Lipschitz partition processes

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We introduce a family of Markov processes on set partitions with a bounded number of blocks, called *Lipschitz partition processes*. We construct these processes explicitly by a Poisson point process on the space of Lipschitz continuous maps on partitions. By this construction, the *Markovian consistency property* is readily satisfied; that is, the finite restrictions of any Lipschitz partition process comprise a compatible collection of finite state space Markov chains. We further characterize the class of exchangeable Lipschitz partition processes by a novel set-valued matrix operation.

Keywords: coalescent process; de Finetti's theorem; exchangeable random partition; iterated random functions; Markov process; paintbox process; Poisson random measure

1. Introduction

Partition-valued Markov processes, particularly coalescent and fragmentation processes, arise as mathematical models in population genetics and mathematical biology. Initially, Ewens [14] derived his celebrated sampling formula while studying neutral allele sampling in population genetics. Extending Ewens's work, Kingman characterized exchangeable partitions of the natural numbers [16,17], which play a larger role in the mathematical study of genetic diversity [18]. Related applications in phylogenetics and the study of ancestral lineages prompted Kingman's coalescent process [19], which arises as the scaling limit of both Wright–Fisher and Moran models under different regimes [23]. Exchangeable coalescent and fragmentation processes have also taken hold in the probability literature because of some beautiful relationships to classical stochastic process theory, for example, Brownian motion and Lévy processes. For specific content in the literature, see [1–4,21]; for recent overviews of this theory, see [5,22].

In this paper, we study a family of Markov processes on labeled partitions with a finite number $k \geq 1$ of classes. By a simple projection, we describe a broad class of processes on the space of partitions with at most k blocks. Processes on this space are cursorily related to composition structures for ordered partitions, for example, [13,15], but our approach more closely follows previous work [8], which is motivated by DNA sequencing applications. In addition to genetics applications, processes on this subspace relate to problems of cluster detection and classification in which the total number of classes is finite, for example, [7,9,20].

Prior to [8], coagulation–fragmentation processes dominated the literature. The processes in [8] do not evolve by fragmentation or coagulation; their jumps involve simultaneous fragmentation and coagulation of all blocks. To describe a broader class of processes, we incorporate ideas from the coagulation–fragmentation literature as well as our previous work. We call these *Lipschitz partition processes*.

Our main theorems are *not* corollaries of the many results for fragmentation and coalescent processes. Instead, our approach extracts fundamental properties of these processes, specifically

their construction from the Coag and Frag operators; see, for example, Bertoin [5], Chapters 3–4. Importantly, these operators are Lipschitz continuous and associative. From these observations, we construct a family of processes by repeated application of random Lipschitz continuous maps that act on the space of partitions.

In the exchangeable case, the random maps are confined to the subspace of *strongly Lipschitz continuous* functions, which we characterize in full by a class of specially structured set-valued matrices (Section 4.2). These set-valued matrices act on labeled partitions similarly to the action of a matrix on a real-valued vector (with obvious modifications to the operations addition and multiplication). They also establish an intimate connection between exchangeable Lipschitz partition processes and random stochastic matrices (Section 4.4).

1.1. General construction: Overview

For now, we regard a labeled partition as a finite collection of non-overlapping, labeled subsets.

Consider the following construction of a discrete-time Markov chain. Let Λ_0 be an initial state and let $F_1, F_2, \ldots$ be independent and identically distributed (i.i.d.) random maps on the space of labeled partitions. Then, for each $t \geq 1$, we define

$$\Lambda_t := F_t(\Lambda_{t-1}) = (F_t \circ F_{t-1} \circ \cdots \circ F_1)(\Lambda_0). \tag{1.1}$$

The collection $\boldsymbol{\Lambda} := (\Lambda_t, t \geq 0)$ is a discrete-time Markov chain.

We study an analogous construction for continuous-time processes. Instead of an i.i.d. sequence of random maps, we construct $\boldsymbol{\Lambda}$ from a Poisson point process on the space of maps. Informally, if $\mathbf{F} := \{(t, F_t)\}$ is a realization of such a Poisson point process (where each F_t is a map), we construct $\boldsymbol{\Lambda}$ by putting

$$\Lambda_t := \begin{cases} F_t(\Lambda_{t-}), & t \text{ is an atom time of } \mathbf{F}, \\ \Lambda_{t-}, & \text{otherwise,} \end{cases} \quad \text{for every } t > 0. \tag{1.2}$$

We are interested in processes $\boldsymbol{\Lambda}$ that exhibit

- *Markovian consistency*, that is, for each $n \in \mathbb{N}$, the restriction of $\boldsymbol{\Lambda}$ to labeled partitions of $[n] := \{1, \ldots, n\}$ is a Markov chain, and
- *exchangeability*, that is, the law of $\boldsymbol{\Lambda}$ is invariant under relabeling of elements of $\mathbb{N}$.

Markovian consistency might also be called the *projective Markov property*, meaning the projection of $\boldsymbol{\Lambda}$ to spaces of finite labeled partitions is also Markov. Throughout the paper, we use the term *consistency* in place of *Markovian consistency*. Consistency plays a central role not only in this paper but also more widely in the study of partition-valued Markov processes. In general, a function of a Markov process need not be Markov, and so consistency is not trivially satisfied; see Example 2.1.

We pay special attention to the exchangeable case, for which we can make some precise statements. In this case, we show that the Poisson point process $\mathbf{F}$ is supported on the space of maps having the *strong Lipschitz property* (Section 4.1).

The general approach outlined in (1.1) and (1.2) can be applied to construct processes on the unrestricted space of set partitions, or even ordered set partitions, but we do not treat these cases.

In our main theorems, we show a correspondence between strongly Lipschitz maps on labeled partitions and $k \times k$ set-valued matrices. Without bounding the number of classes, we cannot obtain such a precise statement.

1.2. Organization of the paper

We organize the paper as follows. In Section 2, we give some preliminaries for partitions and labeled k-partitions. In Section 3, we introduce the general class of Lipschitz partition processes; and in Section 4, we specialize to exchangeable Lipschitz partition processes. In Section 5, we discuss discrete-time Markov chains. In Section 6, we make some concluding remarks about projections to unlabeled set partitions and more general issues concerning partition-valued Markov processes.

2. Preliminaries

2.1. Partitions

For $n \in \mathbb{N} = \{1, 2, \ldots\}$, a *partition* π of $[n] := \{1, \ldots, n\}$ is a collection $\{b_1, \ldots, b_r\}$ of non-empty, disjoint subsets (blocks) satisfying $\bigcup_{i=1}^{r} b_i = [n]$. Alternatively, π can be regarded as an equivalence relation $\sim_\pi$, where

$$i \sim_\pi j \quad \Longleftrightarrow \quad i \text{ and } j \text{ are in the same block of } \pi. \tag{2.1}$$

We write $\#\pi$ to denote the number of blocks of π. Unless otherwise stated, we assume that the blocks of π are listed in increasing order of their least element. We write $\mathcal{P}_{[n]}$ to denote the space of partitions of $[n]$.

Writing $\mathscr{S}_n$ to denote the symmetric group acting on $[n]$, we define the *relabeling* $\pi \in \mathcal{P}_{[n]}$ by $\sigma \in \mathscr{S}_n, \pi \mapsto \pi^\sigma$, where

$$i \sim_{\pi^\sigma} j \quad \Longleftrightarrow \quad \sigma(i) \sim_\pi \sigma(j).$$

Furthermore, for $m \leq n$, we define the *restriction* of $\pi \in \mathcal{P}_{[n]}$ to $\mathcal{P}_{[m]}$ by

$$\pi_{|[m]} = \mathbf{D}_{m,n}\pi := \{b \cap [m] : b \in \pi\} \setminus \{\varnothing\},$$

the restriction of each block of π to $[m]$ after removal of any empty sets. In general, to any injective map $\psi : [m] \to [n]$, we associate a projection $\psi' : \mathcal{P}_{[n]} \to \mathcal{P}_{[m]}$, where

$$i \sim_{\psi'(\pi)} j \quad \Longleftrightarrow \quad \psi(i) \sim_\pi \psi(j).$$

We write $\mathcal{P}_\mathbb{N}$ to denote the space of partitions of $\mathbb{N}$, which are defined as compatible sequences $(\pi_n, n \in \mathbb{N})$ of finite set partitions. For $m \leq n$, we say $\pi \in \mathcal{P}_{[n]}$ and $\pi' \in \mathcal{P}_{[m]}$ are *compatible* if $\pi_{|[m]} = \pi'$; and we call $(\pi_n, n \in \mathbb{N})$ a *compatible sequence* if $\pi_n \in \mathcal{P}_{[n]}$ and $\pi_m = \mathbf{D}_{m,n}\pi_n$, for all $m \leq n$, for every $n \in \mathbb{N}$.

Writing $\mathbf{n}(\pi, \pi') := \max\{n \in \mathbb{N} : \pi_{|[n]} = \pi'_{|[n]}\}$, we equip $\mathcal{P}_\mathbb{N}$ with the ultrametric

$$d_{\mathcal{P}_\mathbb{N}}(\pi, \pi') := 2^{-\mathbf{n}(\pi, \pi')}, \qquad \pi, \pi' \in \mathcal{P}_\mathbb{N}, \tag{2.2}$$

under which $(\mathcal{P}_\mathbb{N}, d_{\mathcal{P}_\mathbb{N}})$ is complete, separable, and naturally endowed with the discrete σ-field $\sigma\langle\bigcup_{n \in \mathbb{N}} \mathcal{P}_{[n]}\rangle$.

2.2. Random partitions

A sequence $(\mu_n, n \in \mathbb{N})$ of measures on the system $(\mathcal{P}_{[n]}, n \in \mathbb{N})$, where μ_n is a measure on $\mathcal{P}_{[n]}$ for each $n \in \mathbb{N}$, is *consistent* if

$$\mu_m = \mu_n \mathbf{D}_{m,n}^{-1} \qquad \text{for every } m \le n; \tag{2.3}$$

that is, μ_m coincides with the law $\mu_n \mathbf{D}_{m,n}^{-1}$ induced by the restriction map. By Kolmogorov's extension theorem, any consistent collection of measures determines a unique measure μ on $\mathcal{P}_\mathbb{N}$. This circle of ideas is central to the theory of random partitions of $\mathbb{N}$ as it permits the explicit construction of a random partition Π through its compatible sequence $(\Pi_n, n \in \mathbb{N})$ of finite random partitions.

A random partition Π of $\mathbb{N}$ is called *exchangeable* if $\Pi^\sigma =_{\mathcal{L}} \Pi$ for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$ that fix all but finitely many elements of $\mathbb{N}$, where $=_{\mathcal{L}}$ denotes *equality in law*. Kingman [17] gives a de Finetti-type characterization of exchangeable random partitions of $\mathbb{N}$ through the paintbox process. Let

$$\Delta^\downarrow := \left\{ (s_1, s_2, \ldots) : s_1 \ge s_2 \ge \cdots \ge 0, \sum_{i=1}^\infty s_i \le 1 \right\}$$

denote the space of *ranked mass partitions*. Given $s \in \Delta^\downarrow$, we write $s_0 := 1 - \sum_{i \ge 1} s_i$ and construct Π as follows. Let $X_1, X_2, \ldots$ be a sequence of independent (but not necessarily identically distributed) random variables with law

$$P_s\{X_i = j\} := \begin{cases} s_j, & j \ge 1, \\ s_0, & j = -i, \\ 0, & \text{otherwise.} \end{cases}$$

Given $X := (X_1, X_2, \ldots)$, we define $\Pi := \Pi(X)$ by the relation

$$i \sim_\Pi j \iff X_i = X_j.$$

We write ϱ_s to denote the law of Π, called a *paintbox process directed by s*. For $n \in \mathbb{N}$, we write $\varrho_s^{(n)}$ to denote the restriction of ϱ_s to a probability measure on $\mathcal{P}_{[n]}$. In this way, $(\varrho_s^{(n)}, n \in \mathbb{N})$ is a consistent collection of finite-dimensional measures determining ϱ_s. More generally, given a measure ν on $\Delta^\downarrow$, the ν-mixture of paintbox processes is defined by

$$\varrho_\nu(\cdot) := \int_{\Delta^\downarrow} \varrho_s(\cdot) \nu(\mathrm{d}s).$$

Kingman's correspondence associates every exchangeable random partition of $\mathbb{N}$ with a unique probability measure on $\Delta^{\downarrow}$.

A widely circulated example of a sequential construction is the Chinese restaurant process. Overall, the Chinese restaurant process produces a compatible collection $(\Pi_n, n \in \mathbb{N})$ of finite partitions for which each Π_n obeys the Ewens distribution on $\mathcal{P}_{[n]}$. The random partition Π determined by $(\Pi_n, n \in \mathbb{N})$ obeys the *Ewens process*, whose directing measure is the two-parameter Poisson–Dirichlet distribution with parameter $(0, \theta)$; see [22] for more information on the distinguishing properties of the Ewens distribution.

2.3. Partition-valued Markov processes

In this paper, we study Markov processes $\mathbf{\Pi} := (\Pi_t, t \geq 0)$ on $\mathcal{P}_{\mathbb{N}}$ that are

- *consistent*: for each $n \in \mathbb{N}$, $\mathbf{\Pi}_{|[n]} := (\Pi_{t|[n]}, t \geq 0)$ is a Markov chain on $\mathcal{P}_{[n]}$; and
- *exchangeable*: $\mathbf{\Pi}^{\sigma} := (\Pi_t^{\sigma}, t \geq 0) =_{\mathcal{L}} \mathbf{\Pi}$ for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$ that fix all but finitely many $n \in \mathbb{N}$.

In this case, exchangeability refers to joint exchangeability in the sense that elements are relabeled according to the same partition at all time points. Consistency refers to a preservation of the Markov property.

A consistent Markov process $\mathbf{\Pi}$ on $\mathcal{P}_{\mathbb{N}}$ can be constructed sequentially through its finite restrictions $(\mathbf{\Pi}_{|[n]}, n \in \mathbb{N})$, but care must be taken to ensure that each of the restrictions $\mathbf{\Pi}_{|[n]}$ has càdlàg sample paths. Perhaps the most well-known example of an exchangeable and consistent Markov process on $\mathcal{P}_{\mathbb{N}}$ is the exchangeable coalescent process.

2.3.1. *Exchangeable coalescent process*

The construction of the coalescent process from the Coag-operator telegraphs our general approach. Let $\pi := \{b_1, b_2, \ldots\}$ be any partition of a finite or infinite set with $\#\pi = k \in \mathbb{N} \cup \{\infty\}$, and let $b' := \{b_1', b_2', \ldots\}$ be a partition of $[k']$, for any $k' \geq k$. We call $\pi'' := \mathrm{Coag}(\pi, \pi') := \{b_1'', b_2'', \ldots\}$ the *coagulation of π by π'*, where

$$b_i'' := \bigcup_{j \in b_i'} b_j, \qquad i \geq 1. \tag{2.4}$$

(To maintain the definition of π'' as a partition, we remove any empty sets that result from this operation.) Essential to definition (2.4) is that the blocks of π are ordered in ascending order of their least element. For example, let $\pi := 1356/2/47/8$ and $\pi' := 135/24$, then

$$\mathrm{Coag}(\pi, \pi') = \mathrm{Coag}(1356/2/47/8, 135/24) = 134567/28.$$

In words: block $\{1, 3, 5\}$ of π' indicates that we merge the *first*, *third*, and *fifth* blocks of π, while block $\{2, 4\}$ indicates that we merge the *second* and *fourth* blocks of π. (We ignore any elements of π' larger than $\#\pi$; for example, there is no fifth block of π and so the position of 5 in π' does not affect $\mathrm{Coag}(\pi, \pi')$.)

The Coag operator has been used extensively in the study of coalescent processes; see Chapter 4 of Bertoin [5]. Let μ be a measure on $\mathcal{P}_\mathbb{N}$ such that

$$\mu\big(\{\mathbf{0}_\mathbb{N}\}\big) = 0 \quad \text{and} \quad \mu\big(\{\pi \in \mathcal{P}_\mathbb{N} : \pi_{|[n]} \neq \mathbf{0}_{[n]}\}\big) < \infty \qquad \text{for every } n \in \mathbb{N}, \qquad (2.5)$$

where $\mathbf{0}_A$ denotes the partition of $A \subseteq \mathbb{N}$ into singletons. Also, let $\mathbf{B} := \{(t, B_t)\} \subset [0, \infty) \times \mathcal{P}_\mathbb{N}$ be a Poisson point process with intensity $\mathrm{d}t \otimes \mu$ (where $\mathrm{d}t$ denotes Lebesgue measure on $[0, \infty)$). Given $\mathbf{B}$, we construct a coalescent process $\mathbf{\Pi} := (\Pi_t, t \geq 0)$ on $\mathcal{P}_\mathbb{N}$ as follows. For each $n \in \mathbb{N}$, we specify $\mathbf{\Pi}^{[n]} := (\Pi_t^{[n]}, t \geq 0)$ on $\mathcal{P}_{[n]}$ by $\Pi_0^{[n]} = \mathbf{0}_{[n]}$ and, for every $t > 0$,

- if $t > 0$ is an atom time of $\mathbf{B}$ such that $B_{t|[n]} \neq \mathbf{0}_{[n]}$, then we put $\Pi_t^{[n]} = \mathrm{Coag}(\Pi_{t-}^{[n]}, B_t)$;
- otherwise, we put $\Pi_t^{[n]} = \Pi_{t-}^{[n]}$.

Note that, by the definition of Coag, $\Pi_t^{[m]} = \mathbf{D}_{m,n} \Pi_t^{[n]}$ for all $t \geq 0$, for all $m \leq n$. Hence, $(\mathbf{\Pi}^{[n]}, n \in \mathbb{N})$ is a compatible collection of processes. Furthermore, by (2.5), each $\mathbf{\Pi}^{[n]}$ is a Markov chain on $\mathcal{P}_{[n]}$ with càdlàg sample paths. Hence, $(\mathbf{\Pi}^{[n]}, n \in \mathbb{N})$ determines a consistent Markov process $\mathbf{\Pi}$ on $\mathcal{P}_\mathbb{N}$. If, in addition, μ is exchangeable, then $\mathbf{\Pi}$ is exchangeable. The process constructed in this way is called a *coalescent process*.

Remark 2.1. The construction of $\mathbf{\Pi}$ from the collection $(\mathbf{\Pi}^{[n]}, n \in \mathbb{N})$ of finite state space processes, rather than directly from the entire process $\mathbf{B}$, is necessary. In general, (2.5) permits $\mathbf{B}$ to have infinitely many atoms in arbitrarily small intervals of $[0, \infty)$; but, by the second half of (2.5), there can be only finitely many atom times $t > 0$ for which $B_{t|[n]} \neq \mathbf{0}_{[n]}$, for each $n \in \mathbb{N}$. Therefore, while the Poisson point process construction cannot be applied directly to construct $\mathbf{\Pi}$ (because the atom times might be dense in $[0, \infty)$), we can construct $\mathbf{\Pi}$ sequentially by building a compatible collection of processes that are consistent in distribution.

An important property of the Coag operator is Lipschitz continuity with respect to (2.2), that is, for every $\pi \in \mathcal{P}_\mathbb{N}$,

$$d_{\mathcal{P}_\mathbb{N}}\big(\mathrm{Coag}(\pi', \pi), \mathrm{Coag}(\pi'', \pi)\big) \leq d_{\mathcal{P}_\mathbb{N}}(\pi', \pi'') \qquad \text{for all } \pi', \pi'' \in \mathcal{P}_\mathbb{N}.$$

Furthermore, $\mathrm{Coag} : \mathcal{P}_\mathbb{N} \times \mathcal{P}_\mathbb{N} \to \mathcal{P}_\mathbb{N}$ is associative in the sense that

$$\mathrm{Coag}\big(\pi, \mathrm{Coag}(\pi', \pi'')\big) = \mathrm{Coag}\big(\mathrm{Coag}(\pi, \pi'), \pi''\big) \qquad \text{for all } \pi, \pi', \pi'' \in \mathcal{P}_\mathbb{N}.$$

Lipschitz continuity is important for the consistency property because it implies that the coagulation of $\pi_{|[n]}$ by π' depends only on $\pi'_{|[n]}$, for every $n \in \mathbb{N}$. Associativity ensures the construction of $\mathbf{\Pi}$ is well-defined.

The Frag operator acts as the dual to Coag in the related study of fragmentation processes. Analogously to the above construction, the Frag operator can be used to construct fragmentation processes on $\mathcal{P}_\mathbb{N}$, but we do not discuss those details. We only acknowledge that the Frag operator is also Lipschitz continuous with respect to (2.2). These operators are important because they characterize the semigroup of coagulation and fragmentation processes. By Lipschitz continuity, the semigroups of these processes are easily shown to fulfill the Feller property (under the additional regularity condition (2.5), or its analog for fragmentation processes).

The coalescent process above need not be exchangeable. Bertoin [5] only considers the exchangeable case and so specializes to the case in which μ in (2.5) is the directing measure of a paintbox process.

2.3.2. *Processes on partitions with a bounded number of blocks* (*Crane* [8], *Section* 4.1)

For $k \in \mathbb{N}$, let $\mathcal{P}_{\mathbb{N}:k} := \{\pi \in \mathcal{P}_{\mathbb{N}} : \#\pi \leq k\}$ be the subcollection of partitions of $\mathbb{N}$ with k or fewer blocks, and let $\Delta_k^{\downarrow} := \{(s_1, \ldots, s_k) : s_1 \geq \cdots \geq s_k \geq 0, \sum_{i=1}^{k} s_i = 1\}$ denote the *ranked k-simplex*. For any probability measure ν on $\Delta_k^{\downarrow}$, the paintbox measure ϱ_ν is supported on $\mathcal{P}_{\mathbb{N}:k}$.

In [8], we studied a family of Markov processes on $\mathcal{P}_{\mathbb{N}:k}$ with the following description. Let ν be a finite measure on Δ_k. Given an initial state $\pi \in \mathcal{P}_{\mathbb{N}:k}$, we construct a Markov process $\boldsymbol{\Pi}$ from a Poisson point process $\mathbf{B} = \{(t, B_t, S_t)\} \subset \mathbb{R}^+ \times \mathcal{P}_{\mathbb{N}:k}^k \times \mathscr{S}_k^k$ with intensity $dt \otimes \varrho_\nu^{\otimes k} \otimes \Upsilon^{\otimes k}$, where Υ is the uniform distribution on $\mathscr{S}_k$ and, for any measure μ, $\mu^{\otimes k} := \mu \otimes \cdots \otimes \mu$ denotes its k-fold product measure. Given a realization of $\mathbf{B}$, we construct $\boldsymbol{\Pi} := (\Pi_t, t \geq 0)$ from its finite restrictions as follows. First, for each $n \in \mathbb{N}$, we put $\Pi_0^{[n]} = \pi_{|[n]}$. Then, for each $t > 0$, we write $\Pi_{t-}^{[n]} = (b_1, \ldots, b_r)$, $r \leq k$, with blocks listed in order of their least element, and

- if $t > 0$ is an atom time of $\mathbf{B}$ with $B_t := (B^1, \ldots, B^k)$ a k-tuple of partitions and $S_t := (S_1, \ldots, S_k)$ a k-tuple of permutations $[k] \to [k]$,
  – we construct the set-valued matrix

$$
\begin{pmatrix}
B_{S_1(1)}^1 \cap b_1 & B_{S_2(1)}^2 \cap b_2 & \cdots & B_{S_r(1)}^r \cap b_r \\
B_{S_1(2)}^1 \cap b_1 & B_{S_2(2)}^2 \cap b_2 & \cdots & B_{S_r(2)}^r \cap b_r \\
\vdots & \vdots & \ddots & \vdots \\
B_{S_1(k)}^1 \cap b_1 & B_{S_2(k)}^2 \cap b_2 & \cdots & B_{S_r(k)}^r \cap b_r
\end{pmatrix},
\tag{2.6}
$$

  and
  – for each $j = 1, \ldots, k$, we put $C_j := \bigcup_{i=1}^{r}(B_{S_i(j)}^i \cap b_i)$, the union of the entries in row j of the above matrix. We then define $\Pi_t^{[n]} := \{C_1, \ldots, C_k\} \setminus \{\varnothing\}$, provided $\Pi_t^{[n]} \neq \Pi_{t-}^{[n]}$;
- otherwise, we put $\Pi_t^{[n]} = \Pi_{t-}^{[n]}$.

We have shown [8] that the finite-dimensional transition rates for this process are

$$
Q_n(\pi, \pi') = k^{\downarrow \#\pi'} \prod_{b \in \pi} \frac{\varrho_\nu^b(\pi_{|b}')}{k^{\downarrow \#\pi_{|b}'}}, \qquad \pi \neq \pi' \in \mathcal{P}_{[n]:k},
$$

where $k^{\downarrow j} := k(k-1) \cdots (k - j + 1)$ and ϱ_ν^b denotes the measure ϱ_ν induces on the space of partitions of $b \subseteq \mathbb{N}$.

The above construction has an easy description as a three step procedure. For $k \geq 1$, let $\pi := \{b_1, \ldots, b_r\}$, $r \leq k$, be a partition of a finite or infinite set. Then we obtain a jump from π to π' as follows.

(i) Independently, for each $i = 1, \ldots, r$, randomly partition b_i according to the paintbox process ϱ_ν restricted to b_i. Write $B_i := \{B_{i,1}, \ldots, B_{i,r_i}\}$ to denote the partition obtained.

(ii) Independently, for each $i = 1, \ldots, r$, randomly label the blocks of B_i by sampling uniformly without replacement from $[k]$. Equivalently, we can draw a uniform random permutation σ_i of $[k]$ and order the blocks of B_i by adding $k - r_i$ empty-sets to the end of B_i and writing $C_i := (B_{i,\sigma_i(1)}, \ldots, B_{i,\sigma_i(k)})$.

(iii) We define π' by merging all subsets assigned the same label in step (ii); that is, we put $B'_l := \bigcup_{j=1}^{k} B_{j,\sigma_j(l)}$ for each $l = 1, \ldots, k$ and then define $\pi' := \{B'_1, \ldots, B'_k\} \setminus \{\varnothing\}$.

This procedure produces an exchangeable Feller process on $\mathcal{P}_{\mathbb{N}:k}$. The next example illustrates that an exchangeable Markov process on $\mathcal{P}_{\mathbb{N}:k}$ need not be consistent.

Example 2.1 (Failure of consistency property). Throughout this example, let $s_0 := (2/3, 1/3) \in \Delta_k^\downarrow$. With initial state $\Pi_0 \sim \varrho_{s_0}$, we define the infinitesimal jump rates of $\boldsymbol{\Pi}$ as follows. For every $t \geq 0$,

- given $\Pi_t \neq \mathbf{1}_{\mathbb{N}}$, the trivial one-block partition of $\mathbb{N}$, Π_t jumps to $\mathbf{1}_{\mathbb{N}}$ at rate 1, and
- given $\Pi_t = \mathbf{1}_{\mathbb{N}}$, Π_t jumps to $B \sim \varrho_{s_0}$ at rate 2.

Clearly, $\boldsymbol{\Pi}$ is Markovian, exchangeable, and has càdlàg sample paths; however, for each $n \in \mathbb{N}$, the restriction $\boldsymbol{\Pi}_{|[n]} := (\Pi_{t|[n]}, t \geq 0)$ is not Markovian because the jump rate at every time $t \geq 0$ depends on whether Π_t is trivial, which depends on the tail of $(\Pi_{t|[n]}, n \in \mathbb{N})$.

We focus on generalizing (i)–(iii). To do so, we work on the space $\mathcal{L}_{\mathbb{N}:k}$ of labeled partitions of $\mathbb{N}$ with k classes. The relationship between $\mathcal{L}_{\mathbb{N}:k}$ and $\mathcal{P}_{\mathbb{N}:k}$ is straightforward, and the added structure of $\mathcal{L}_{\mathbb{N}:k}$ enables a cleaner exposition.

2.4. Labeled partitions

For fixed $k \in \mathbb{N}$, a *k-partition* is a labeled set partition with k classes. Specifically, for any $A \subseteq \mathbb{N}$, a k-partition λ of A is a length k set-valued vector $(\lambda_1, \ldots, \lambda_k)$ with $\lambda_i \subseteq A$ for each $i \in [k]$, $\lambda_i \cap \lambda_{i'} = \varnothing$ for $i \neq i'$, and $\bigcup_{i=1}^{k} \lambda_i = A$. Alternatively, for $A = [n]$, λ can be regarded as

- a sequence $\lambda = \lambda^1 \lambda^2 \cdots \lambda^n$ in $[k]^{[n]}$, where

$$\lambda^i = j \iff i \in \lambda_j, \quad \text{or}$$

- a map $\lambda : [n] \to [k]$, where $\lambda(i) = \lambda^i$ for each $i \in [n]$.

Note that all three specifications of λ are equivalent and can be used interchangeably. In general, we write $\mathcal{L}_{A:k}$ to denote the space of k-partitions of $A \subseteq \mathbb{N}$.

Any $\lambda \in \mathcal{L}_{[n]:k}$ induces a partition of $[n]$ through the map $\mathcal{B}_n : \mathcal{L}_{[n]:k} \to \mathcal{P}_{[n]:k}$, defined by

$$\mathcal{B}_n(\lambda) := \{\lambda_1, \ldots, \lambda_k\} \setminus \{\varnothing\},$$

the unordered collection of classes of λ with empty sets removed. Permutations and injection maps act on $(\mathcal{L}_{[n]:k}, n \in \mathbb{N})$ similarly to their action on $(\mathcal{P}_{[n]:k}, n \in \mathbb{N})$. In general, let $\psi : [m] \to$

$[n]$, $m \le n$, be an injection. Then we define $\psi^* : \mathcal{L}_{[n]:k} \to \mathcal{L}_{[m]:k}$ by

$$\psi^*(\lambda) := \lambda \circ \psi \qquad \text{for every } \lambda \in \mathcal{L}_{[n]:k},$$

where $\lambda \in \mathcal{L}_{[n]:k}$ is treated as a map $[n] \to [k]$.

The restriction map $\mathcal{L}_{[n]:k} \to \mathcal{L}_{[m]:k}$ is defined by

$$\lambda_{|[m]} := \big(\lambda_1 \cap [m], \ldots, \lambda_k \cap [m]\big),$$

and the notion of compatibility for sequences of labeled partitions carries over from unlabeled partitions. We define $\mathcal{L}_{\mathbb{N}:k}$ as the space of k-partitions of $\mathbb{N}$, whose elements can be represented by a compatible sequence of finite k-partitions. Finally, we equip $\mathcal{L}_{\mathbb{N}:k}$ with ultrametric

$$d\big(\lambda, \lambda'\big) := 2^{-\mathbf{n}(\lambda, \lambda')}, \qquad \lambda, \lambda' \in \mathcal{L}_{\mathbb{N}:k}, \tag{2.7}$$

where $\mathbf{n}(\lambda, \lambda') := \max\{n \in \mathbb{N} : \lambda_{|[n]} = \lambda'_{|[n]}\}$, and σ-field $\sigma\langle \bigcup_{n \in \mathbb{N}} \mathcal{L}_{[n]:k}\rangle$.

The projections $(\mathcal{B}_n, n \in \mathbb{N})$, ψ', and ψ^* cooperate with one another; that is, the diagram in (2.8) commutes: $\mathcal{B}_m \circ \psi^* = \psi' \circ \mathcal{B}_n$. By this natural correspondence, we can study processes on $\mathcal{L}_{\mathbb{N}:k}$ and later project to $\mathcal{P}_{\mathbb{N}:k}$. Under mild conditions, the projection into $\mathcal{P}_{\mathbb{N}:k}$ preserves most, and sometimes all, of the properties of a process on $\mathcal{L}_{\mathbb{N}:k}$. Using this correspondence, we principally study processes on $\mathcal{L}_{\mathbb{N}:k}$ with the intention to later project into $\mathcal{P}_{\mathbb{N}:k}$. We discuss this procedure briefly in Section 6.1, but, by that time, most of its implications should be obvious.

$$
\begin{array}{ccc}
[n] & \mathcal{L}_{[n]:k} \xrightarrow{\ \mathcal{B}_n\ } \mathcal{P}_{[n]:k} & \\
\psi \uparrow & \psi^* \downarrow \qquad\qquad \downarrow \psi' & \\
[m] & \mathcal{L}_{[m]:k} \xrightarrow{\ \mathcal{B}_m\ } \mathcal{P}_{[m]:k} &
\end{array}
\tag{2.8}
$$

2.5. Exchangeable random k-partitions

A random k-partition $\Lambda := (\Lambda_i, 1 \le i \le k)$ of $\mathbb{N}$ is called *exchangeable* if, regarded as a $[k]$-valued sequence $\Lambda := \Lambda^1 \Lambda^2 \cdots$, it satisfies

$$\Lambda^\sigma := \Lambda^{\sigma(1)} \Lambda^{\sigma(2)} \cdots =_{\mathcal{L}} \Lambda,$$

for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$ fixing all but finitely many $n \in \mathbb{N}$.

By de Finetti's theorem, the law of an exchangeable k-partition is determined by a unique probability measure ν on the $(k-1)$-dimensional simplex

$$\Delta_k := \left\{ (s_1, \ldots, s_k) : s_i \ge 0 \text{ and } \sum_{i=1}^{k} s_i = 1 \right\}.$$

For $s \in \Delta_k$, we let $\Lambda^1, \Lambda^2, \ldots$ be i.i.d. from

$$P_s\big\{\Lambda^1 = j\big\} = s_j, \qquad j = 1, \ldots, k,$$

and define $\Lambda := \Lambda^1 \Lambda^2 \cdots$, whose distribution we denote ζ_s. For a measure ν on Δ_k, we write

$$\zeta_\nu(\cdot) := \int_{\Delta_k} \zeta_s(\cdot) \nu(\mathrm{d}s)$$

to denote the ν-mixture of ζ_s-measures.

3. Lipschitz partition processes

A random collection $\boldsymbol{\Lambda} = (\Lambda_t, t \geq 0)$ in $\mathcal{L}_{\mathbb{N}:k}$ is a *Markov process* if, for every $t > 0$, the σ-fields $\sigma\langle \Lambda_s, s < t \rangle$ and $\sigma\langle \Lambda_s, s > t \rangle$ are conditionally independent given Λ_t. We are interested in *consistent* Markov processes on $\mathcal{L}_{\mathbb{N}:k}$. We specialize to exchangeable processes in Section 4.

In Section 2.3.1, we showed a construction of exchangeable coalescent processes by an iterated application of the Coag operator at the atom times of a Poisson point process. Fundamental properties of the Coag operator endow the coalescent process with consistency and the Feller property. Of utmost importance is Lipschitz continuity, without which the process restricted to, say, $[n]$ could depend on indices $\{n+1, n+2, \ldots\}$ and the restrictions need not be Markovian, as in Example 2.1.

3.1. Poissonian construction

Let $\Phi := \{F : \mathcal{L}_{\mathbb{N}:k} \to \mathcal{L}_{\mathbb{N}:k}\}$ be the collection of all maps $\mathcal{L}_{\mathbb{N}:k} \to \mathcal{L}_{\mathbb{N}:k}$ and, for each $n \in \mathbb{N}$, let $\Phi_n \subseteq \Phi$ be the subcollection of maps so that the restriction of $F(\lambda)$ to $\mathcal{L}_{[n]:k}$ depends on λ only through $\lambda_{|[n]}$, that is

$$\Phi_n := \big\{ F \in \Phi : \lambda_{|[n]} = \lambda'_{|[n]} \implies F(\lambda)_{|[n]} = F(\lambda')_{|[n]} \text{ for all } \lambda, \lambda' \in \mathcal{L}_{\mathbb{N}:k} \big\}.$$

These collections satisfy

$$\Phi \supset \cdots \supset \Phi_{n-1} \supset \Phi_n \supset \Phi_{n+1} \supset \cdots,$$

whose limit $\bigcap_{n \in \mathbb{N}} \Phi_n = \Phi_\infty$ exists and is non-empty. (For example, the identity map $\mathbf{Id} : \mathcal{L}_{\mathbb{N}:k} \to \mathcal{L}_{\mathbb{N}:k}$ is in Φ_n for every $n \in \mathbb{N}$ and, hence, $\mathbf{Id} \in \Phi_\infty$.) For all $F \in \Phi_\infty$, the restriction $F(\lambda)_{|[n]}$ depends only on $\lambda_{|[n]}$, for every $n \in \mathbb{N}$.

Lemma 3.1. *The collection Φ_∞ is in one-to-one correspondence with*

$$\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k}) := \big\{ F \in \Phi : d\big(F(\lambda), F(\lambda')\big) \leq d\big(\lambda, \lambda'\big) \text{ for all } \lambda, \lambda' \in \mathcal{L}_{\mathbb{N}:k} \big\},$$

Lipschitz continuous maps $\mathcal{L}_{\mathbb{N}:k} \to \mathcal{L}_{\mathbb{N}:k}$ with Lipschitz constant 1.

Proof. First, suppose $F \in \mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$. Then $d(F(\lambda), F(\lambda')) \leq d(\lambda, \lambda')$ for every $\lambda, \lambda' \in \mathcal{L}_{\mathbb{N}:k}$. By definition of the metric (2.7), $\lambda_{|[r]} = \lambda'_{|[r]}$ for all $r \leq -\log_2 d(\lambda, \lambda')$ and $d(F(\lambda), F(\lambda')) \leq 2^{-r}$; hence, for every $n \in \mathbb{N}$, $\lambda_{|[n]} = \lambda'_{|[n]}$ implies $F(\lambda)_{|[n]} = F(\lambda')_{|[n]}$ and $F \in \Phi_\infty$. The converse is immediate by the definition of the sets $(\Phi_n, n \in \mathbb{N})$ above. $\qquad \square$

As $\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$ is exactly the set

$$\left\{ F \in \Phi : \forall n \in \mathbb{N}, \lambda_{|[n]} = \lambda'_{|[n]} \Longrightarrow F(\lambda)_{|[n]} = F(\lambda')_{|[n]} \text{ for all } \lambda, \lambda' \in \mathcal{L}_{\mathbb{N}:k} \right\},$$

any $F \in \mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$ can be written as the compatible sequence $(F_{[1]}, F_{[2]}, \ldots)$ of its restrictions to $\mathrm{Lip}(\mathcal{L}_{[n]:k})$ for each $n \in \mathbb{N}$. Specifically, the restriction $F_{[n]}$ of $F \in \mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$ to $\mathrm{Lip}(\mathcal{L}_{[n]:k})$ is defined, for every $\lambda \in \mathcal{L}_{[n]:k}$, by $F_{[n]}(\lambda) = F(\lambda^*)_{|[n]}$, for any choice of $\lambda^* \in \mathcal{L}_{\mathbb{N}:k}$ such that $\lambda^*_{|[n]} = \lambda$. In this sense, $\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$ is a projective limit space which we can equip with the ultra-metric

$$d_\Phi(F, F') := 2^{-\mathbf{n}(F, F')}, \tag{3.1}$$

where $\mathbf{n}(F, F') := \max\{n \in \mathbb{N} : F_{[n]} = F'_{[n]}\}$, and σ-field $\mathcal{F} = \sigma\langle \bigcup_{n \in \mathbb{N}} \mathrm{Lip}(\mathcal{L}_{[n]:k}) \rangle$. It follows that any measure φ on $(\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k}), \mathcal{F})$ determines a measure φ_n on $\mathrm{Lip}(\mathcal{L}_{[n]:k})$ through

$$\varphi_n(F) := \varphi(\{F^* \in \mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k}) : F^*_{[n]} = F\}), \qquad F \in \mathrm{Lip}(\mathcal{L}_{[n]:k}). \tag{3.2}$$

For any $n \in \mathbb{N} \cup \{\infty\}$, let $\mathbf{Id}_n$ denote the identity map $\mathcal{L}_{[n]:k} \to \mathcal{L}_{[n]:k}$.[1] Then, for a measure φ on $(\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k}), \mathcal{F})$ satisfying

$$\varphi(\{\mathbf{Id}\}) = 0 \quad \text{and} \quad \varphi_n(\mathrm{Lip}(\mathcal{L}_{[n]:k}) \setminus \{\mathbf{Id}_n\}) < \infty \qquad \text{for every } n \in \mathbb{N}, \tag{3.3}$$

let $\mathbf{F} := \{(t, F^t)\} \subset \mathbb{R}^+ \times \mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$ be a Poisson point process with intensity $dt \otimes \varphi$. Given $\mathbf{F}$ and some (possibly random) initial state $\lambda_0 \in \mathcal{L}_{\mathbb{N}:k}$, we construct a Markov process $\mathbf{\Lambda}$ on $\mathcal{L}_{\mathbb{N}:k}$ as follows. For each $n \in \mathbb{N}$, we define $\mathbf{\Lambda}^{[n]} = (\Lambda_t^{[n]}, t \geq 0)$ on $\mathcal{L}_{[n]:k}$ by $\Lambda_0^{[n]} = \lambda_{0|[n]}$ and

- if $t > 0$ is an atom time of $\mathbf{F}$ such that $F_{[n]}^t \neq \mathbf{Id}_n$, we put $\Lambda_t^{[n]} = F_{[n]}^t(\Lambda_{t-}^{[n]})$;
- otherwise, we put $\Lambda_t^{[n]} = \Lambda_{t-}^{[n]}$. $\qquad\qquad$ (3.4)

Proposition 3.1. *For every $n \in \mathbb{N}$, $\mathbf{\Lambda}^{[n]}$ is a càdlàg finite state space Markov process, and $(\mathbf{\Lambda}^{[n]}, n \in \mathbb{N})$ determines a unique consistent Markov process $\mathbf{\Lambda}$ on $\mathcal{L}_{\mathbb{N}:k}$.*

Proof. That each $\mathbf{\Lambda}^{[n]}$ is càdlàg follows from (3.3) since $\varphi_n(\mathrm{Lip}(\mathcal{L}_{[n]:k}) \setminus \{\mathbf{Id}_n\}) < \infty$ ensures that, within any bounded interval of $[0, \infty)$, there are at most finitely many atom times of $\mathbf{F}$ for which $F_{[n]}^t \neq \mathbf{Id}_n$. Furthermore, for each $n \in \mathbb{N}$, $\mathbf{\Lambda}^{[n]}$ is Markov by the construction in (3.4). The collection $(\mathbf{\Lambda}^{[n]}, n \in \mathbb{N})$ is compatible by construction and therefore, for every $t \geq 0$, $(\Lambda_t^{[n]}, n \in \mathbb{N})$ determines a unique k-partition Λ_t of $\mathbb{N}$. It follows that $(\mathbf{\Lambda}^{[n]}, n \in \mathbb{N})$ determines a unique consistent Markov process $\mathbf{\Lambda} = (\Lambda_t, t \geq 0)$ on $\mathcal{L}_{\mathbb{N}:k}$. $\qquad \square$

---

[1] To maintain consistent notation, we also define $[\infty] := \mathbb{N}$ so that $\mathcal{L}_{[n]:k} = \mathcal{L}_{\mathbb{N}:k}$ for $n = \infty$.

Some remarks about the above construction:

(i) $\mathbf{\Lambda}$ need not be exchangeable; we treat exchangeable processes in Section 4 and give an explicit example of a non-exchangeable process in Section 4.5.

(ii) Each restriction $\mathbf{\Lambda}_{|[n]} := (\Lambda_{t|[n]}, t \geq 0)$ has a Poisson point process construction based on $\mathbf{F}^{(n)} \subset \mathbb{R}^+ \times \mathrm{Lip}(\mathcal{L}_{[n]:k})$ with intensity $dt \otimes \varphi_n$.

(iii) The second half of (3.3) is needed so that the finite restrictions $(\mathbf{\Lambda}_{|[n]}, n \in \mathbb{N})$ are càdlàg. Also, φ must put all its support on $\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$, or else the construction in (3.4) would not result in a compatible collection of finite state space processes.

The second half of (3.3) corresponds to the second half of (2.5) in the following precise sense. In (3.3), we exclude the identity map $\mathbf{Id}_n$ since it does not result in a jump in the restricted process $\mathbf{\Lambda}^{[n]}$, for each $n \in \mathbb{N}$. Similarly, for each $n \in \mathbb{N}$, $\mathbf{0}_{[n]}$ is the neutral element for Coag, that is, $\mathrm{Coag}(\pi, \mathbf{0}_{[n]}) = \pi$ for all $\pi \in \mathcal{P}_{[n]}$. Hence, $\mathbf{0}_{[n]}$ determines the identity map $\mathcal{P}_{[n]} \to \mathcal{P}_{[n]}$ by way of the coagulation operator.

3.2. The Feller property

Alternatively, we can construct $\mathbf{\Lambda}$ from $\mathbf{F}$ by first constructing a Markov process ϕ^∞ on $\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$. For each $n \in \mathbb{N}$, we construct $\phi^{[n]} := (\phi_t^{[n]}, t \geq 0)$ on $\mathrm{Lip}(\mathcal{L}_{[n]:k})$ by $\phi_0^{[n]} = \mathbf{Id}_n$ and

- if $t > 0$ is an atom time of $\mathbf{F}$ such that $F_{[n]}^t \neq \mathbf{Id}_n$, we put $\phi_t^{[n]} = F_{[n]}^t \circ \phi_{t-}^{[n]}$;
- otherwise, we put $\phi_t^{[n]} = \phi_{t-}^{[n]}$. $\qquad (3.5)$

Corollary 3.1. *The collection* $(\phi^{[n]}, n \in \mathbb{N})$ *is consistent on* $(\mathrm{Lip}(\mathcal{L}_{[n]:k}), n \in \mathbb{N})$ *and determines a unique Markov process* ϕ^∞ *on* $\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$. *Moreover,* $\mathbf{\Lambda}^\infty = (\Lambda_t^\infty, t \geq 0)$ *defined by*

$$\Lambda_t^\infty = \phi_t^\infty(\Lambda_0) \qquad \text{for every } t \geq 0, \qquad (3.6)$$

is a version of $\mathbf{\Lambda}$ *in* (3.4).

Proof. The first claim follows immediately by the arguments in Proposition 3.1.

To establish the second claim, let $\mathbf{F}$ be the Poisson point process with intensity $dt \otimes \varphi$ and, for every $n \in \mathbb{N}$, let J_n be the set of atom times of $\mathbf{F}$ such that $F_{[n]}^t \neq \mathbf{Id}_n$. By (3.3), $J_n \cap [0, t]$ is almost surely finite for every $n \in \mathbb{N}$ and $t < \infty$. We construct $\mathbf{\Lambda}$ from $\mathbf{F}$ as in (3.4) and ϕ^∞ from $\mathbf{F}$ as in (3.5).

For fixed $n \in \mathbb{N}$ and $t > 0$, write $t \geq t_1 > \cdots > t_r > 0$ to be the ranked atom times of J_n before time t. Each $F^{t_i} \in \mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$, $i = 1, \ldots, r$, and so

$$\Lambda_{t|[n]} = \big(F_{[n]}^{t_1} \circ \cdots \circ F_{[n]}^{t_r}\big)(\Lambda_{0|[n]}) = \phi_{t,[n]}^\infty(\Lambda_{0|[n]}) = \phi_t^\infty(\Lambda_0)_{|[n]} = \Lambda_{t|[n]}^\infty,$$

where $\phi_{t,[n]}^\infty$ denotes the restriction of ϕ_t^∞ to $\mathrm{Lip}(\mathcal{L}_{[n]:k})$. Hence, $\mathbf{\Lambda}_{|[n]}^\infty = \mathbf{\Lambda}_{|[n]}$ almost surely for every $n \in \mathbb{N}$; whence, $\mathbf{\Lambda}^\infty = \mathbf{\Lambda}$ almost surely. The conclusion follows. $\qquad \square$

Remark 3.1. In essence, representation (3.6) entails the application of a flow ($\phi_{s,t}, 0 \le s < t < \infty$) on the space $\text{Lip}(\mathcal{L}_{\mathbb{N}:k})$, for which we apply $\phi_t := \phi_{0,t}$ to Λ_0, for each $t \ge 0$. This can be compared to constructions of coalescent processes by flows of bridges [6].

Representation (3.6) is convenient for studying the semigroup of $\boldsymbol{\Lambda}$. For every bounded, continuous function $g : \mathcal{L}_{\mathbb{N}:k} \to \mathbb{R}$, the semigroup ($\mathbf{P}_t, t \ge 0$) of $\boldsymbol{\Lambda}$ is defined by

$$\mathbf{P}_t g(\lambda) := \mathbb{E}_\lambda g(\Lambda_t), \qquad t \ge 0, \lambda \in \mathcal{L}_{\mathbb{N}:k},$$

the expectation of $g(\Lambda_t)$ given $\Lambda_0 = \lambda$. In addition, ($\mathbf{P}_t, t \ge 0$) is called a *Feller semigroup*, and the process $\boldsymbol{\Lambda}$ is called a *Feller process*, if, for every bounded, continuous $g : \mathcal{L}_{\mathbb{N}:k} \to \mathbb{R}$,

- $\lambda \mapsto \mathbf{P}_t g(\lambda)$ is continuous for every $t > 0$, and
- $\lim_{t \downarrow 0} \mathbf{P}_t g(\lambda) = g(\lambda)$ for all $\lambda \in \mathcal{L}_{\mathbb{N}:k}$.

Corollary 3.2. *The semigroup* ($\mathbf{P}_t, t \ge 0$) *of* $\boldsymbol{\Lambda}$ *satisfies*

$$\mathbf{P}_t g(\lambda) := \mathbb{E}g\big(\phi_t^\infty(\lambda)\big), \tag{3.7}$$

for every bounded, continuous map $g : \mathcal{L}_{\mathbb{N}:k} \to \mathbb{R}$ *and every* $\lambda \in \mathcal{L}_{\mathbb{N}:k}$, *where* ($\phi_t^\infty, t \ge 0$) *is the process in Corollary* 3.1.

The proof follows immediately from Corollary 3.1.

Theorem 3.1. *The process* $\boldsymbol{\Lambda}$ *constructed in* (3.4) *fulfills the Feller property.*

Proof. Continuity of the map $\lambda \mapsto \mathbf{P}_t g(\lambda)$ is an immediate consequence of continuity of g, the description of $\mathbf{P}_t$ in (3.7), and the fact that $\phi_t^\infty \in \text{Lip}(\mathcal{L}_{\mathbb{N}:k})$ for all $t > 0$ almost surely.

That $\lim_{t \downarrow 0} \mathbf{P}_t g(\lambda) = g(\lambda)$ for all $\lambda \in \mathcal{L}_{\mathbb{N}:k}$ follows by continuity of g and (3.3), which ensures that the time of the initial jump out of $\lambda_{|[n]}$ is strictly positive, for every $n \in \mathbb{N}$. $\qquad\square$

By the Feller property, any $\boldsymbol{\Lambda}$ with the construction in (3.4) has a càdlàg version. For the rest of the paper, we implicitly assume $\boldsymbol{\Lambda}$ has càdlàg paths.

Definition 3.1 (Lipschitz partition process). *We call the Markov process* $\boldsymbol{\Lambda}$ *constructed in* (3.5) *a* Lipschitz partition process *directed by* φ.

4. Exchangeable Lipschitz partition processes

A process $\boldsymbol{\Lambda}$ on $\mathcal{L}_{\mathbb{N}:k}$ is called *exchangeable* if $\boldsymbol{\Lambda} =_{\mathcal{L}} \boldsymbol{\Lambda}^\sigma$ for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$ fixing all but finitely many elements of $\mathbb{N}$. We have already shown (Proposition 3.1 and Theorem 3.1) that Lipschitz partition processes are consistent and possess the Feller property. We now consider *exchangeable Lipschitz partition processes* on $\mathcal{L}_{\mathbb{N}:k}$.

Provided its rate measure μ is exchangeable, a coalescent process (Section 2.3.1) is exchangeable. In the exchangeable case, the directing measure μ in (2.5) need only satisfy $\mu(1 \sim 2) < \infty$. Furthermore, if we describe μ by a paintbox measure ϱ_ν on $\mathcal{P}_\mathbb{N}$, (2.5) implies

$$\nu\big(\{(0, 0, \ldots)\}\big) = 0 \quad \text{and} \quad \int_{\Delta^\downarrow} (1 - s_1)\nu(\mathrm{d}s) < \infty.$$

For Lipschitz partition processes constructed in (3.4), φ must be restricted to the space of *strongly Lipschitz maps* to ensure exchangeability. We introduce strongly Lipschitz maps in Section 4.1 and show some of their properties in Section 4.2.

4.1. Strongly Lipschitz maps

In this section, we see that any exchangeable Markov process $\boldsymbol{\Lambda}$ with construction (3.4) must be directed by a measure φ whose support is contained in the proper subset of *strongly Lipschitz maps* on $\mathcal{L}_{\mathbb{N}:k}$.

For any $A \subseteq \mathbb{N}$ and $\lambda, \lambda' \in \mathcal{L}_{A:k}$, we define the *overlap* of λ and λ' by

$$\lambda \cap \lambda' := \bigcup_{i=1}^{k} (\lambda_i \cap \lambda_i'), \tag{4.1}$$

and let

$$\Sigma_n := \big\{ F \in \mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k}) : F_{[n]}(\lambda) \cap F_{[n]}(\lambda') \supseteq \lambda \cap \lambda' \text{ for all } \lambda, \lambda' \in \mathcal{L}_{[n]:k} \big\} \tag{4.2}$$

be the subset of functions $F \in \mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$ for which the overlap of the image of any $\lambda, \lambda' \in \mathcal{L}_{[n]:k}$ by the restriction $F_{[n]}$ contains the overlap of λ and λ'. By definition of the ultrametric (2.7) on $\mathcal{L}_{\mathbb{N}:k}$, if $d(\lambda, \lambda') \leq 2^{-n}$ for some $n \in \mathbb{N}$, then $[n] \subseteq \lambda \cap \lambda'$; thus, $\Sigma_n \subseteq \Phi_n$ for all $n \in \mathbb{N}$. We write $\Sigma := \bigcap_{n \in \mathbb{N}} \Sigma_n$ to denote the collection of Lipschitz continuous maps satisfying

$$F(\lambda) \cap F(\lambda') \supseteq \lambda \cap \lambda' \qquad \text{for all } \lambda, \lambda' \in \mathcal{L}_{\mathbb{N}:k}, \tag{4.3}$$

and we call any $F \in \Sigma$ *strongly Lipschitz continuous*. In the following proposition, let $\boldsymbol{\Lambda}$ be a Lipschitz partition process directed by φ.

Proposition 4.1. *If $\boldsymbol{\Lambda}$ is exchangeable, then φ is supported on $\mathcal{F} \cap \Sigma$, the trace σ-field of $\mathcal{F} = \sigma \langle \bigcup_{n \in \mathbb{N}} \mathrm{Lip}(\mathcal{L}_{[n]:k}) \rangle$.*

Proof. Suppose $\boldsymbol{\Lambda}$ is exchangeable and fix $n \in \mathbb{N}$. Then $\boldsymbol{\Lambda}_{|[n]}^\sigma =_{\mathcal{L}} \boldsymbol{\Lambda}_{|[n]}$ for all $\sigma \in \mathscr{S}_n$. Hence, we can construct $\boldsymbol{\Lambda}^\sigma$ and $\boldsymbol{\Lambda}$ from the same Poisson point process $\mathbf{F} := \{(t, F^t)\}$ by putting

$$\Lambda_{t|[n]}^\sigma = \sigma^* F_{[n]}^t (\Lambda_{t-|[n]}) = \sigma^* F_{[n]}^t \sigma^{*^{-1}} \big(\Lambda_{t-|[n]}^\sigma\big) \tag{4.4}$$

for every $t \in J_n := \{t > 0 : (t, F^t) \in \mathbf{F} \text{ and } F_{[n]}^t \neq \mathbf{Id}_n\}$, the jump times of $\boldsymbol{\Lambda}_{|[n]}$. By (4.4) and the construction of $\boldsymbol{\Lambda}$ in (3.4), $\mathbf{F}^\sigma := \{(t, \sigma^* F^t \sigma^{*^{-1}})\}$ has the same law as a Poisson point process

on $[0, \infty) \times \text{Lip}(\mathcal{L}_{\mathbb{N}:k})$ with intensity $dt \otimes \varphi$, for all $\sigma \in \mathscr{S}_n$. Since we have assumed that the support of φ is a subset of $\text{Lip}(\mathcal{L}_{\mathbb{N}:k})$ and the set J_∞ of atom times of $\mathbf{F}$ is at most countable by (3.3), we have $\sigma^* F^t \sigma^{*^{-1}} \in \text{Lip}(\mathcal{L}_{\mathbb{N}:k})$ for all $t \in J_\infty$ almost surely. It follows that φ must be supported on

$$\Xi := \left\{ F \in \text{Lip}(\mathcal{L}_{\mathbb{N}:k}) : \sigma^* F \sigma^{*^{-1}} \in \text{Lip}(\mathcal{L}_{\mathbb{N}:k}) \text{ for every finite permutation } \sigma : \mathbb{N} \to \mathbb{N} \right\},$$

which is non-empty. To see that $\Xi \subset \Sigma$, we need the following lemma.

Lemma 4.1. *For $n \in \mathbb{N}$, let $\lambda, \lambda' \in \mathcal{L}_{[n]:k}$ have overlap of size $\#(\lambda \cap \lambda') = r \in [n]$. Then there exists $\sigma \in \mathscr{S}_n$ such that σ^2 is the identity $[n] \to [n]$ and $\lambda^\sigma \cap \lambda'^\sigma = [r]$.*

Proof. For $m, m' \leq r$, let $r < i_1 < i_2 < \cdots < i_m$ be the elements of $(\lambda \cap \lambda') \setminus [r]$ and let $j_1 < \cdots < j_{m'} \leq r$ be the elements of $(\lambda \cap \lambda')^c \cap [r]$. Note that $m' = r - (r - m) = m$; so we can define $\sigma \in \mathscr{S}_n$ by $\sigma(i_l) = j_l$ and $\sigma(j_l) = i_l$ for every $l = 1, \ldots, m$, and $\sigma(i) = i$ otherwise. Clearly, σ^2 is the identity and $i \in \lambda \cap \lambda'$ implies $\sigma(i) \in [r]$. $\square$

Now, fix $n \in \mathbb{N}$ and take $F \in \Xi$. For any $\sigma \in \mathscr{S}_n$, we write $F^\sigma_{[n]} := \sigma^* F_{[n]} \sigma^{*^{-1}}$. Take any $\lambda, \lambda' \in \mathcal{L}_{[n]:k}$ and let σ be the permutation of $[n]$ from the preceding lemma. Then $\sigma^* = \sigma^{*^{-1}}$, $F^\sigma_{[n]} := \sigma^* F_{[n]} \sigma^* \in \text{Lip}(\mathcal{L}_{[n]:k})$, and $F_{[n]} = \sigma^* F^\sigma_{[n]} \sigma^*$. Let d_n denote the restriction of the metric d in (2.7) to $\mathcal{L}_{[n]:k}$. By Lipschitz continuity and Lemma 4.1,

$$d_n \left(F^\sigma_{[n]} \sigma^*(\lambda), F^\sigma_{[n]} \sigma^*(\lambda') \right) = d_n \left(F^\sigma_{[n]}(\lambda^\sigma), F^\sigma_{[n]}(\lambda'^\sigma) \right) \leq 2^{-r};$$

hence, $\lambda^\sigma(j) = \lambda'^\sigma(j)$ and $[F^\sigma_{[n]} \sigma^*(\lambda)](j) = [F^\sigma_{[n]} \sigma^*(\lambda')](j)$ for all $j \in [r]$. Finally, take $i \in \lambda \cap \lambda'$. Then $\sigma(i) \in [r]$ by Lemma 4.1, which implies

$$
\begin{aligned}
\left[F_{[n]}(\lambda) \right](i) &= \sigma^* \left[F^\sigma_{[n]} \sigma^*(\lambda) \right](i) \\
&= \left[F^\sigma_{[n]} \sigma^*(\lambda) \right](\sigma(i)) \\
&= \left[F^\sigma_{[n]} \sigma^*(\lambda') \right](\sigma(i)) \\
&= \sigma^* \left[F^\sigma_{[n]} \sigma^*(\lambda') \right](i) = \left[F_{[n]}(\lambda') \right](i),
\end{aligned}
$$

and $i \in F(\lambda) \cap F(\lambda')$. It follows that $\Xi \subset \Sigma$. $\square$

Remark 4.1. The converse of Proposition 4.1 does not hold.

Proposition 4.1 shows that the directing measure of an exchangeable Lipschitz partition process can only assign positive measure to events in the trace σ-field $\mathcal{F} \cap \Sigma$. In the next section, we use condition (4.3) to characterize the space Σ.

4.2. Strongly Lipschitz maps and set-valued matrix multiplication

A $k \times k$ matrix M over $S \subset \mathbb{N}$ is a collection $(M_{ij}, 1 \le i, j \le k)$ of subsets of S for which we define the operation *multiplication* by

$$\left(M * M'\right)_{ij} \equiv \left(M M'\right)_{ij} := \bigcup_{l=1}^{k}\left(M_{il} \cap M'_{lj}\right), \qquad 1 \le i, j \le k. \tag{4.5}$$

The operation in (4.5) mimics multiplication of real-valued matrices, but for matrices taking values in a distributive lattice. Here, the lattice operations $\cap$ and $\cup$ correspond to multiplication and addition, respectively.

We are particularly interested in *partition operators*, matrices M over $[n]$ with each $M^j \in \mathcal{L}_{[n]:k}$, $j = 1, \ldots, k$, where M^j denotes the jth column of M. We write $\mathcal{M}_{[n]:k}$ to denote the set of $k \times k$ partition operators over $[n]$.

Some observations about the operation (4.5):

(i) For $m \le n$, we can define the restriction of $M \in \mathcal{M}_{[n]:k}$ to $\mathcal{M}_{[m]:k}$. First, we let $I_m^k :=$ diag$([m], \ldots, [m])$ be the $k \times k$ matrix with diagonal entries $[m]$ and off-diagonal entries the empty set. Then, for any $M \in \mathcal{M}_{[n]:k}$, the product $M_{[m]} := I_m^k M = M I_m^k \in \mathcal{M}_{[m]:k}$ is well-defined as the restriction of M to $\mathcal{M}_{[m]:k}$. It follows that $(\mathcal{M}_{[n]:k}, n \in \mathbb{N})$ is a projective system with limit space $\mathcal{M}_{\mathbb{N}:k}$, partition operators on $\mathcal{L}_{\mathbb{N}:k}$.

(ii) For any injection $\psi := (\psi_1, \ldots, \psi_k) : [m]^k \to [n]^k$, $m \le n$, we define the projection $\psi^{**} : \mathcal{M}_{[n]:k} \to \mathcal{M}_{[m]:k}$ by

$$\psi^{**}(M) := \left(\psi_1^*\left(M^1\right), \ldots, \psi_k^*\left(M^k\right)\right), \qquad \text{for every } M \in \mathcal{M}_{[n]:k},$$

where we write $M := (M^1, \ldots, M^k)$ as the vector of its columns. In particular, for $\sigma \in \mathscr{S}_n$, we write $\sigma^{**}M = M^\sigma = (\sigma^*M^1, \ldots, \sigma^*M^k)$, the image of M under relabeling each of its columns by σ.

(iii) We can equip $\mathcal{M}_{\mathbb{N}:k}$ with the ultrametric d_Φ in (3.1) restricted to $\mathcal{M}_{\mathbb{N}:k}$; in particular,

$$d_\Phi\left(M, M'\right) := 2^{-\mathbf{n}(M, M')},$$

where $\mathbf{n}(M, M') := \max\{n \in \mathbb{N} : M I_n^k = M' I_n^k\}$.

We record some facts about partition operators.

Lemma 4.2. *Let $n \in \mathbb{N} \cup \{\infty\}$ and $m \le n$.*

(i) *Any $M \in \mathcal{M}_{[m]:k}$ determines a map $M : \mathcal{M}_{[n]:k} \to \mathcal{M}_{[m]:k}$, $M' \mapsto M M'$.*

(ii) *Any $M \in \mathcal{M}_{[m]:k}$ determines a map $M : \mathcal{L}_{[n]:k} \to \mathcal{L}_{[m]:k}$ by*

$$(M\lambda)_i := \bigcup_{j=1}^{k}(M_{ij} \cap \lambda_j), \qquad i = 1, \ldots, k, \lambda \in \mathcal{L}_{[n]:k}. \tag{4.6}$$

(iii) *The operation* (4.5) *is associative, that is,* $M(M'M'') = (MM')M''$ *for all* $M, M', M'' \in \mathcal{M}_{[n]:k}$.

(iv) *Each* $M \in \mathcal{M}_{[n]:k}$ *determines a Lipschitz continuous map* $M : \mathcal{M}_{[n]:k} \to \mathcal{M}_{[n]:k}$ *through* (4.5) *and* $M : \mathcal{L}_{[n]:k} \to \mathcal{L}_{[n]:k}$ *through* (4.6).

Proof. The proof is routine, but we include the proof of (iv) because it is crucial to the paper. Note that the restriction of any $\lambda \in \mathcal{L}_{\mathbb{N}:k}$ to $n \in \mathbb{N}$ can be expressed as $\lambda_{|[n]} = I_n^k \lambda$. Let $\lambda, \lambda' \in \mathcal{L}_{\mathbb{N}:k}$ be such that $I_r^k \lambda = I_r^k \lambda'$ for some $r \in \mathbb{N}$. Then $d(\lambda, \lambda') \leq 2^{-r}$ and, for every $M \in \mathcal{M}_{\mathbb{N}:k}$,

$$I_r^k(M\lambda) = (I_r^k M)\lambda = (M I_r^k)\lambda = M(I_r^k \lambda) = M(I_r^k \lambda') = I_r^k(M\lambda'),$$

implying $d(M\lambda, M\lambda') \leq d(\lambda, \lambda')$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 4.1 (Partition operator). Fix $n = 6$, $k = 2$, and let $\lambda = (\{1, 3, 4, 5\}, \{2, 6\})$. Then the image of λ by

$$M := \begin{pmatrix} \{2, 3\} & \{2, 4, 5, 6\} \\ \{1, 4, 5, 6\} & \{1, 3\} \end{pmatrix}$$

is

$$M\lambda := \begin{pmatrix} \{2, 3\} & \{2, 4, 5, 6\} \\ \{1, 4, 5, 6\} & \{1, 3\} \end{pmatrix} \begin{pmatrix} \{1, 3, 4, 5\} \\ \{2, 6\} \end{pmatrix}$$

$$= \begin{pmatrix} (\{2, 3\} \cap \{1, 3, 4, 5\}) \cup (\{2, 4, 5, 6\} \cap \{2, 6\}) \\ (\{1, 4, 5, 6\} \cap \{1, 3, 4, 5\}) \cup (\{1, 3\} \cap \{2, 6\}) \end{pmatrix}$$

$$= \begin{pmatrix} \{2, 3, 6\} \\ \{1, 4, 5\} \end{pmatrix}.$$

Remark 4.2 (Partition operators and the Coag operator). There is a relationship between partition operators and the coagulation operator $\text{Coag} : \mathcal{P}_{\mathbb{N}} \times \mathcal{P}_{\mathbb{N}} \to \mathcal{P}_{\mathbb{N}}$ from Section 2.3.1. For $k \in \mathbb{N}$, let $\pi := \{b_1, \ldots, b_k\} \in \mathcal{P}_{\mathbb{N}:k}$ and define $\lambda := (b_1, \ldots, b_k)$, the k-partition obtained by listing the blocks of π in ascending order of their least element. Now, given $\pi' = \{b'_1, \ldots, b'_{r'}\} \in \mathcal{P}_{[k]}$, we define $M := M_{\pi'}$ by

$$M_{ij} := \begin{cases} \mathbb{N}, & j \in b'_i, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Then $\mathcal{B}_\infty(M_{\pi'}\lambda) = \text{Coag}(\pi, \pi')$. For example, let $\pi = 123/45/678/9$ so that $\lambda = (123, 45, 678, 9)$, and let $\pi' = 12/34$. In this case, $\text{Coag}(\pi, \pi') = 12345/6789$ and

$$M_{\pi'}\lambda = \begin{pmatrix} \mathbb{N} & \mathbb{N} & \varnothing & \varnothing \\ \varnothing & \varnothing & \mathbb{N} & \mathbb{N} \\ \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix} \begin{pmatrix} 123 \\ 45 \\ 678 \\ 9 \end{pmatrix} = \begin{pmatrix} 12345 \\ 6789 \\ \varnothing \\ \varnothing \end{pmatrix}.$$

Note that, in general, partition operators cannot be used instead of the coagulation operator in the construction of the coalescent process because, in the standard coalescent, the initial state

$\Pi_0 := \mathbf{0}_{\mathbb{N}}$ has infinitely many blocks, but partition operators are defined as $k \times k$ matrices for finite $k \geq 1$.

Proposition 4.2. *The space $\mathcal{M}_{\mathbb{N}:k}$ of partition operators is in one-to-one correspondence with Σ defined in* (4.2).

Proof. Let $F \in \Sigma$ and $n \in \mathbb{N}$. Then $F \in \Sigma_n$ and, for each $i \in [n]$, if $\lambda(i) = \lambda'(i)$ then $F_{[n]}(\lambda_{|[n]})(i) = F_{[n]}(\lambda'_{|[n]})(i)$. For $j = 1, \ldots, k$, let $E_j^{(n)} \in \mathcal{L}_{[n]:k}$ be the k-partition of $[n]$ satisfying $E_j^{(n)}(i) = j$ for every $i \in [n]$. Construct $M_{[n]} \in \mathcal{M}_{[n]:k}$ by setting its jth column $M_{[n]}^j$ equal to the image of $E_j^{(n)}$ by $F_{[n]}$. So $M_{[n]} := (F_{[n]}(E_1^{(n)}), F_{[n]}(E_2^{(n)}), \ldots, F_{[n]}(E_k^{(n)}))$. By definition of Σ_n in (4.2), it is clear that $M_{[n]}\lambda = F_{[n]}(\lambda)$, for every $\lambda \in \mathcal{L}_{[n]:k}$. The collection $(M_{[n]}, n \in \mathbb{N})$ is compatible with respect to the restriction maps on $(\mathcal{M}_{[n]:k}, n \in \mathbb{N})$ and therefore determines a unique $M \in \mathcal{M}_{\mathbb{N}:k}$ satisfying

$$M\lambda = F(\lambda) \qquad \text{for every } \lambda \in \mathcal{L}_{\mathbb{N}:k}.$$

The opposite morphism $\mathcal{M}_{\mathbb{N}:k} \to \Sigma$ follows from definition (4.6) and definition of the metric in (2.7). $\qquad \square$

From Proposition 4.2, we can assume, without loss of generality, that any exchangeable process with construction (3.4) is directed by a measure μ on $(\mathcal{M}_{\mathbb{N}:k}, \sigma \langle \bigcup_{n \in \mathbb{N}} \mathcal{M}_{[n]:k} \rangle)$ for which

$$\mu(\{I_\infty^k\}) = 0 \quad \text{and} \quad \mu_n(\mathcal{M}_{[n]:k} \setminus \{I_n^k\}) < \infty \qquad \text{for all } n \in \mathbb{N}, \tag{4.7}$$

where I_n^k is the partition operator with diagonal entries $[n]$ and off-diagonal entries the empty set, and μ_n denotes the restriction of μ to $\mathcal{M}_{[n]:k}$. Note that (4.7) agrees with (3.3).

Theorem 4.1. *Let $\Lambda := (\Lambda_t, t \geq 0)$ be a Lipschitz partition process on $\mathcal{L}_{\mathbb{N}:k}$. Then Λ is exchangeable if and only if its directing measure μ*

- *is supported on $\mathcal{M}_{\mathbb{N}:k}$,*
- *satisfies*

$$\mu(\{I_\infty^k\}) = 0 \quad \text{and} \quad \mu_2(\{M \in \mathcal{M}_{[2]:k} : M \neq I_2^k\}) < \infty, \tag{4.8}$$

 and
- *for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$ fixing all but finitely many $n \in \mathbb{N}$ and every measurable subset $A \subseteq \mathcal{M}_{\mathbb{N}:k}$,*

$$\mu(A) = \mu(\{M^\sigma : M \in A\}). \tag{4.9}$$

Proof. Support of μ on $\mathcal{M}_{\mathbb{N}:k}$ follows from Proposition 4.2, and (4.9) is a consequence of exchangeability and the fact that, for any $M \in \mathcal{M}_{[n]:k}$, $\lambda \in \mathcal{L}_{[n]:k}$, and $\sigma \in \mathscr{S}_n$, $(M\lambda)^\sigma = M^\sigma \lambda^\sigma$. Condition (4.8) follows from (4.7).

To show the converse, we need only show that (4.8) implies (4.7). Indeed, let $n \in \mathbb{N}$ and note that the event $\mathcal{M}_{[n]:k} \setminus \{I_n^k\} = \{M \in \mathcal{M}_{[n]:k} : M \neq I_n^k\}$ implies that there is some permutation $\sigma \in \mathscr{S}_n$ such that $M^\sigma I_2^k \neq I_2^k$; hence,

$$
\mu_n\big(\{M : M \neq I_n^k\}\big) = \mu_n\bigg(\bigcup_{\sigma \in \mathscr{S}_n} \{M : M^\sigma I_2^k \neq I_2^k\} \bigg)
$$

$$
\leq \sum_{\sigma \in \mathscr{S}_n} \mu_2\big(\{M \in \mathcal{M}_{[2]:k} : M \neq I_2^k\}\big)
$$

$$
= n! \mu_2\big(\{M \in \mathcal{M}_{[2]:k} : M \neq I_2^k\}\big)
$$

$$
< \infty.
$$

The rest is immediate.   □

Proposition 4.2 and Theorem 4.1 suggest a construction of arbitrary partition operators. In brief, take any collection $\lambda^{(1)}, \ldots, \lambda^{(k)}$ in $\mathcal{L}_{\mathbb{N}:k}$ and, for each $j = 1, \ldots, k$, put the jth column of $M \in \mathcal{M}_{\mathbb{N}:k}$ equal to $\lambda^{(j)}$. Likewise, a measure μ on $\mathcal{M}_{\mathbb{N}:k}$ can be defined by a measure on the product space $\mathcal{L}_{\mathbb{N}:k}^k$. Furthermore, using the above observation, we can construct a measure φ with support in $\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$ but not in $\mathcal{M}_{\mathbb{N}:k}$, leading to an explicit construction of a non-exchangeable Lipschitz process whose semigroup is not determined by strongly Lipschitz functions. We show such a process in Section 4.5.

4.3. Examples: Exchangeable Lipschitz partition processes

Example 4.2 (Self-similar exchangeable Markov process on $\mathcal{L}_{\mathbb{N}:k}$). For any probability measure ν on Δ_k, recall the definition of ζ_ν (Section 2.5). Given a measure ν on Δ_k, we write $\mu_{\nu^{\otimes k}}$ to denote the measure on $\mathcal{M}_{\mathbb{N}:k}$ coinciding with the product measure $\zeta_\nu \otimes \cdots \otimes \zeta_\nu$ on $\mathcal{L}_{\mathbb{N}:k}^k$. More generally, for measures $\nu_1, \ldots, \nu_k$ on Δ_k, $\mu_{\nu_1 \otimes \cdots \otimes \nu_k}$ is the measure on $\mathcal{M}_{\mathbb{N}:k}$ coinciding with $\zeta_{\nu_1} \otimes \cdots \otimes \zeta_{\nu_k}$ on $\mathcal{L}_{\mathbb{N}:k}^k$.

Let $\nu_1, \ldots, \nu_k$ be measures on Δ_k such that

$$
\int_{\Delta_k} (1 - s_i) \nu_i(\mathrm{d}s) < \infty \qquad \text{for all } i = 1, \ldots, k.
$$

Then the second half of (3.3) is satisfied for $\mu_{\nu_1 \otimes \cdots \otimes \nu_k}$ and we can construct a process $\mathbf{\Lambda} := (\Lambda_t, t \geq 0)$ from a Poisson point process $\mathbf{M} := \{(t, M_t)\} \subset \mathbb{R}^+ \times \mathcal{M}_{\mathbb{N}:k}$ with intensity $\mathrm{d}t \otimes \mu_{\nu_1 \otimes \cdots \otimes \nu_k}$, just as in (3.4). The infinitesimal jump rates of this process are given explicitly by

$$
Q_n(\lambda, \lambda') := \prod_{i=1}^k \zeta_{\nu_i}^{\lambda_i}(\lambda'_{|\lambda_i}), \qquad \lambda \neq \lambda' \in \mathcal{L}_{[n]:k},
$$

for each $n \in \mathbb{N}$, where ζ_ν^b denotes the measure induced on $\mathcal{L}_{b:k}$ by ζ_ν for any $b \subseteq \mathbb{N}$. This process is the analog of the self-similar processes in Section 2.3.2.

Example 4.3. Similar to the above example, let ν be a measure on Δ_k so that

$$\zeta_\nu^{(n)}\big(\mathcal{L}_{[n]:k}\backslash\{E_i^{(n)}\}\big) < \infty, \qquad \text{for every } n \in \mathbb{N} \text{ and all } i = 1,\ldots,k,$$

where $E_i^{(n)} \in \mathcal{L}_{[n]:k}$ is the k-partition of $[n]$ with all elements labeled i. With $\mathcal{U}_k$ denoting the uniform distribution on $[k]$, the Poisson point process $\mathbf{F} = \{(t,\lambda_t,U_t)\} \subset [0,\infty) \times \mathcal{L}_{\mathbb{N}:k} \times [k]$, with intensity $dt \otimes \zeta_\nu \otimes \mathcal{U}_k$, determines a random subset $\mathbf{M} \subset [0,\infty) \times \mathcal{M}_{\mathbb{N}:k}$, where for each atom time $t > 0$ of $\mathbf{F}$ we define $M_t \in \mathcal{M}_{\mathbb{N}:k}$ by putting

$$M_t^i = \begin{cases} \lambda_t, & i = U_t, \\ E_i, & \text{otherwise}; \end{cases}$$

that is, writing $\lambda_t = (\lambda_{t,1},\ldots,\lambda_{t,k})$, we put

$$M_t := \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & \cdots & U_t & \cdots & k \end{array} \\ \begin{pmatrix} \mathbb{N} & \varnothing & \cdots & \lambda_{t,1} & \cdots & \varnothing \\ \varnothing & \mathbb{N} & \cdots & \lambda_{t,2} & \cdots & \varnothing \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \varnothing & \varnothing & \cdots & \lambda_{t,k} & \cdots & \mathbb{N} \end{pmatrix} \end{array}.$$

Given $\mathbf{F}$ and an initial state $\Lambda_0 \in \mathcal{L}_{\mathbb{N}:k}$, we construct the process Λ as in (3.4) by putting $\Lambda_t = M_t \Lambda_{t-}$ whenever $t > 0$ is an atom time of $\mathbf{F}$. Variations of this description, for example, for which at most one class of the current state Λ_t is broken apart in any single jump, are possible and straightforward. For example, the rates at which different classes experience jumps need not be identical.

Example 4.4 (Group action on $\mathcal{L}_{\mathbb{N}:k}$). For any $\lambda \in \mathcal{L}_{\mathbb{N}:k}$, we define $\mathbf{M}_\lambda \in \mathcal{M}_{\mathbb{N}:k}$ by

$$\mathbf{M}_\lambda := \begin{pmatrix} \lambda_1 & \lambda_k & \lambda_{k-1} & \cdots & \lambda_2 \\ \lambda_2 & \lambda_1 & \lambda_k & \cdots & \lambda_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_k & \lambda_{k-1} & \lambda_{k-2} & \cdots & \lambda_1 \end{pmatrix}. \tag{4.10}$$

In words, $\mathbf{M}_\lambda$ is the $k \times k$ matrix whose jth column is the jth cyclic shift of the classes of λ. Note that $\mathbf{M}_\lambda \lambda' = \mathbf{M}_{\lambda'} \lambda$ for all $\lambda, \lambda' \in \mathcal{L}_{\mathbb{N}:k}$.

Any measure ζ on $\mathcal{L}_{\mathbb{N}:k}$ determines a measure μ_ζ on $\mathcal{M}_{\mathbb{N}:k}$ as follows. Let $A \subset \mathcal{L}_{\mathbb{N}:k}$ be any measurable subset, then we define

$$\mu_\zeta\big(\{\mathbf{M}_\lambda : \lambda \in A\}\big) = \zeta(A).$$

Let ζ be a measure on $\mathcal{L}_{\mathbb{N}:k}$ so that μ_ζ satisfies (4.7) and let $\mathbf{G} := \{(t,G_t)\} \subset \mathbb{R}^+ \times \mathcal{L}_{\mathbb{N}:k}$ be a Poisson point process with intensity $dt \otimes \zeta$. Then the construction of the process Λ on $\mathcal{L}_{\mathbb{N}:k}$ with initial state $\Lambda_0 \in \mathcal{L}_{\mathbb{N}:k}$ proceeds as in (3.4) where, for every atom time $t > 0$ of $\mathbf{G}$, we define $M_t := \mathbf{M}_{G_t}$. Note that the exchangeability condition from Theorem 4.1 (in coordination with de Finetti) implies that if the process Λ constructed from $\mathbf{G}$ is exchangeable, then the measure ζ directing $\mathbf{G}$ must coincide with ζ_ν for some measure ν on Δ_k.

Corollary 4.1. *The process* $\mathbf{\Lambda}$ *based on* $\mathbf{G}$ *and* Λ_0 *is exchangeable if and only if* $\zeta = \zeta_\nu$, *for some measure* ν *on* Δ_k, *and* Λ_0 *is an exchangeable* k-*partition of* $\mathbb{N}$.

By regarding the elements of $\mathbb{N}$ as labeled balls and the classes of $\lambda \in \mathcal{L}_{\mathbb{N}:k}$ as labeled boxes, the action $\mathbf{M}_\lambda \lambda'$ can be interpreted as the reassignment of each of the balls to a new class via a cyclic shift by one less than their class assignment in λ (modulo k). The commutative property of this class of maps also implies that the collection $\{\mathbf{M}_\lambda : \lambda \in \mathcal{L}_{\mathbb{N}:k}\} \subset \mathcal{M}_{\mathbb{N}:k}$ is a special subspace of $\mathcal{M}_{\mathbb{N}:k}$. For instance, for every $\lambda \in \mathcal{L}_{\mathbb{N}:k}$, $\mathbf{M}_\lambda \mathbf{M}_\lambda^T = I_\infty^k$, where $M^T \in \mathcal{M}_{\mathbb{N}:k}$ denotes the usual matrix transpose of M.

4.4. Associated Δ_k-valued Markov process

We define the *asymptotic frequency* of any $A \subseteq \mathbb{N}$ by the limit

$$|A| := \lim_{n \to \infty} \frac{\#(A \cap [n])}{n}, \qquad \text{if it exists.} \tag{4.11}$$

Furthermore, we say $\lambda \in \mathcal{L}_{\mathbb{N}:k}$ possesses asymptotic frequency $|\lambda| := (|\lambda_j|, 1 \le j \le k) \in \Delta_k$, provided $|\lambda_j|$ exists for every $j = 1, \ldots, k$. By de Finetti's theorem, any exchangeable k-partition of $\mathbb{N}$ possesses an asymptotic frequency almost surely. In particular, for $s \in \Delta_k$, $\Lambda \sim \zeta_s$ has $|\Lambda| = s$ with probability one.

Given a process $\mathbf{\Lambda}$ on $\mathcal{L}_{\mathbb{N}:k}$, its associated Δ_k-valued process is defined by $|\mathbf{\Lambda}| := (|\Lambda_t|, t \ge 0)$, provided $|\Lambda_t|$ exists for all $t \ge 0$ simultaneously. In this section, we show that the associated Δ_k-valued process of any exchangeable Lipschitz partition process $\mathbf{\Lambda}$ exists almost surely and is a Feller process.

Let μ be the directing measure of an exchangeable Lipschitz partition process $\mathbf{\Lambda}$ and let $\mathbf{M} := \{(t, M_t)\}$ be a Poisson point process with intensity $dt \otimes \mu$. For $M \in \mathcal{M}_{\mathbb{N}:k}$, we define the asymptotic frequency of any $M \in \mathcal{M}_{\mathbb{N}:k}$ as the (column) stochastic matrix $S := |M|_k$ with (i, j)-entry $S_{ij} := |M_{ij}|$, provided $|M_{ij}|$ exists for all $i, j = 1, \ldots, k$. We have the following lemmas.

Lemma 4.3. *For every atom time* $t > 0$ *of* $\mathbf{M}$, $|M_t|_k$ *exists almost surely.*

Proof. This is a consequence of Theorem 4.1, by which, for any atom time $t > 0$ of $\mathbf{M}$, each column of M_t is an exchangeable k-partition. By de Finetti's theorem, the asymptotic frequency of each column of M_t exists almost surely. Since $k < \infty$, $|M_t|_k$ exists a.s. $\qquad \square$

For each atom time t of $\mathbf{M}$, we write $S_t := |M_t|_k$. We also augment the map $|\cdot|_k$ on $\mathcal{M}_{\mathbb{N}:k}$ by including the cemetery state ∂ in the codomain of $|\cdot|_k$ and defining $|M|_k = \partial$ if $|M|_k$ does not exist. This makes $|\cdot|_k : \mathcal{M}_{\mathbb{N}:k} \to \mathcal{S}_k \cup \{\partial\}$ a measurable map, where $\mathcal{S}_k$ is the space of $k \times k$ column stochastic matrices, that is, $S = (S_{ij}, 1 \le i, j \le k) \in \mathcal{S}_k$ satisfies $S_{ij} \ge 0$ and $S_{1j} + \cdots + S_{kj} = 1$ for all $j = 1, \ldots, k$.

Lemma 4.4. *The image* $\mathbf{S} := \{(t, S_t)\} \subseteq \mathbb{R}^+ \times \mathcal{S}_k$ *of* $\mathbf{M} := \{(t, M_t)\} \subset \mathbb{R}^+ \times \mathcal{M}_{\mathbb{N}:k}$ *by* $|\cdot|_k$, *that is,* $S_t := |M_t|_k$ *for all atom times* $t > 0$ *of* $\mathbf{M}$, *is almost surely a Poisson point process with intensity* $dt \otimes |\mu|_k$, *where* $|\mu|_k$ *denotes the image measure of* μ *by* $|\cdot|_k$.

Proof. Let $J \subset [0, \infty)$ denote the subset of atom times of $\mathbf{M}$. By condition (3.3), J is at most countable almost surely. By Lemma 4.3, $|M_t|_k$ exists μ-almost everywhere for every $t \in J$. Therefore,

$$\mu\left(\bigcup_{t \in J} \{|M_t|_k = \partial\}\right) \leq \sum_{t \in J} \mu\left(\{|M_t|_k = \partial\}\right) = 0,$$

and $\mathbf{S}$ is almost surely a subset of $\mathbb{R}^+ \times \mathcal{S}_k$. That $\mathbf{S}$ is a Poisson point process with the appropriate intensity is clear as it is the image of the Poisson point process $\mathbf{M}$ by the measurable function $|\cdot|_k$. $\qquad\square$

Theorem 4.2. *The associated* Δ_k-*valued process* $|\mathbf{\Lambda}| := (|\Lambda_t|, t \geq 0)$ *of an exchangeable Lipschitz partition process* $\mathbf{\Lambda}$ *exists almost surely and is a Feller process on* Δ_k.

Proof. By exchangeability of $\mathbf{\Lambda}$, the asymptotic frequency $|\Lambda_t|$ exists for all fixed times $t > 0$, with probability one. In order for $|\mathbf{\Lambda}|$ to exist on Δ_k, we must show that, with probability one, $|\Lambda_t|$ exists for all $t > 0$ simultaneously. Let $\mathbf{M}$ be a Poisson point process that determines the jumps of $\mathbf{\Lambda}$ by (3.4). For each $n \in \mathbb{N}$, let $\mathcal{D}_n$ be the dyadic rationals in $[0, 1]$. Then $|\Lambda_t|$ exists almost surely on $\bigcup_{n \in \mathbb{N}} \mathcal{D}_n$, which is dense in $[0, 1]$. Existence of $|\mathbf{\Lambda}|$ now follows by density, càdlàg paths of $\mathbf{\Lambda}$, the Poisson point process construction of $\mathbf{\Lambda}$ via $\mathbf{M}$, and Lemmas 4.3 and 4.4.

The Feller property follows from Corollary 3.1 and Lemma 4.4. As in the general case, in which ϕ^∞ is a Feller process on $\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$, we can construct a Feller process $\mathbf{Q} := (Q_t, t \geq 0)$ on $\mathcal{M}_{\mathbb{N}:k}$ such that $\mathbf{Q}^\sigma := (Q_t^\sigma, t \geq 0) =_{\mathcal{L}} \mathbf{Q}$, for all $\sigma : \mathbb{N} \to \mathbb{N}$ fixing all but finitely many $n \in \mathbb{N}$. By Corollary 3.1, the semigroup of $\mathbf{\Lambda}$ satisfies $\mathbf{P}_t g(\lambda) := \mathbb{E}_\lambda g(Q_t \lambda)$. Furthermore, by Lemma 4.4 and the argument to show that $|\mathbf{\Lambda}|$ exists, the projection $|\mathbf{Q}| := (|Q_t|_k, t \geq 0)$ into $\mathcal{S}_k$ exists almost surely and $|\mathbf{\Lambda}|$ satisfies $\Lambda_t := |Q_t|_k |\Lambda_0|$ for all $t > 0$. The Feller property is a consequence of Lipschitz continuity of the linear map $S : \Delta_k \to \Delta_k$ determined by any $S \in \mathcal{S}_k$. $\qquad\square$

Remark 4.3. A detailed proof of Theorem 4.2 is technical and provides no new insights. Essentially, existence of $|\mathbf{\Lambda}|$ is a consequence of regularity of the paths of $\mathbf{\Lambda}$ and density of the countable set of dyadic rationals. The Feller property follows by Lipschitz continuity of maps determined by stochastic matrices. For a blueprint of the proof, we point the reader to [12].

4.5. A non-exchangeable Lipschitz process

The processes in the above examples are exchangeable Lipschitz partition processes. We now show an example of a Lipschitz partition process that is not exchangeable, and whose directing measure is not confined to the subspace $\mathcal{M}_{\mathbb{N}:k}$.

Let $\mathbf{A} := (M_j^i, i \in [k], j \geq 0)$ be an array of elements in $\mathcal{L}_{\mathbb{N}:k}$. Given $\mathbf{A}$, we define $F_{\mathbf{A}} = F \in \Phi$ by $F(\lambda) = \mathbf{A}_\lambda \lambda$, where $\mathbf{A}_\lambda \in \mathcal{M}_{\mathbb{N}:k}$ is defined as follows. For every $i \in [k]$, we put

$$m_i := \begin{cases} \min\{n \in \mathbb{N} : n \in \lambda_i\}, & \lambda_i \neq \varnothing, \\ 0, & \text{otherwise}. \end{cases} \tag{4.12}$$

For each $i = 1, \ldots, k$, we put $\mathbf{A}_\lambda^i = M_{m_i}^i$ and let $\mathbf{A}_\lambda := (\mathbf{A}_\lambda^1, \ldots, \mathbf{A}_\lambda^k) \in \mathcal{M}_{\mathbb{N}:k}$. It should be clear that, as specified, F need not be *strongly* Lipschitz.

Proposition 4.3. *The map $F_{\mathbf{A}}$ defined above is Lipschitz continuous.*

Proof. Take any $\lambda, \lambda' \in \mathcal{L}_{\mathbb{N}:k}$ and let $r = -\log_2 d(\lambda, \lambda') \in \mathbb{N} \cup \{\infty\}$. Let $0 < m_{(1)} < m_{(2)} < \cdots < m_{(k')} \leq r$ and $0 < m'_{(1)} < m'_{(2)} < \cdots < m'_{(k'')} \leq r$ be the minima (4.12) of λ and λ' (respectively) that are greater than zero but not greater than r. Since $I_r^k \lambda = I_r^k \lambda'$ by definition (2.7), we must have $k'' = k'$ and $m_{(i)} = m'_{(i)}$ for all $1 \leq i \leq k'$. It follows that $\mathbf{A}_\lambda I_r^k = \mathbf{A}_{\lambda'} I_r^k$ and

$$F_{\mathbf{A}}(\lambda)_{|[r]} = \left(I_r^k \mathbf{A}_\lambda\right)\lambda = \left(\mathbf{A}_\lambda I_r^k\right)\lambda = \mathbf{A}_\lambda\left(I_r^k \lambda\right) = \mathbf{A}_\lambda\left(I_r^k \lambda'\right) = \left(\mathbf{A}_\lambda I_r^k\right)\lambda' = \left(\mathbf{A}_{\lambda'} I_r^k\right)\lambda' = F_{\mathbf{A}}\left(\lambda'\right)_{|[r]}.$$

As this must hold for all $\lambda, \lambda' \in \mathcal{L}_{\mathbb{N}:k}$, it follows that $F_{\mathbf{A}}$ is Lipschitz continuous. $\qquad\square$

Now, we construct a measure on $\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$ using the above observation. In particular, for every $j \geq 0$, let ν_j be a measure on Δ_k such that

$$\zeta_{\nu_j}^{(n)}\left(\mathcal{L}_{[n]:k} \setminus \left\{E_i^{(n)}\right\}\right) < \infty \qquad \text{for all } i = 1, \ldots, k, \text{ for all } n \in \mathbb{N}, \tag{4.13}$$

where $E_i^{(n)} \in \mathcal{L}_{[n]:k}$ is the k-partition of $[n]$ with all elements labeled i, as in the proof of Proposition 4.2. We define the measure μ on $k \times \infty$ arrays of independent k-partitions (as $\mathbf{A}$ above) for which the partition in the ith row and jth column has distribution ζ_{ν_j}. We then define φ_μ as the measure on $\mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$ induced by the random array $\mathbf{A}$ with distribution μ and the map $F \in \mathrm{Lip}(\mathcal{L}_{\mathbb{N}:k})$ associated to $\mathbf{A}$ by the above discussion. We let $\mathbf{F}$ be a Poisson point process with intensity $\mathrm{d}t \otimes \varphi_\mu$ and construct $\mathbf{\Lambda}$ on $\mathcal{L}_{\mathbb{N}:k}$ as in (3.4).

In the following proposition, let $\mathbf{F} := \{(t, F^t)\}$ be a Poisson point process with intensity $\mathrm{d}t \otimes \varphi_\mu$, which is determined by a Poisson point process $\mathbf{A} := \{(t, A_t)\}$ with intensity $\mathrm{d}t \otimes \mu$, where each A_t is a random $k \times \infty$ array. In particular, for each atom time $t > 0$ of $\mathbf{F}$, we put $F^t := F_{A_t}$, as defined above.

Proposition 4.4. $\mathbf{\Lambda}$ *constructed from* $\mathbf{F}$ *is a Feller process on* $\mathcal{L}_{\mathbb{N}:k}$. *If, in addition,* $\nu_i \neq \nu_j$ *for some* $1 \leq i < j < \infty$, *then* $\mathbf{\Lambda}$ *is not exchangeable.*

Proof. For every $n \in \mathbb{N}$ and atom time $t > 0$ of $\mathbf{A}$, the restriction $\mathbf{\Lambda}_{t|[n]}$ depends only on the first $n + 1$ columns of any A_t. By assumption (4.13) on the underlying directing measures ζ_{ν_j}, φ_μ satisfies (3.3). Theorem 3.1 and Proposition 4.3 now imply that $\mathbf{\Lambda}$ is a Feller process.

Non-exchangeability of $\mathbf{\Lambda}$ under the stated condition is clear: since $\nu_i \neq \nu_j$ implies $\zeta_{\nu_i}^{(n)} \neq \zeta_{\nu_j}^{(n)}$ for all $n \in \mathbb{N}$, then the jump rates from a state with $m_{i'} = i$ and $m_{j'} = j$ differ from the jump rates from a state with $m_{i'} = j$ and $m_{j'} = i$, for any $1 \leq i' \neq j' \leq k$. $\qquad\square$

5. Discrete-time processes

From our previous discussion of continuous-time processes, we need not prove anything further for discrete-time chains; but we make some observations specific to the discrete-time case. Throughout this section, all measures on $\text{Lip}(\mathcal{L}_{\mathbb{N}:k})$ and/or $\mathcal{M}_{\mathbb{N}:k}$ are *probability measures*.

First, given a probability measure φ on $\text{Lip}(\mathcal{L}_{\mathbb{N}:k})$, we construct a Markov chain $\boldsymbol{\Lambda} := (\Lambda_m, m \geq 0)$ with initial state $\Lambda_0 \in \mathcal{L}_{\mathbb{N}:k}$ by taking $F_1, F_2, \ldots$ i.i.d. with law φ and defining

$$\Lambda_m = F_m(\Lambda_{m-1}) = (F_m \circ F_{m-1} \circ \cdots \circ F_1)(\Lambda_0), \qquad \text{for each } m \geq 1. \tag{5.1}$$

Constructed this way, $\boldsymbol{\Lambda}$ is a Markov chain on $\mathcal{L}_{\mathbb{N}:k}$. Furthermore, by Lipschitz continuity of the maps $F_1, F_2, \ldots$, the finite restrictions $(\boldsymbol{\Lambda}_{|[n]}, n \in \mathbb{N})$ are finite state space Markov chains. The following corollary follows from arguments in the continuous-time case.

Corollary 5.1. *Let $\boldsymbol{\Lambda}$ constructed in (5.1) be exchangeable. Then we have the following.*

- *φ is supported on $\mathcal{F} \cap \Sigma$ and we can assume, without loss of generality, that φ is a probability measure on $\mathcal{M}_{\mathbb{N}:k}$.*
- *The Δ_k-valued Markov chain $|\boldsymbol{\Lambda}| = (|\Lambda_m|, m \geq 0)$ exists almost surely and can be constructed as in (5.1) from $S_1, S_2, \ldots$ i.i.d. $|\varphi|_k$, the measure induced by φ on $\mathcal{S}_k$ through the map $|\cdot|_k$. In particular, $|\boldsymbol{\Lambda}| =_{\mathcal{L}} \mathbf{D} := (D_m, m \geq 0)$, where*

$$D_m := S_m \cdots S_1 D_0, \qquad m \geq 1,$$

for $D_0 := |\Lambda_0|$ and $S_1, S_2, \ldots$ i.i.d. $|\varphi|_k$.

6. Concluding remarks

To conclude, we remark about the projection of Lipschitz partition processes into $\mathcal{P}_{\mathbb{N}:k}$ and discuss more general aspects of partition-valued processes.

6.1. Associated Lipschitz partition processes on $\mathcal{P}_{\mathbb{N}:k}$

Let φ be the directing measure of a Lipschitz partition process $\boldsymbol{\Lambda}$ on $\mathcal{L}_{\mathbb{N}:k}$. Intuitively, the projection $\mathcal{B}_\infty(\boldsymbol{\Lambda}) := (\mathcal{B}_\infty(\Lambda_t), t \geq 0)$ into $\mathcal{P}_{\mathbb{N}:k}$ is, itself, a Markov process as long as φ treats the classes of every $\lambda \in \mathcal{L}_{\mathbb{N}:k}$ "symmetrically." In particular, for any permutation $\gamma : [k] \to [k]$, let us define $\Gamma \in \mathcal{M}_{\mathbb{N}:k}$ as the $k \times k$ partition operator with entries

$$\Gamma_{ij} = \begin{cases} \mathbb{N}, & \gamma(i) = j, \\ \varnothing, & \text{otherwise.} \end{cases}$$

The matrix Γ acts on $\mathcal{L}_{\mathbb{N}:k}$ by relabeling classes; that is, for any $\lambda := (\lambda_i, 1 \leq i \leq k) \in \mathcal{L}_{\mathbb{N}:k}$, $\Gamma\lambda := (\lambda_{\gamma(i)}, 1 \leq i \leq k)$. Therefore, the projection $\boldsymbol{\Pi} := \mathcal{B}_\infty(\boldsymbol{\Lambda})$ into $\mathcal{P}_{\mathbb{N}:k}$ is a Markov process if and only if, for every $\lambda \in \mathcal{L}_{\mathbb{N}:k}$ and every measurable subset $C \subseteq \mathcal{P}_{\mathbb{N}:k}$, φ assigns equal measure to

the events $\{F \in \Phi : F(\lambda) \in \mathcal{B}_{\infty}^{-1}(C)\}$ and $\{F \in \Phi : F(\Gamma\lambda) \in \mathcal{B}_{\infty}^{-1}(C)\}$, for all $\gamma \in \mathscr{S}_k$. Moreover, if $\mathbf{\Pi}$ is a Markov process, then it fulfills the Feller property.

By the preceding discussion, we can generate a Lipschitz partition process on $\mathcal{P}_{\mathbb{N}:k}$ by projecting a process $\mathbf{\Lambda}$ that treats labels symmetrically. The projection $\mathcal{B}_{\infty}(\mathbf{\Lambda})$ is a Feller process; and, if $\mathbf{\Lambda}$ is exchangeable, then so is $\mathcal{B}_{\infty}(\mathbf{\Lambda})$.

6.2. Existence and related notions

Sections 4.3 and 4.5 contain explicit examples of exchangeable and non-exchangeable Lipschitz partition processes. These examples confirm that Lipschitz partition processes exist, and their Poisson point process construction lends insight into their behavior. The Poisson point process construction is also useful in simulation and Markov chain Monte Carlo sampling.

There remain broader questions surrounding existence of measures satisfying (4.7), as well as more general partition-valued Markov processes. We undertake some of these questions elsewhere: we characterize exchangeable Feller processes on $\mathcal{P}_{\mathbb{N}:k}$ in [10]; we show the cutoff phenomenon for a class of these chains in [11]; and we study exchangeable processes without the Feller property in [12].

Acknowledgements

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