformly regularly varying in strips in $\mathbb{R}_+^2$ of index β with limit function C.

The assumption $q<1$ ensures

$$\lim_{t\to\infty}v(t)/h(t)^2=\infty,$$

hence $\mathbb{E}[X(t)^2]\sim v(t)$, in particular, $v(t)$ is regularly varying of index β which must be nonnegative. Further, $\lim_{t\to\infty}\mathbb{E}[X(ut)]\mathbb{E}[X(wt)]/v(t)=0$ because

$$\frac{\mathbb{E}[X(ut)]\mathbb{E}[X(wt)]}{v(t)}\leq\frac{(\mathbb{E}[X(wt)])^2}{v(wt)}\frac{v(wt)}{v(t)}$$

for $0<u<w$ by monotonicity. More importantly, $\lim_{t\to\infty}\mathbb{E}[X(ut)X(wt)]/v(t)=C(u,w)$, and the function C is continuous in $\mathbb{R}_+^2$ (see Lemma 2 in [9]) because, for each $t>0$, the function $(u,w)\mapsto\mathbb{E}[X(ut)X(wt)]$ is increasing in each variable. Recall that convergence of monotone functions to a continuous limit is necessarily locally uniform. Therefore, in both limit relations above the convergence is locally uniform in $\mathbb{R}_+^2$. Hence

$$\lim_{t\to\infty}\frac{f(ut,wt)}{v(t)}=C(u,w)$$

locally uniformly in $\mathbb{R}_+^2$ which entails the uniformity in strips, as desired.

Recall that $\beta\geq 0$ and note that whenever h is regularly varying of index ρ we must have $\rho\geq 0$. Putting

$$Q_{\alpha,\rho}(u):=\int_{[0,u]}(u-y)^\rho\,\mathrm{d}W_\alpha^{\leftarrow}(y),\qquad u\geq 0$$

we observe that $Q_{\alpha,\rho}:=(Q_{\alpha,\rho}(u))_{u\geq 0}$ is a.s. continuous on $[0,\infty)$ with $Q_{\alpha,\rho}(0)=0$ (in the case $\rho=0$ the process is just $W_\alpha^{\leftarrow}$ and the random function $u\mapsto W_\alpha^{\leftarrow}(u)$ is a.s. continuous as the generalized inverse of the a.s. strictly increasing random function $t \mapsto W_\alpha(t)$). Now we check that, by continuity, we can define $Z_{\alpha,\beta}(0)$ to be equal to 0. To this end, observe that

$$\mathbb{E}\big[Z_{\alpha,\beta}(u)^2\big] = \mathbb{E}\bigg[\int_{[0,u]} (u-y)^\beta \, \mathrm{d}W_\alpha^{\leftarrow}(y)\bigg] = \frac{\Gamma(\beta+1)}{\Gamma(1-\alpha)\Gamma(\alpha+\beta+1)} u^{\beta+\alpha} \to 0, \qquad u \downarrow 0$$

having used (63) for the second equality. Hence, $\lim_{u\downarrow 0} Z_{\alpha,\beta}(u) = 0$ in probability. An important consequence of the fact that the limit processes are equal to zero at the origin is that the weak convergence of the finite-dimensional distributions proved in Theorem 2.5 for $u > 0$ can be extended to $u \geq 0$.

Since, for each $t > 0$, the process $(Y(ut))_{u\geq 0}$ is a.s. nondecreasing, according to Theorem 3 in [6] it remains to show that the limit processes are continuous in probability. This is obvious for $Q_{\alpha,\rho}$. Further,

$$\begin{aligned}
&\mathbb{P}\big\{\big|Z_{\alpha,\beta}(w) - Z_{\alpha,\beta}(u)\big| > \varepsilon | W_\alpha^{\leftarrow}\big\} \\
&\quad \leq \frac{1}{\varepsilon^2}\bigg(\int_{[0,u]} (u-y)^\beta \, \mathrm{d}W_\alpha^{\leftarrow}(y) \\
&\qquad + \int_{[0,w]} (w-y)^\beta \, \mathrm{d}W_\alpha^{\leftarrow}(y) - 2\int_{[0,u]} C(u-y, w-y) \, \mathrm{d}W_\alpha^{\leftarrow}(y)\bigg)
\end{aligned}$$

for $0 < u < w$ and $\varepsilon > 0$, by Chebyshev's inequality. As $w \downarrow u$, the second term converges a.s. to $\int_{[0,u]}(u-y)^\beta \, \mathrm{d}W_\alpha^{\leftarrow}(y)$ in view of the aforementioned a.s. continuity. By Fatou's lemma

$$\begin{aligned}
\liminf_{w\downarrow u} \int_{[0,u]} C(u-y, w-y) \, \mathrm{d}W_\alpha^{\leftarrow}(y) &\geq \int_{[0,u]} C(u-y, u-y) \, \mathrm{d}W_\alpha^{\leftarrow}(y) \\
&= \int_{[0,u]} (u-y)^\beta \, \mathrm{d}W_\alpha^{\leftarrow}(y)
\end{aligned}$$

as C is continuous in $\mathbb{R}_+^2$. Hence, $\lim_{w\downarrow u} \mathbb{P}\{|Z_{\alpha,\beta}(u) - Z_{\alpha,\beta}(w)| \geq \varepsilon | W_\alpha^{\leftarrow}\} = 0$ a.s. The proof of this convergence when $w \uparrow u$ is analogous. Applying now the Lebesgue dominated convergence theorem, we conclude that $(Z_{\alpha,\beta}(u))_{u\geq 0}$ is continuous in probability. The proof of Corollary 2.6 is complete. □

Proof of Proposition 2.7. Let Z_α^* be a version of $Z_{\alpha,-\alpha}$. We show that for every interval $[a,b]$ with $0 < a < b$,

$$\mathbb{E}\Big[\sup_{t\in[W_\alpha(a), W_\alpha(b)]} Z_\alpha^*(t)^2 \big| W_\alpha^{\leftarrow}\Big] = \infty \qquad \text{a.s.} \tag{48}$$

To prove this, first notice that according to Theorem 2 in [10] there exists an event Ω' with $\mathbb{P}(\Omega') = 1$ such that for any $\omega \in \Omega'$

$$\limsup_{y\uparrow s} \frac{W_\alpha(s,\omega) - W_\alpha(y,\omega)}{(s-y)^{1/\alpha}} \leq r$$

for some deterministic constant $r \in (0,\infty)$ and some $s := s(\omega) \in [a,b]$. Fix any $\omega \in \Omega'$. There exists $s_1 := s_1(\omega)$ such that

$$\big(W_\alpha(s,\omega) - W_\alpha(y,\omega)\big)^{-\alpha} \geq (s-y)^{-1} r^{-\alpha}/2$$

whenever $y \in (s_1, s)$. Set $t := t(\omega) = W_\alpha(s,\omega)$ and write

$$\begin{aligned}
\mathbb{E}\big[Z_\alpha^*(t)^2 | W_\alpha^{\leftarrow}\big](\omega) &= \int_{[0,t(\omega)]} \big(t(\omega) - y\big)^{-\alpha} \, \mathrm{d}W_\alpha^{\leftarrow}(y,\omega) \\
&= \int_{[0, W_\alpha(s,\omega)]} \big(W_\alpha(s,\omega) - y\big)^{-\alpha} \, \mathrm{d}W_\alpha^{\leftarrow}(y,\omega) \\
&= \int_0^s \big(W_\alpha(s,\omega) - W_\alpha(y,\omega)\big)^{-\alpha} \, \mathrm{d}y \\
&\geq \int_{s_1}^s \big(W_\alpha(s,\omega) - W_\alpha(y,\omega)\big)^{-\alpha} \, \mathrm{d}y \\
&\geq \frac{1}{2r^\alpha} \int_{s_1}^s (s-y)^{-1} \, \mathrm{d}y = +\infty.
\end{aligned}$$

This proves (48), for $t(\omega) \in [W_\alpha(a,\omega), W_\alpha(b,\omega)]$ for all $\omega \in \Omega'$.

Now observe that if Z_α^* has paths in $D(0,\infty)$ a.s., then, for any $0 < a < b$,

$$\mathbb{P}\Big\{\Big|\sup_{t \in [W_\alpha(a), W_\alpha(b)]} Z_\alpha^*(t)\Big| < \infty \Big| W_\alpha^{\leftarrow}\Big\} = 1. \tag{49}$$

Note that the process W_α is measurable with respect to the σ-field generated by $W_\alpha^{\leftarrow}$ and that, given $W_\alpha^{\leftarrow}$, the process Z_α^* is centered Gaussian. Hence, from Theorem 3.2 on page 63 in [1] (applied to $(Z_\alpha^*(t))_{t \in [W_\alpha(a), W_\alpha(b)]}$ and $(-Z_\alpha^*(t))_{t \in [W_\alpha(a), W_\alpha(b)]}$ both conditionally given $W_\alpha^{\leftarrow}$), we conclude that (49) is equivalent to

$$\mathbb{E}\Big[\sup_{t \in [W_\alpha(a), W_\alpha(b)]} Z_\alpha^*(t)^2 \Big| W_\alpha^{\leftarrow}\Big] < \infty \qquad \text{a.s.} \tag{50}$$

which cannot hold due to (48). Hence Z_α^* has paths in $D(0,\infty)$ with probability less than 1.

Finally, suppose that $C(u,w) = 0$ for all $u \neq w$, $u, w > 0$. Then, given $W_\alpha^{\leftarrow}$, the Gaussian process Z_α^* has uncorrelated, hence independent values. For any fixed $t > 0$ and any decreasing sequence $(h_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} h_n = 0$ we infer

$$\mathbb{P}\big\{Z_\alpha^* \text{ is right-continuous at } t | W_\alpha^{\leftarrow}\big\} \leq \mathbb{P}\Big\{\limsup_{n\to\infty} Z_\alpha^*(t+h_n) = Z_\alpha^*(t) \Big| W_\alpha^{\leftarrow}\Big\} = 0 \qquad \text{a.s.} \tag{51}$$

which proves that Z_α^* has paths in the Skorokhod space with probability 0. To justify (51) observe that, given $W_\alpha^{\leftarrow}$, the distribution of $Z_\alpha^*(t)$ is Gaussian, hence continuous, while $\limsup_{n\to\infty} Z_\alpha^*(t+h_n)$ is equal to a constant (possibly $\pm\infty$) a.s. by the Kolmogorov zero–

one law which is applicable because $Z^*_\alpha(t+h_1)$, $Z^*_\alpha(t+h_2), \dots$ are (conditionally) independent. The proof of Proposition 2.7 is complete. □

5. Auxiliary results

In this section, we collect technical results some of which are known and stated here for the reader's convenience. Others are extensions of known results or important technical steps used more than once in the derivations of our main results. We begin with a series of known results: Lemma 5.1 is Theorem 4.2 in [4], Lemma 5.2(a) is Theorem 1.5.2 from [7], Lemma 5.2(b) is a consequence of Theorem 1.5.3 in [7], Lemma 5.2(c) is Karamata's theorem (Proposition 1.5.8 in [7]), Lemma 5.3 is Lemma 3.2 in [22].

Lemma 5.1. *Let (S,d) be an arbitrary metric space. Suppose that (Z_{un}, Z_n) are random elements on $S \times S$. If $Z_{un} \Rightarrow_n Z_u \Rightarrow_u Z$ on (S,d) and*

$$\lim_u \limsup_n \mathbb{P}\big\{d(Z_{un}, Z_n) > \varepsilon\big\} = 0$$

for every $\varepsilon > 0$, then $Z_n \Rightarrow Z$ on (S,d), as $n \to \infty$.

Lemma 5.2. *Let g be regularly varying at ∞ of index ρ and locally bounded outside zero.*

(a) *Then, for all $0 < a < b < \infty$,*

$$\lim_{t\to\infty} \sup_{a \le s \le b} \left| \frac{g(st)}{g(t)} - s^\rho \right| = 0.$$

(b) *Suppose $\rho \ne 0$. Then there exists a monotone function such that $g(t) \sim u(t)$ as $t \to \infty$.*
(c) *Let $\rho > -1$ and $a > 0$. Then $\int_a^t g(y)\,\mathrm{d}y \sim (\rho+1) t g(t)$ as $t \to \infty$.*

Lemma 5.3. *$c(t)$ appearing in (11) is regularly varying at ∞ of index $1/\alpha$.*

Lemma 5.4 follows from Lemma A.5 in [21] in combination with the continuous mapping theorem. We note in passing that [34] and Chapter VI, Section 6c in [25] are classical references concerning the convergence of stochastic integrals.

Lemma 5.4. *Let $0 \le a < b < \infty$.*

(a) *Suppose that, for each $t > 0$, $f_t \in D$ and that the random process $(\mathcal{X}_t(y))_{a \le y \le b}$ has almost surely increasing path. Assume further that $\lim_{t\to\infty} f_t(y) = f(y)$ uniformly in $y \in [a,b]$ and that $\mathcal{X}_t \Rightarrow \mathcal{X}$, $t \to \infty$ in the J_1-topology on $D[a,b]$, the paths of $(\mathcal{X}(y))_{a\le y\le b}$ being almost surely continuous. Then*

$$\int_{[a,b]} f_t(y)\,\mathrm{d}\mathcal{X}_t(y) \xrightarrow{\mathrm{d}} \int_{[a,b]} f(y)\,\mathrm{d}\mathcal{X}(y), \qquad t \to \infty.$$

(b) *Assume that* $\mathcal{X}_t \Rightarrow \mathcal{X}$, $t \to \infty$, *in the* J_1- *or* M_1-*topology on* $D[a,b]$ *and that, as* $t \to \infty$, *finite measures* ρ_t *converge weakly on* $[a,b]$ *to* δ_c, *the Dirac measure concentrated at* c. *If* $\mathcal{X}$ *is almost surely continuous at* c, *then*

$$\int_{[a,b]} \mathcal{X}_t(y)\rho_t(\mathrm{d}y) \xrightarrow{d} \mathcal{X}(c), \qquad t \to \infty.$$

Lemma 5.5. *Let* W *be a nonnegative random variable with Laplace transform* $\varphi(s) := \mathbb{E}[e^{-sW}]$, $s \ge 0$. *Then, for* $\theta \in (0,1)$,

$$\mathbb{E}\bigl[W^{-\theta}\bigr] = \frac{1}{\Gamma(\theta)} \int_0^\infty s^{\theta-1}\varphi(s)\,\mathrm{d}s. \tag{52}$$

Proof. Let R be a random variable with the standard exponential law which is independent of W. Then, for $s \ge 0$, $\varphi(s) = \mathbb{P}\{R/W > s\}$. Hence, $\Gamma(1+\theta)\mathbb{E}[W^{-\theta}] = \mathbb{E}[(R/W)^\theta] = \theta\int_0^\infty s^{\theta-1}\varphi(s)\,\mathrm{d}s$. □

Lemma 5.6. *Let* $(Z_{k,t})_{k\in\mathbb{N},t>0}$ *be a family of random variables defined on some probability space* $(\Omega, \mathcal{R}, \mathbb{P})$ *and let* $\mathcal{G}$ *be a sub-*σ*-algebra of* $\mathcal{R}$. *Assume that, given* $\mathcal{G}$ *and for each fixed* $t > 0$, *the* $Z_{k,t}$, $k \in \mathbb{N}$ *are independent. If*

$$\sum_{k\ge0} \mathbb{E}_{\mathcal{G}}\bigl[Z_{k+1,t}^2\bigr] \xrightarrow{d} D, \qquad t \to \infty \tag{53}$$

for a random variable D *and*

$$\sum_{k\ge0} \mathbb{E}_{\mathcal{G}}\bigl[Z_{k+1,t}^2 \mathbb{1}_{\{|Z_{k+1,t}|>y\}}\bigr] \xrightarrow{\mathbb{P}} 0, \qquad t \to \infty \tag{54}$$

for all $y > 0$, *then, for each* $z \in \mathbb{R}$,

$$\mathbb{E}_{\mathcal{G}}\biggl[\exp\biggl(\mathrm{i}z\sum_{k\ge0} Z_{k+1,t}\biggr)\biggr] \xrightarrow{d} \exp\bigl(-Dz^2/2\bigr), \qquad t \to \infty, \tag{55}$$

$$\mathbb{E}\biggl[\exp\biggl(\mathrm{i}z\sum_{k\ge0} Z_{k+1,t}\biggr)\biggr] \to \mathbb{E}\bigl[\exp\bigl(-Dz^2/2\bigr)\bigr], \qquad t \to \infty \tag{56}$$

and

$$\mathbb{E}_{\mathcal{G}}\biggl[\exp\biggl(\mathrm{i}z\sum_{k\ge0} Z_{k+1,t}\biggr)\biggr] - \mathbb{E}_{\mathcal{G}}\biggl[\exp\biggl(\mathrm{i}z\sum_{k\ge0} \widehat{Z}_{k+1,t}\biggr)\biggr] \xrightarrow{\mathbb{P}} 0, \qquad t \to \infty, \tag{57}$$

where, given $\mathcal{G}$, $\widehat{Z}_{1,t}, \widehat{Z}_{2,t}, \ldots$ *are conditionally independent normal random variables with mean* 0 *and variance* $\mathbb{E}_{\mathcal{G}}[Z_{k+1,t}^2]$, *that is*,

$$\mathbb{E}_{\mathcal{G}}\bigl[\exp(\mathrm{i}z\widehat{Z}_{k+1,t})\bigr] = \exp\bigl(-\mathbb{E}_{\mathcal{G}}\bigl[Z_{k+1,t}^2\bigr]z^2/2\bigr), \qquad k \in \mathbb{N}_0.$$

Proof. Apart from minor modifications, the following argument can be found in the proof of Theorem 4.12 in [28] in which the weak convergence of the row sums in triangular arrays to a normal law is investigated. For any $\varepsilon > 0$,

$$\sup_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\big] \leq \varepsilon^2 + \sup_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\mathbb{1}_{\{|Z_{k+1,t}|>\varepsilon\}}\big] \leq \varepsilon^2 + \sum_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\mathbb{1}_{\{|Z_{k+1,t}|>\varepsilon\}}\big].$$

Using (54) and letting first $t \to \infty$ and then $\varepsilon \downarrow 0$, we infer

$$\sup_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\big] \xrightarrow{\mathbb{P}} 0. \tag{58}$$

In view of (53)

$$\mathbb{E}_{\mathcal{G}}\bigg[\exp\bigg(\mathrm{i}z\sum_{k\geq 0}\widehat{Z}_{k+1,t}\bigg)\bigg] = \exp\bigg(-\sum_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\big]z^2/2\bigg) \xrightarrow{\mathrm{d}} \exp\big(-Dz^2/2\big) \tag{59}$$

for each $z \in \mathbb{R}$. Next, we show that $\sum_{k\geq 0} Z_{k+1,t}$ has the same distributional limit as $\sum_{k\geq 0}\widehat{Z}_{k+1,t}$ as $t \to \infty$. To this end, for $z \in \mathbb{R}$, consider

$$\begin{aligned}
&\bigg|\mathbb{E}_{\mathcal{G}}\bigg[\exp\bigg(\mathrm{i}z\sum_{k\geq 0}Z_{k+1,t}\bigg)\bigg] - \mathbb{E}_{\mathcal{G}}\bigg[\exp\bigg(\mathrm{i}z\sum_{k\geq 0}\widehat{Z}_{k+1,t}\bigg)\bigg]\bigg| \\
&\quad = \bigg|\prod_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[\exp(\mathrm{i}zZ_{k+1,t})\big] - \prod_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[\exp(\mathrm{i}z\widehat{Z}_{k+1,t})\big]\bigg| \\
&\quad \leq \sum_{k\geq 0}\big|\mathbb{E}_{\mathcal{G}}\big[\exp(\mathrm{i}zZ_{k+1,t})\big] - \mathbb{E}_{\mathcal{G}}\big[\exp(\mathrm{i}z\widehat{Z}_{k+1,t})\big]\big| \\
&\quad \leq \sum_{k\geq 0}\bigg|\mathbb{E}_{\mathcal{G}}\big[\exp(\mathrm{i}zZ_{k+1,t})\big] - 1 + \frac{z^2}{2}\mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\big]\bigg| \\
&\qquad + \sum_{k\geq 0}\bigg|\mathbb{E}_{\mathcal{G}}\big[\exp(\mathrm{i}z\widehat{Z}_{k+1,t})\big] - 1 + \frac{z^2}{2}\mathbb{E}_{\mathcal{G}}\big[\widehat{Z}_{k+1,t}^2\big]\bigg| \\
&\quad \leq z^2\sum_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\big(1\wedge 6^{-1}|zZ_{k+1,t}|\big)\big] + z^2\sum_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[\widehat{Z}_{k+1,t}^2\big(1\wedge 6^{-1}|z\widehat{Z}_{k+1,t}|\big)\big],
\end{aligned}$$

where, to arrive at the last line, we have utilized $|\mathbb{E}_{\mathcal{G}}[\cdot]| \leq \mathbb{E}_{\mathcal{G}}[|\cdot|]$ and the inequality

$$\big|e^{\mathrm{i}z} - 1 - \mathrm{i}z + z^2/2\big| \leq z^2 \wedge 6^{-1}|z|^3, \qquad z \in \mathbb{R},$$

which can be found, for instance, in Lemma 4.14 of [28]. For any $\varepsilon \in (0, 1)$ and $z \neq 0$

$$\sum_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\big(1\wedge 6^{-1}|zZ_{k+1,t}|\big)\big] \leq \varepsilon\sum_{k\geq 0}\mathbb{E}\big[Z_{k+1,t}^2\big] + \sum_{k\geq 0}\mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\mathbb{1}_{\{|Z_{k+1,t}|>6\varepsilon/|z|\}}\big].$$

Recalling (54) and letting first $t \to \infty$ and then $\varepsilon \downarrow 0$ give

$$\sum_{k\geq 0} \mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\big(1 \wedge 6^{-1}|zZ_{k+1,t}|\big)\big] \xrightarrow{\mathbb{P}} 0.$$

Further,

$$\begin{aligned}
\sum_{k\geq 0} \mathbb{E}_{\mathcal{G}}\big[\widehat{Z}_{k+1,t}^2\big(1 \wedge 6^{-1}|z\widehat{Z}_{k+1,t}|\big)\big] &\leq \frac{|z|}{6}\sum_{k\geq 0} \mathbb{E}_{\mathcal{G}}\big[|\widehat{Z}_{k+1,t}|^3\big] \\
&= \frac{\sqrt{2}|z|}{3\sqrt{\pi}} \sum_{k\geq 0} \big(\mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\big]\big)^{3/2} \\
&\leq \frac{\sqrt{2}|z|}{3\sqrt{\pi}} \Big(\sup_{k\geq 0} \mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\big]\Big)^{1/2} \sum_{k\geq 0} \mathbb{E}_{\mathcal{G}}\big[Z_{k+1,t}^2\big].
\end{aligned}$$

Here, (53) and (58) yield

$$\sum_{k\geq 0} \mathbb{E}_{\mathcal{G}}\big[\widehat{Z}_{k+1,t}^2\big(1 \wedge 6^{-1}|z\widehat{Z}_{k+1,t}|\big)\big] \xrightarrow{\mathbb{P}} 0.$$

Thus, we have already proved (57) which together with (59) implies (55). Relation (56) follows from (55) by taking expectations and using uniform integrability. The proof of Lemma 5.6 is complete. $\square$

Lemma 5.7. *Let $\rho > -1$, $\alpha \in (0,1)$ and C denote the limit function for $f(u,w) = \mathbb{E}[X(u)X(w)] - \mathbb{E}[X(u)]\mathbb{E}[X(w)]$ wide-sense regularly varying in $\mathbb{R}_+^2$ of index β for some $\beta \geq -\alpha$. Then the integrals*

$$\begin{aligned}
&\int_{[0,s]} C(s-y,t-y)\,\mathrm{d}W_\alpha^{\leftarrow}(y), \qquad 0 < s < t < \infty \quad \text{and} \\
&\int_{[0,s]} (s-y)^\rho\,\mathrm{d}W_\alpha^{\leftarrow}(y), \qquad s > 0
\end{aligned} \tag{60}$$

exist as Lebesgue–Stieltjes integrals and are almost surely finite. Furthermore, the process $Z_{\alpha,\beta}$ is well-defined.

Proof. To begin with, we intend to show that

$$\mathbb{E}\bigg[\int_{[0,s]} (s-y)^\rho\,\mathrm{d}W_\alpha^{\leftarrow}(y)\bigg] < \infty, \qquad s > 0. \tag{61}$$

To this end, we first derive the following identity

$$\mathbb{E}\big[W_\alpha^{\leftarrow}(y)\big] = \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} y^\alpha =: d_\alpha y^\alpha, \qquad y \geq 0. \tag{62}$$

Indeed,

$$\mathbb{E}\big[W_\alpha^{\leftarrow}(y)\big] = \int_0^\infty \mathbb{P}\big\{W_\alpha^{\leftarrow}(y) > t\big\}\,\mathrm{d}t = \int_0^\infty \mathbb{P}\big\{W_\alpha(t) \le y\big\}\,\mathrm{d}t = \int_0^\infty \mathbb{P}\big\{t^{1/\alpha}W_\alpha(1) \le y\big\}\,\mathrm{d}t$$
$$= \int_0^\infty \mathbb{P}\big\{W_\alpha(1)^{-\alpha}y^\alpha \ge t\big\}\,\mathrm{d}t = \mathbb{E}\big[W_\alpha(1)^{-\alpha}\big]y^\alpha = \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)}y^\alpha,$$

where the last equality follows from (52) with $\theta = \alpha$ and $\varphi(s) = \exp(-\Gamma(1-\alpha)s^\alpha)$. Hence

$$\mathbb{E}\bigg[\int_{[0,s]}(s-y)^\rho\,\mathrm{d}W_\alpha^{\leftarrow}(y)\bigg] = \frac{\alpha}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\int_0^s (s-y)^\rho y^{\alpha-1}\,\mathrm{d}y$$
$$= \frac{\Gamma(\rho+1)}{\Gamma(1-\alpha)\Gamma(\rho+\alpha+1)}s^{\rho+\alpha} \qquad (63)$$

which proves (61).

Passing to the proof of

$$\mathbb{E}\bigg[\int_{[0,s]}C(s-y,t-y)\,\mathrm{d}W_\alpha^{\leftarrow}(y)\bigg] < \infty, \qquad 0 < s < t \qquad (64)$$

we assume that $C(u,w) > 0$ for some $u \ne w$, $u,w > 0$ and then observe that in view of (62) and (14),

$$\mathbb{E}\bigg[\int_{[0,s]}C(s-y,t-y)\,\mathrm{d}W_\alpha^{\leftarrow}(y)\bigg] = \alpha d_\alpha \int_0^s C(s-y,t-y)y^{\alpha-1}\,\mathrm{d}y$$
$$\le \frac{\alpha d_\alpha}{2}\int_0^s \big((s-y)^\beta + (t-y)^\beta\big)y^{\alpha-1}\,\mathrm{d}y < \infty$$

since $\alpha > 0$ and $\beta \ge -\alpha > -1$. This proves (64).

Further, we check that the process $Z_{\alpha,\beta}$ is well-defined. To this end, we show that the function $\Pi(s,t)$ defined by

$$\Pi(s,t) := \int_{[0,s]}C(s-y,t-y)\,\mathrm{d}W_\alpha^{\leftarrow}(y), \qquad 0 < s \le t$$

is nonnegative definite, that is, for any $m \in \mathbb{N}$, any $\gamma_1,\ldots,\gamma_m \in \mathbb{R}$ and any $0 < u_1 < \cdots < u_m < \infty$

$$\sum_{j=1}^m \gamma_j^2 \Pi(u_j,u_j) + 2\sum_{1\le r<l\le m}\gamma_r\gamma_l\Pi(u_r,u_l)$$
$$= \sum_{i=1}^{m-1}\int_{(u_{i-1},u_i]}\bigg(\sum_{k=i}^m \gamma_k^2 C(u_k-y,u_k-y) + 2\sum_{i\le r<l\le m}\gamma_r\gamma_l C(u_r-y,u_l-y)\bigg)\mathrm{d}W_\alpha^{\leftarrow}(y)$$
$$+ \gamma_m^2\int_{(u_{m-1},u_m]}C(u_m-y,u_m-y)\,\mathrm{d}W_\alpha^{\leftarrow}(y) \ge 0 \qquad \text{a.s.},$$

where $u_0 := 0$. Since the second term is nonnegative a.s., it suffices to prove that so is the first. The function $(u, w) \mapsto C(u, w)$, $0 < u \le w$ is nonnegative definite as a limit of nonnegative definite functions. Hence, for each $1 \le i \le m-1$ and $y \in (u_{i-1}, u_i)$,

$$\sum_{k=i}^{m} \gamma_k^2 C(u_k - y, u_k - y) + 2 \sum_{i \le r < l \le m} \gamma_r \gamma_l C(u_r - y, u_l - y) \ge 0.$$

Thus, the process $Z_{\alpha,\beta}$ does exist as a conditionally Gaussian process with covariance function $\Pi(s, t)$, $0 < s \le t$. □

Lemma 5.8 is designed to facilitate the proofs of Proposition 2.2 and Theorem 2.5.

Lemma 5.8. *Suppose that condition (6) holds for some $\alpha \in (0, 1)$ and some ℓ^*, and that $f(u, w) = \mathrm{Cov}[X(u)X(w)]$ is either uniformly regularly varying in strips in $\mathbb{R}_+^2$ or fictitious regularly varying in $\mathbb{R}_+^2$, in either of the cases, of index β for some $\beta \ge -\alpha$ and with limit function C. If*

$$\lim_{\rho \uparrow 1} \limsup_{t \to \infty} \frac{\mathbb{P}\{\xi > t\}}{v(t)} \int_{(\rho z, z]} v\big(t(z - y)\big)\, \mathrm{d}U(ty) = 0 \tag{65}$$

for all $z > 0$, then

$$\begin{aligned}
&\lambda_1 \sqrt{\frac{\mathbb{P}\{\xi > t\}}{v(t)}} \int_{[0,u_m]} \sum_{j=1}^{m} \gamma_j h\big((u_j - y)t\big) \mathbb{1}_{[0,u_j]}(y)\, \mathrm{d}\nu(ty) \\
&\quad + \lambda_2 \frac{\mathbb{P}\{\xi > t\}}{v(t)} \int_{[0,u_m]} \Bigg(\sum_{j=1}^{m} \gamma_j^2 v\big((u_j - y)t\big) \mathbb{1}_{[0,u_j]}(y) \\
&\quad + 2 \sum_{1 \le i < j \le m} \gamma_i \gamma_j f\big((u_i - y)t, (u_j - y)t\big) \mathbb{1}_{[0,u_i]}(y) \Bigg)\, \mathrm{d}\nu(ty) \\
&\overset{d}{\to} \lambda_1 b^{-1/2} \sum_{j=1}^{m} \gamma_j \int_{[0,u_j]} (u_j - y)^{(\beta-\alpha)/2}\, \mathrm{d}W_\alpha^{\leftarrow}(y) \\
&\quad + \lambda_2 \Bigg(\sum_{j=1}^{m} \gamma_j^2 \int_{[0,u_j]} (u_j - y)^{\beta}\, \mathrm{d}W_\alpha^{\leftarrow}(y) \\
&\quad + 2 \sum_{1 \le i < j \le m} \gamma_i \gamma_j \int_{[0,u_i]} C(u_i - y, u_j - y)\, \mathrm{d}W_\alpha^{\leftarrow}(y) \Bigg)
\end{aligned} \tag{66}$$

for any $m \in \mathbb{N}$, *any real* $\gamma_1, \dots, \gamma_m$, *any* $0 < u_1 < \dots < u_m < \infty$ *and any real* λ_1 *and* λ_2 *provided that whenever* $\lambda_1 > 0$

$$\lim_{t\to\infty} \frac{v(t)\mathbb{P}\{\xi > t\}}{h(t)^2} = b \in (0, \infty) \tag{67}$$

and

$$\lim_{\rho\uparrow 1} \limsup_{t\to\infty} \sqrt{\frac{\mathbb{P}\{\xi > t\}}{v(t)}} \int_{(\rho z, z]} h\big((z-y)t\big)\, \mathrm{d}U(ty) = 0 \tag{68}$$

for all $z > 0$.

Proof. We only prove the lemma in the case that $\lambda_2 \neq 0$ and $C(u, w) > 0$ for some $u \neq w$. Fix any $\rho \in (0, 1)$ such that $\rho u_m > u_{m-1}$ ($u_0 := 0$).

Since v is regularly varying at ∞ of index β, we infer

$$\lim_{t\to\infty} v\big((u-y)t\big)/v(t) = (u-y)^{\beta},$$
$$\lim_{t\to\infty} h\big((u-y)t\big)/\sqrt{v(t)\mathbb{P}\{\xi > t\}} = b^{-1/2}(u-y)^{(\beta-\alpha)/2}$$

for each $y \in [0, u)$, respectively, having utilized (67) for the second relation. Furthermore, the convergence in each of these limit relations is uniform in $y \in [0, \rho u]$ by Lemma 5.2(a). Since $f(u, w)$ is uniformly regularly varying in strips in $\mathbb{R}_+^2$ we conclude that for $r < l$ the convergence $\lim_{t\to\infty} f((u_r - y)t, (u_l - y)t)/v(t) = C(u_r - y, u_l - y)$ is uniform in $y \in [0, \rho u_r]$, too. Hence,

$$\begin{aligned}
\lim_{t\to\infty} \Bigg(& \lambda_1 \frac{\sum_{j=1}^m \gamma_j h((u_j - y)t) \mathbb{1}_{[0,\rho u_j]}(y)}{\sqrt{v(t)\mathbb{P}\{\xi > t\}}} + \lambda_2 \frac{\sum_{j=1}^m \gamma_j^2 v((u_j - y)t) \mathbb{1}_{[0,\rho u_j]}(y)}{v(t)} \\
& + 2\lambda_2 \frac{\sum_{1\le r<l\le m} \gamma_r \gamma_l f((u_r - y)t, (u_l - y)t) \mathbb{1}_{[0,\rho u_r]}(y)}{v(t)} \Bigg) \\
= {} & \lambda_1 b^{-1/2} \sum_{j=1}^m \gamma_j (u_j - y)^{(\beta-\alpha)/2} \mathbb{1}_{[0,\rho u_j]}(y) \\
& + \lambda_2 \Bigg(\sum_{j=1}^m \gamma_j^2 (u_j - y)^{\beta} \mathbb{1}_{[0,\rho u_j]}(y) \\
& + 2 \sum_{1\le r<l\le m} \gamma_r \gamma_l C(u_r - y, u_l - y) \mathbb{1}_{[0,\rho u_r]}(y) \Bigg)
\end{aligned}$$

uniformly in $y \in [0, \rho u_m]$. The random function $W_\alpha^{\leftarrow}$ is a.s. continuous as has already been explained in the proof of Corollary 2.6. Thus,[6] in view of (10)

$$\begin{aligned}
&\lambda_1 \sqrt{\frac{\mathbb{P}\{\xi > t\}}{v(t)}} \int_{[0,\rho u_m]} \sum_{j=1}^m \gamma_j h\big((u_j - y)t\big) \mathbb{1}_{[0,\rho u_j]}(y)\, \mathrm{d}\nu(ty) \\
&\quad + \lambda_2 \frac{\mathbb{P}\{\xi > t\}}{v(t)} \int_{[0,\rho u_m]} \Bigg(\sum_{j=1}^m \gamma_j^2 v\big((u_j - y)t\big) \mathbb{1}_{[0,\rho u_j]}(y) \\
&\quad + 2 \sum_{1 \le r < l \le m} \gamma_r \gamma_l f\big((u_r - y)t, (u_l - y)t\big) \mathbb{1}_{[0,\rho u_r]}(y) \Bigg) \mathrm{d}\nu(ty) \\
&\xrightarrow{\mathrm{d}} \lambda_1 b^{-1/2} \sum_{j=1}^m \gamma_j \int_{[0,\rho u_j]} (u_j - y)^{(\beta-\alpha)/2} \, \mathrm{d}W_\alpha^{\leftarrow}(y) \\
&\quad + \lambda_2 \Bigg(\sum_{j=1}^m \gamma_j^2 \int_{[0,\rho u_j]} (u_j - y)^{\beta} \, \mathrm{d}W_\alpha^{\leftarrow}(y) \\
&\quad + 2 \sum_{1 \le r < l \le m} \gamma_r \gamma_l \int_{[0,\rho u_r]} C(u_r - y, u_l - y) \, \mathrm{d}W_\alpha^{\leftarrow}(y) \Bigg)
\end{aligned} \tag{69}$$

by Lemma 5.4(a). For later use note that

$$\lim_{\rho \uparrow 1} \limsup_{t \to \infty} \frac{\mathbb{P}\{\xi > t\}}{v(t)} \int_{(\rho u_r, u_r]} v\big((u_l - y)t\big) \, \mathrm{d}U(ty) = 0, \qquad r < l \tag{70}$$

which can be proved by the same argument as before (since $v(t(u_l - y))/v(t)$ converges uniformly to $(u_l - y)^\beta$ on $(\rho u_r, u_r]$ as $t \to \infty$), though appealing to

$$\lim_{t \to \infty} \mathbb{P}\{\xi > t\} U(ty) = \frac{y^\alpha}{\Gamma(1-\alpha)\Gamma(1+\alpha)}$$

(see formula (8.6.4) on page 361 in [7]) rather than (10).

According to Lemma 5.1, relation (66) follows if we can verify that, as $\rho \uparrow 1$, the right-hand side of (69) converges in distribution to the right-hand side of (66) and that

$$\begin{aligned}
&\lim_{\rho \uparrow 1} \limsup_{t \to \infty} \mathbb{P}\Bigg\{ \Bigg| \lambda_1 \sqrt{\frac{\mathbb{P}\{\xi > t\}}{v(t)}} \sum_{j=1}^m \gamma_j \int_{(\rho u_j, u_j]} h\big(t(u_j - y)\big) \, \mathrm{d}\nu(ty) \\
&\quad + \lambda_2 \frac{\mathbb{P}\{\xi > t\}}{v(t)} \Bigg(\sum_{j=1}^m \gamma_j^2 \int_{(\rho u_j, u_j]} v\big(t(u_j - y)\big) \, \mathrm{d}\nu(ty)
\end{aligned} \tag{71}$$

[6] Since $y \mapsto \nu(y)$ has a.s. increasing paths, so does $y \mapsto \mathbb{P}\{\xi > t\}\nu(ty)$ for each $t > 0$.

$$+2 \sum_{1\le r<l\le m} \gamma_r\gamma_l \int_{(\rho u_r, u_r]} f\big((u_r-y)t, (u_l-y)t\big)\,\mathrm{d}\nu(ty)\Bigg)\Bigg| > \delta\Bigg\} = 0$$

for all $\delta > 0$. The first of these (even with distributional convergence replaced by a.s. convergence) is a consequence of the monotone convergence theorem and the a.s. finiteness of the integrals in (66) which follows from Lemma 5.7. Left with proving (71) use (13) to observe that (65) together with (70) leads to

$$\lim_{\rho\uparrow 1}\limsup_{t\to\infty} \frac{\mathbb{P}\{\xi > t\}}{v(t)} \int_{(\rho u_r, u_r]} \big|f\big((u_r-y)t, (u_l-y)t\big)\big|\,\mathrm{d}U(ty) = 0, \qquad r < l.$$

Applying Markov's inequality, we conclude that (71) follows from the last asymptotic relation, (65) and (68). This completes the proof of (66). □

Lemma 5.9. *Let* $0 \le r_1 < r_2 \le 1$. *Suppose that* $\phi : [0,\infty) \to [0,\infty)$ *is either increasing and* $\lim_{t\to\infty}(\phi(t)/\int_0^t \phi(y)\,\mathrm{d}y) = 0$, *or decreasing and, if* $r_2 = 1$, *locally integrable. If* $\mathbb{E}\xi < \infty$ *and* $\lim_{t\to\infty}\int_{(1-r_2)t}^{(1-r_1)t}\phi(y)\,\mathrm{d}y = \infty$, *then*

$$\int_{[r_1t, r_2t]} \phi(t-y)\,\mathrm{d}U(y) \sim \frac{1}{\mathbb{E}\xi}\int_{(1-r_2)t}^{(1-r_1)t}\phi(y)\,\mathrm{d}y, \qquad t\to\infty.$$

If ϕ is decreasing this is Lemma 8.2 in [24], the case of increasing ϕ and $r_1 = 0$, $r_2 = 1$ is covered by Lemma A.4 in [20]. In the general case the proof goes along the lines of the proof of Theorem 4 in [46], we omit the details.

In [20], Lemma A.4, it is shown that a particular case of Lemma 5.9 also holds for functions ϕ of bounded variation. We now give an example which demonstrates that the result of Lemma 5.9 may fail to hold for ill-behaved ϕ. Let, for instance, $\phi(t) = \mathbb{1}_{\mathbb{Q}_+^{\mathrm{c}}}(t)$, where $\mathbb{Q}_+^{\mathrm{c}}$ is the set of positive irrational numbers. Then $\int_0^t \phi(y)\,\mathrm{d}y = t$. Now suppose the law of ξ is concentrated at rational points in (0, 1). Note that choosing these points properly, the law of ξ can be made lattice as well as nonlattice. The points of increase of the renewal function $U(y)$ are rational points only. Hence $\int_{[0,t]}\phi(t-y)\,\mathrm{d}U(y) = 0$ for rational t.

Lemma 5.10. *Let* $\mathbb{E}\xi < \infty$ *and* $\phi : [0,\infty) \to [0,\infty)$ *be a locally bounded and measurable function.*

(a) *Let* $0 \le r_1 < r_2 \le 1$. *If there exists a monotone function* $\psi : [0,\infty) \to [0,\infty)$ *such that* $\phi(t) \sim \psi(t)$ *as* $t \to \infty$, *then*

$$\int_{[r_1t, r_2t]} \phi(t-y)\,\mathrm{d}U(y) \sim \int_{[r_1t, r_2t]} \psi(t-y)\,\mathrm{d}U(y), \qquad t\to\infty$$

provided that, when $r_2 = 1$, $\lim_{t\to\infty}\int_0^t \phi(y)\,\mathrm{d}y = \infty$ *and* $\lim_{t\to\infty}(\phi(t)/\int_0^t \phi(y)\,\mathrm{d}y) = 0$.

(b) *If there exists a locally bounded and measurable function* $\psi : [0,\infty) \to [0,\infty)$ *such that* $\phi(t) = o(\psi(t))$ *as* $t \to \infty$ *and* $\lim_{t\to\infty} \int_0^t \psi(t-y)\,\mathrm{d}U(y) = +\infty$, *then*

$$\int_{[0,t]} \phi(t-y)\,\mathrm{d}U(y) = o\bigg(\int_{[0,t]} \psi(t-y)\,\mathrm{d}U(y)\bigg), \qquad t\to\infty.$$

Proof. (a) For any $\delta \in (0,1)$ there exists a $t_0 > 0$ such that

$$1-\delta \le \phi(t)/\psi(t) \le 1+\delta \tag{72}$$

for all $t \ge t_0$.

Case $r_2 < 1$. We have, for $t \ge (1-r_2)^{-1} t_0$,

$$(1-\delta)\int_{[r_1t,r_2t]} \psi(t-y)\,\mathrm{d}U(y) \le \int_{[r_1t,r_2t]} \phi(t-y)\,\mathrm{d}U(y) \le (1+\delta)\int_{[r_1t,r_2t]} \psi(t-y)\,\mathrm{d}U(y).$$

Dividing both sides by $\int_{[r_1t,r_2t]} \psi(t-y)\,\mathrm{d}U(y)$ and sending $t\to\infty$ and then $\delta \downarrow 0$ gives the result.

Case $r_2 = 1$. Since ψ is monotone, it is locally integrable. Further, $\lim_{t\to\infty}\int_0^t \psi(y)\,\mathrm{d}y = \infty$ and $\lim_{t\to\infty}(\psi(t)/\int_0^t \psi(y)\,\mathrm{d}y) = 0$. Hence, Lemma 5.9 applies and yields

$$\lim_{t\to\infty} \int_{[r_1t,t]} \psi(t-y)\,\mathrm{d}U(y) = \infty.$$

In view of (72), we have

$$\begin{aligned}\int_{[r_1t,t]} \phi(t-y)\,\mathrm{d}U(y) &\le (1+\delta)\int_{[r_1t,t-t_0]} \psi(t-y)\,\mathrm{d}U(y) + \int_{(t-t_0,t]} \phi(t-y)\,\mathrm{d}U(y)\\ &\le (1+\delta)\int_{[r_1t,t]} \psi(t-y)\,\mathrm{d}U(y) + U(t_0)\sup_{0\le y\le t_0}\phi(y)\end{aligned}$$

for $t \ge (1-r_1)^{-1}t_0$, the last inequality following from the subadditivity of U. Dividing both sides by $\int_{[r_1t,t]} \psi(t-y)\,\mathrm{d}U(y)$ and sending $t\to\infty$ yields

$$\limsup_{t\to\infty} \frac{\int_{[r_1t,t]} \phi(t-y)\,\mathrm{d}U(y)}{\int_{[r_1t,t]} \psi(t-y)\,\mathrm{d}U(y)} \le 1+\delta.$$

The converse inequality for the lower limit follows analogously.

(b) For any $\delta \in (0,1)$ there exists a $t_0 > 0$ such that $\phi(t)/\psi(t) \le \delta$ for all $t \ge t_0$. The rest of the proof is the same as for the case $r_2 = 1$ of part (a). □

In the main text we have used the following corollary of Lemma 5.9.

Lemma 5.11. *Let $\mathbb{E}\xi < \infty$ and $\phi : [0,\infty) \to [0,\infty)$ be locally bounded, measurable and regularly varying at $+\infty$ of index $\beta \in (-1,\infty)$. If $\beta = 0$, assume further that there exists a monotone function u such that $\phi(t) \sim u(t)$ as $t \to \infty$. Then, for $0 \le r_1 < r_2 \le 1$,*

$$\int_{[r_1 t, r_2 t]} \phi(t-y)\,\mathrm{d}U(y) \sim \frac{t\phi(t)}{(1+\beta)\mathbb{E}\xi}\big((1-r_1)^{1+\beta} - (1-r_2)^{1+\beta}\big), \qquad t \to \infty.$$

The result is well known in the case where ϕ is increasing, $r_1 = 0$, $r_2 = 1$, $\beta \neq 0$, and the law of ξ is nonlattice, see Theorem 2.1 in [39].

Proof of Lemma 5.11. If $\beta \neq 0$, Lemma 5.2(b) ensures the existence of a positive monotone function u such that $\phi(t) \sim u(t)$ as $t \to \infty$. If $\beta = 0$ such a function exists by assumption. Modifying u if needed in the right vicinity of zero we can assume that u is monotone and locally integrable. Therefore,

$$\int_{[r_1 t, r_2 t]} \phi(t-y)\,\mathrm{d}U(y) \sim \int_{[r_1 t, r_2 t]} u(t-y)\,\mathrm{d}U(y) \sim \frac{1}{\mathbb{E}\xi}\int_{(1-r_2)t}^{(1-r_1)t} u(y)\,\mathrm{d}y,$$

where the first equivalence follows from Lemma 5.10(a) and the second is a consequence of Lemma 5.9 (observe that, with $g = \phi$ or $g = u$, $\lim_{t\to\infty}(g(t)/\int_0^t g(y)\,\mathrm{d}y) = 0$ and $\lim_{t\to\infty}\int_{(1-r_2)t}^{(1-r_1)t} g(y)\,\mathrm{d}y = \infty$ hold by Lemma 5.2(c) because g is regularly varying of index $\beta > -1$). Finally, using Lemma 5.2(c) we obtain

$$\begin{aligned}\frac{1}{\mathbb{E}\xi}\int_{(1-r_2)t}^{(1-r_1)t} u(y)\,\mathrm{d}y &\sim \frac{tu(t)}{(1+\beta)\mathbb{E}\xi}\big((1-r_1)^{1+\beta} - (1-r_2)^{1+\beta}\big)\\ &\sim \frac{t\phi(t)}{(1+\beta)\mathbb{E}\xi}\big((1-r_1)^{1+\beta} - (1-r_2)^{1+\beta}\big).\end{aligned}$$

The proof is complete. □

Part (a) of the next lemma is a slight extension of Lemma 5.2 in [22].

Lemma 5.12. *Suppose that (6) holds. Let $\phi : [0,\infty) \to \mathbb{R}$ be a locally bounded and measurable function satisfying $\phi(t) \sim t^{\gamma}\ell(t)$ as $t \to \infty$ for some $\gamma \ge -\alpha$ and some ℓ. If $\gamma = -\alpha$, assume additionally that there exists a positive increasing function q such that $\lim_{t\to\infty}\frac{\phi(t)}{\mathbb{P}\{\xi>t\}q(t)} = 1$. Then*:

(a)

$$\lim_{\rho\uparrow 1}\limsup_{t\to\infty}\frac{\mathbb{P}\{\xi > t\}}{\phi(t)}\int_{[\rho t, t]} \phi(t-y)\,\mathrm{d}U(y) = 0;$$

in particular,

$$\lim_{t\to\infty}\frac{\mathbb{P}\{\xi > t\}}{\phi(t)}\int_{[0,t]} \phi(t-y)\,\mathrm{d}U(y) = \frac{\Gamma(1+\gamma)}{\Gamma(1-\alpha)\Gamma(1+\alpha+\gamma)};$$

(b) $\int_{[0,t]} \phi_1(t-x)\,\mathrm{d}U(x) = o(\phi(t)/\mathbb{P}\{\xi > t\})$ *as* $t \to \infty$ *for any positive locally bounded function* ϕ_1 *such that* $\phi_1(t) = o(\phi(t))$, $t \to \infty$.

Proof. (a) In the case $\gamma \in [-\alpha, 0]$ this is just Lemma 5.2 of [22]. In the case $\gamma > 0$ exactly the same proof applies.

(b) For any $\delta > 0$ there exists a $t_0 > 0$ such that $\phi_1(t)/\phi(t) \le \delta$ for all $t \ge t_0$. Hence,

$$\int_{[0,t]} \phi_1(t-y)\,\mathrm{d}U(y) \le \delta \int_{[0,t]} \phi(t-y)\,\mathrm{d}U(y) + \big(U(t) - U(t-t_0)\big) \sup_{0\le y\le t_0} \phi_1(y)$$

for $t \ge t_0$. According to part (a) the first term on the right-hand side grows like $\mathrm{const}\,\phi(t)/\mathbb{P}\{\xi > t\}$. By Blackwell's renewal theorem, $\lim_{t\to\infty}(U(t) - U(t-t_0)) = 0$. Dividing the inequality above by $\phi(t)/\mathbb{P}\{\xi > t\}$ and sending first $t \to \infty$ and then $\delta \downarrow 0$ finishes the proof. □

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