

Of copulas, quantiles, ranks and spectra: An L_1 -approach to spectral analysis

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In this paper, we present an alternative method for the spectral analysis of a univariate, strictly stationary time series $\{Y_t\}_{t \in \mathbb{Z}}$. We define a “new” spectrum as the Fourier transform of the differences between copulas of the pairs (Y_t, Y_{t-k}) and the independence copula. This object is called a *copula spectral density kernel* and allows to separate the marginal and serial aspects of a time series. We show that this spectrum is closely related to the concept of quantile regression. Like quantile regression, which provides much more information about conditional distributions than classical location-scale regression models, copula spectral density kernels are more informative than traditional spectral densities obtained from classical autocovariances. In particular, copula spectral density kernels, in their population versions, provide (asymptotically provide, in their sample versions) a complete description of the copulas of all pairs (Y_t, Y_{t-k}) . Moreover, they inherit the robustness properties of classical quantile regression, and do not require any distributional assumptions such as the existence of finite moments. In order to estimate the copula spectral density kernel, we introduce rank-based Laplace periodograms which are calculated as bilinear forms of weighted L_1 -projections of the ranks of the observed time series onto a harmonic regression model. We establish the asymptotic distribution of those periodograms, and the consistency of adequately smoothed versions. The finite-sample properties of the new methodology, and its potential for applications are briefly investigated by simulations and a short empirical example.

Keywords: copulas; periodogram; quantile regression; ranks; spectral analysis; time reversibility; time series

1. Introduction

1.1. The location-scale paradigm

Whether linear or not, most traditional time series models are of the conditional location/scale type: conditionally on past values Y_{t-1}, Y_{t-2}, \dots , the random variable Y_t is of the form

$$Y_t = \psi(Y_{t-1}, Y_{t-2}, \dots) + \sigma(Y_{t-1}, Y_{t-2}, \dots)\varepsilon_t, \quad t \in \mathbb{Z}, \quad (1.1)$$

where $\{\varepsilon_t\}_{t \in \mathbb{Z}}$ is white noise (either strong or weak, depending on the authors – here, by white noise we throughout mean strong, i.e., independent white noise), and ε_t is independent of (in the case of weak white noise, orthogonal to) Y_{t-1}, Y_{t-2}, \dots . The $(Y_{t-1}, Y_{t-2}, \dots)$ -measurable

functions ψ and σ are (conditional) location and scale functions, possibly parametrized by some ϑ . Equation (1.1) may characterize a data-generating process – in which case “=” in (1.1) is to be considered as “almost sure equality” – or, more generally, it simply describes Y_t 's conditional (on Y_{t-1}, Y_{t-2}, \dots) distribution – and “=” is to be interpreted as “equality in (conditional) distribution”. Such distinction is, however, irrelevant from a statistical point of view, as it has no impact on likelihoods.

In model (1.1), the distribution of Y_t conditional on Y_{t-1}, Y_{t-2}, \dots is nothing but the distribution of ε_t , rescaled by the conditional scale parameter $\sigma(Y_{t-1}, Y_{t-2}, \dots)$ and shifted by the conditional location parameter $\psi(Y_{t-1}, Y_{t-2}, \dots)$. Sophisticated as they may be, the mappings

$$(Y_{t-1}, Y_{t-2}, \dots) \mapsto (\psi(Y_{t-1}, Y_{t-2}, \dots), \sigma(Y_{t-1}, Y_{t-2}, \dots))$$

only can account for a very limited type of dynamics for the process $\{Y_t\}_{t \in \mathbb{Z}}$. The volatility dynamics for such models, for instance, are quite poor, being of a pure rescaling nature. In particular, no impact of past values on skewness, kurtosis, tails, can be reflected. All standardized conditional distributions strictly coincide with that of ε , and all conditional τ -quantiles, hence all values at risk, follow, irrespectively of τ , from those of ε via one single linear transformation.

Note that the interpretation of ψ and σ depends on the identification constraints on ε : if ε is assumed to have mean zero and variance one, then ψ and σ are a conditional mean and a conditional standard error, respectively. In this case, models of the form (1.1) clearly belong to the L_2 -Gaussian legacy. If ε is assumed to have median zero and expected absolute deviation or median absolute deviation one, ψ and σ are a conditional median and a conditional expected or median absolute deviation.

On the basis of these “remarks”, the following questions naturally arise: Can we do better? Can we go beyond that (conditional) “location-scale paradigm”? Can we model richer dynamics under which the conditional quantiles of Y are not just a shifted and rescaled version of those of ε , and under which the whole conditional distribution of Y_t , not just its location and scale, can be affected by the past? And, can we achieve this in a statistically tractable way?

1.2. Marginal and serial features

Another drawback of models of the form (1.1) is their sensitivity to nonlinear marginal transformations. If two statisticians observe the same time series, but measure it on different scales, Y_t and Y_t^3 or e^{Y_t} , for instance, and both adjust a model of the form (1.1) to their measurements, they will end up with drastically different analyses and predictions. The only way to avoid this problem consists in disentangling the marginal (viz., related to the scale of measurement) aspects of the series under study from its serial aspects, that is, basing the description of serial dependence features on quantities such as the $F(Y_t)$'s, where F is Y_t 's marginal distribution function. Those quantities do not depend on the measurement scale since they are invariant under continuous strictly increasing transformations.

This point of view is closely related to the concept of copulas (see Nelsen [35] or Genest and Favre [14]). Consider, for instance, a strictly stationary Markovian process $\{Y_t\}_{t \in \mathbb{Z}}$ of order one. This process is fully characterized by the joint distribution of (Y_t, Y_{t-1}) or, equivalently, by the marginal distribution function F (equivalently, the quantile function F^{-1}) of Y_t , along with the

joint distribution of $(U_t, U_{t-1}) := (F(Y_t), F(Y_{t-1}))$, a “serial copula of order one”. In such a description, the marginal features of the process $\{Y_t\}_{t \in \mathbb{Z}}$ are entirely described by F , independently of the serial features, that are accounted for by the serial copula. Two statisticians observing the same phenomenon but recording Y_t and e^{Y_t} , respectively, would use distinct quantile functions, but they would agree on serial features.

In more general cases, serial copulas of order one are not sufficient, and higher-order or multiple copulas may be needed. Note that the description of the model in this context is clearly “in distribution”: U_t is not related to U_{t-1} through any direct interpretable “almost sure relation” reflecting some “physical” data-generating mechanism.

1.3. A new nonparametric approach

The objective of this paper is to show how to overcome the limitations of conditional location-scale modelling described in Sections 1.1 and 1.2, and to provide statistical tools for a fully general approach to time series modelling. Not surprisingly, those tools are essentially related to copulas, quantiles and ranks. The traditional nonparametric techniques, such as spectral analysis (in its usual L_2 -form), which only account for second-order serial features, cannot handle such objects, and we therefore propose and develop an original, flexible and fully nonparametric L_1 -spectral analysis method.

While classical spectral densities are obtained as Fourier transforms of classical covariance functions, we rather define spectral density *kernels*, associated with covariance *kernels* of the form (for $(\tau_1, \tau_2) \in (0, 1)^2$)

$$\gamma_k(x_1, x_2) := \text{Cov}(I\{Y_t \leq x_1\}, I\{Y_{t-k} \leq x_2\}) \tag{1.2}$$

(Laplace cross-covariance kernels) or

$$\gamma_k^U(\tau_1, \tau_2) := \text{Cov}(I\{U_t \leq \tau_1\}, I\{U_{t-k} \leq \tau_2\}) \tag{1.3}$$

(copula cross-covariance kernels), where $U_t := F(Y_t)$ and F denotes the marginal distribution of the strictly stationary process $\{Y_t\}_{t \in \mathbb{Z}}$ and $I\{A\}$ stands for the indicator function of A . Contrary to covariance functions, the *kernels* $\{\gamma_k(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$ and $\{\gamma_k^U(\tau_1, \tau_2) | \tau_1, \tau_2 \in (0, 1)\}$ allow for a complete description of arbitrary bivariate distributions for the couples (Y_t, Y_{t-k}) and the corresponding copulas, respectively, and thus escape the conditional location-scale paradigm discussed in Section 1.1. They are able to account for sophisticated dependence features that covariance-based methods are unable to detect, such as time-irreversibility, tail dependence, varying conditional skewness or kurtosis, etc. And, in view of the desired separation between marginal and serial features expressed in Section 1.2, special virtues, such as invariance/equivariance (with respect to continuous order-preserving marginal transformations), can be expected from the copula covariance kernels defined in (1.3).

Classical nonparametric spectral-based inference methods have proven quite effective (see, e.g., Granger [16], Bloomfield [4]), essentially in a Gaussian context, where dependencies are fully characterized by autocovariance functions. Therefore, it can be anticipated that similar methods, based on estimated versions of Laplace or copula spectral kernels (associated with

Laplace and copula covariance kernels, respectively) would be quite useful in the study of series exhibiting those features that classical covariance-related spectra cannot account for.

Estimation of Laplace and copula spectral kernels, however, calls for a substitute to the *ordinary periodogram* concept considered in the classical approach. We therefore introduce Laplace and copula *periodogram kernels*. While ordinary periodograms are defined via least squares regression of the observations on the sines and cosines of the harmonic basis, our periodogram kernels are obtained via quantile regression in the Koenker and Bassett [27] sense. A study of their asymptotic properties shows that, just as ordinary periodograms, they produce asymptotically unbiased estimates (more precisely, the mean of their asymptotic distribution is 2π times the corresponding spectrum), and we therefore also consider smoothed versions that yield consistency. Asymptotic results show that copula periodograms, as anticipated, are preferable to the Laplace ones, as their asymptotic behavior only depends on the bivariate copulas of the pairs (U_t, U_{t-k}) , not on the (in general unknown) marginal distribution F of the Y_t 's.

Unfortunately, copula periodogram kernels are not statistics, since their definition involves the transformation of Y_t into U_t , hence the knowledge of the marginal distribution function F . We therefore introduce a third periodogram kernel, based on the empirical version \hat{F}_n of F , that is, on the *ranks* of the random variables Y_1, \dots, Y_n , and establish, under mild assumptions, the asymptotic equivalence of that rank-based Laplace periodogram with the copula one. Smoothed rank-based Laplace periodogram kernels, accordingly, seem to be the adequate tools in this context. We conclude with a brief numerical illustration – simulations and an empirical application – of their potential use in practical problems.

1.4. Review of related literature

Quantities of the form (1.2) and (1.3) naturally come into the picture when the *clipped processes* $(I\{Y_t \leq x\})_{t \in \mathbb{Z}}$ and $(I\{U_t \leq \tau\})_{t \in \mathbb{Z}}$ are investigated. Such clipped processes have been considered earlier in the literature (see, for instance, Kedem [24]). In the field of signal processing, the idea to replace the quadratic loss by other loss functions has been discussed by Katkovnik [23], who proposes using L_p -distances and analyzes the properties of the resulting *M-periodograms*. Hong [21] used the Laplace covariances corresponding to positive lags to construct a test for serial dependence. Linton and Whang [33] considered sequences of Laplace autocorrelations $\gamma_k(\tau, \tau)/\gamma_0(\tau, \tau)$ (called *quantilogram* by these authors) in order to test for directional predictability. Mikosch and Zhao [34] define a periodogram generated from a suitable sequence of indicator functions of rare events.

In a pioneering paper, Li [30] suggested least absolute deviation estimators in a harmonic regression model assuming that the median of the random variables Y_t is zero. The focus of this author is on the quantities of the form (for $\omega \in (0, \pi)$; throughout, i stands for the root of -1)

$$f_{0,0}(\omega) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \gamma_k(0, 0) \exp(ik\omega), \quad \omega \in (0, \pi),$$

the collection of which he calls the *Laplace spectrum*. He constructs an asymptotically unbiased estimator for a quantity which differs from $f_{0,0}(\omega_j)$ (ω_j the j th Fourier frequency) by a factor involving $1/(F'(0))^2$ and, in Li [31], extends his results to arbitrary quantiles. An important

drawback of Li's method is that it requires estimates of the quantity $F'(0)$ in order to obtain an estimate of the Laplace spectrum; moreover, the consistency of a smoothed version of his estimates is not established. More recently, Hagemann [17] proposed an alternative method to estimate the Laplace spectrum (called *quantile spectrum* by this author), which is based on the estimation of a linearization of Li [30]'s statistic. This approach does not suffer from the drawbacks of Li's method, and yields consistent estimates avoiding estimation of the marginal density; on the other hand, it does not allow a direct interpretation in terms of (weighted) absolute deviation estimates.

In order to obtain a complete description of the two-dimensional distributions at lag k , Hong [20] introduced a *generalized spectrum* defined as the covariance $\text{Cov}(e^{ix_1 Y_t}, e^{ix_2 Y_{t+k}})$; this concept was used by Chung and Hong [10] to test for directional predictability. Recently, Lee and Rao [29] considered a Fourier transform of the differences between the joint density of the pairs (Y_t, Y_{t-k}) and the product of their marginal densities to investigate serial dependence. Unlike ours, these methods are not invariant with respect to transformations of the marginal distributions.

Finally, there exist some recent proposals using pair-copula constructions to describe dependencies in the time-domain. Domma, Giordano and Perri [11] assume first-order Markov dependence, so that only distributions of pairs (Y_t, Y_{t+1}) at lag $k = 1$ need to be considered. Smith *et al.* [39] decompose the distribution at a point in time, conditional upon the past, into the product of a sequence of bivariate copula densities and the marginal density, known as D-vine (Bedford and Cooke [2]).

The approach presented in this paper differs from these references in many important aspects. Essentially, it combines their attractive features while avoiding some of their drawbacks. It shares the quantile-based flavor of Kedem [24], Linton and Whang [33], Li [30,31] and Hagemann [17]. In contrast to the latter, however, we do not focus on a particular quantile, and consider copula cross-covariances $\gamma_k^U(\tau_1, \tau_2)$ for *all* pairs (τ_1, τ_2) , while Li [30,31] and Hagemann [17] restrict to the case $\tau_1 = \tau_2$. As a consequence, we obtain, as in the characteristic function approach of Hong [20], a complete characterization of the dependencies among the pairs (Y_t, Y_{t-k}) . This allows to address such important features as time reversibility (see Proposition 2.1) or tail dependence in general. By replacing the original observations with their ranks, we furthermore achieve an attractive invariance property with respect to modifications of marginal distributions, which is not satisfied in the case of Hong [20]'s method. Moreover, we also avoid the scaling problem of Li's estimates and establish the consistency of a smoothed version of periodograms. Finally, because our method is related to the concept of copulas, it allows to separate the marginal and serial aspects of a time series, which should make it attractive for practitioners.

1.5. Outline of the paper

The paper is organized as follows. In Section 2.1, we introduce the concepts of *Laplace* and *copula cross-covariance kernels* which, in this quantile-based approach, are to replace the ordinary autocovariance function. The corresponding spectra and periodograms are introduced in Sections 2.2 and 2.3, respectively. Section 3 deals with the asymptotic properties of the Laplace, copula, and rank-based Laplace periodograms. In Section 4, smoothed periodograms are considered, and the smoothed rank-based Laplace periodogram kernel is shown to be a consistent

estimator of the copula spectral density. Some numerical illustration is provided in Section 5, and most of the technical details are concentrated in an appendix.

2. An L_1 -approach to spectral analysis

2.1. The Laplace and copula cross-covariance kernels

Covariances clearly are not sufficient for describing a serial copula. We therefore introduce the following concept, which will be convenient for that purpose. Let $\{Y_t\}_{t \in \mathbb{Z}}$ be a strictly stationary process and define its *copula cross-covariance kernel* of lag $k \in \mathbb{Z}$ of $\{Y_t\}_{t \in \mathbb{Z}}$ as

$$\gamma_k^U := \{\gamma_k^U(\tau_1, \tau_2) | (\tau_1, \tau_2) \in (0, 1)^2\},$$

where $\gamma_k^U(\tau_1, \tau_2)$ is defined in (1.3). Similarly, define the *Laplace cross-covariance kernel* of lag $k \in \mathbb{Z}$ of $\{Y_t\}_{t \in \mathbb{Z}}$ as

$$\gamma_k := \{\gamma_k(x_1, x_2) | (x_1, x_2) \in \mathbb{R}^2\},$$

where $\gamma_k(x_1, x_2)$ is defined in (1.2). Contrary to traditional cross-covariances, copula and Laplace cross-covariance kernels exist for all k (no finite variance assumption needed). The words ‘‘covariance’’ and ‘‘cross-covariance’’ are used out of time series classical terminology; but we only consider covariances of indicators, which always exist, and provide a canonical description of their joint distributions. The copula cross-covariance kernel of order k indeed entirely characterizes the joint distribution of (U_t, U_{t-k}) , and conversely, without requiring any information on the distribution function F of Y_t . Along with F , the copula cross-covariance kernel of order k entirely characterizes the Laplace cross-covariance kernel of order k and the joint distribution of (Y_t, Y_{t-k}) , and conversely. If $\int x^2 dF < \infty$, the distribution function F of Y_t and the collection of copula cross-covariance kernels of all orders jointly characterize the autocovariance function of $\{Y_t\}_{t \in \mathbb{Z}}$.

2.2. The Laplace and copula spectral density kernels

Assume that the Laplace cross-covariance kernels γ_k (equivalently, the copula cross-covariance kernels γ_k^U), $k \in \mathbb{Z}$ are absolutely summable, that is, assume that they satisfy

$$\sum_{k=-\infty}^{\infty} |\gamma_k(x_1, x_2)| < \infty \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.$$

Then, γ_k admits the representation

$$\gamma_k(x_1, x_2) = \int_{-\pi}^{\pi} e^{ik\omega} f_{x_1, x_2}(\omega) d\omega, \quad (x_1, x_2) \in \mathbb{R}^2$$

with

$$f_{x_1, x_2}(\omega) := \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k(x_1, x_2) e^{-ik\omega}, \quad (x_1, x_2) \in \mathbb{R}^2. \tag{2.1}$$

The collection $\{\omega \mapsto \mathfrak{f}_{x_1, x_2}(\omega) | (x_1, x_2) \in \mathbb{R}^2\}$, call it the *Laplace spectral density kernel*, is such that each mapping $\omega \in (-\pi, \pi] \mapsto \mathfrak{f}_{x_1, x_2}(\omega), (x_1, x_2) \in \mathbb{R}^2$, is continuous and satisfies (writing \bar{z} for the complex conjugate of $z \in \mathbb{C}$)

$$\mathfrak{f}_{x_1, x_2}(-\omega) = \mathfrak{f}_{x_2, x_1}(\omega) = \overline{\mathfrak{f}_{x_1, x_2}(\omega)}. \tag{2.2}$$

Similarly define the *copula spectral density kernel* as

$$\mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k^U(\tau_1, \tau_2) e^{-ik\omega}, \quad (\tau_1, \tau_2) \in (0, 1)^2, \tag{2.3}$$

where $q_{\tau_i} := F^{-1}(\tau_i)$ ($i = 1, 2$). Note that $\mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}$ is the Fourier transform of the differences between copulas of the pairs (Y_t, Y_{t-k}) and the independence copula. Clearly, the same identity (2.2) holds for $\mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega)$ as for $\mathfrak{f}_{x_1, x_2}(\omega)$.

Throughout this paper, we denote by $\stackrel{d}{=}$ equality in distribution and define $\Im z$ and $\Re z$ as the imaginary and real part of $z \in \mathbb{C}$, respectively. Obviously, we have $\Im \mathfrak{f}_{x_1, x_2}(\omega) = 0$ for all ω if and only if $\gamma_k(x_1, x_2) = \gamma_{-k}(x_1, x_2)$ for all k , and we obtain the following result.

Proposition 2.1. *The following statements are equivalent:*

- (1) $(Y_t, Y_{t+k}) \stackrel{d}{=} (Y_t, Y_{t-k})$ for all $k \in \mathbb{Z}$ (pairwise time-reversibility);
- (2) $\Im \mathfrak{f}_{x_1, x_2}(\omega) = 0$ for all $\omega \in (0, \pi)$ and $(x_1, x_2) \in \mathbb{R}^2$;
- (3) $\Im \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) = 0$ for all $\omega \in (0, \pi)$ and $(\tau_1, \tau_2) \in (0, 1)^2$.

2.3. The Laplace, copula and rank-based Laplace periodogram kernels

The copula cross-covariance kernels describe the serial behavior of Y_t 's quantiles. If quantiles are to be considered, it seems intuitively reasonable that the traditional L_2 -tools, which are closely related with the concepts of mean and variance, be abandoned in favor of quantile-related ones. In particular, traditional L_2 -projections should be replaced with (weighted) L_1 -projections. Recall that, in traditional spectral analysis, estimation is usually based on the *ordinary periodogram*

$$I_n(\omega_{j,n}) := \frac{1}{n} \left| \sum_{t=1}^n Y_t e^{-it\omega_{j,n}} \right|^2,$$

where $\omega_{j,n} = 2\pi j/n \in \mathcal{F}_n := \{2\pi j/n | j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor - 1, \lfloor \frac{n-1}{2} \rfloor\}$ denote the positive *Fourier frequencies*. A straightforward calculation shows that this can be expressed as

$$I_n(\omega_{j,n}) = \frac{n}{4} \|\hat{\mathbf{b}}_{n, \text{OLS}}(\omega_{j,n})\|^2 := \frac{n}{4} \hat{\mathbf{b}}'_{n, \text{OLS}}(\omega_{j,n}) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \hat{\mathbf{b}}_{n, \text{OLS}}(\omega_{j,n}),$$

where $\|\cdot\|$ denotes the euclidian norm, and

$$(\hat{a}_{n, \text{OLS}}(\omega_{j,n}), \hat{\mathbf{b}}'_{n, \text{OLS}}(\omega_{j,n})) := \underset{(a, \mathbf{b}') \in \mathbb{R}^3}{\text{Argmin}} \sum_{t=1}^n (Y_t - (a, \mathbf{b}') \mathbf{c}_t(\omega_{j,n}))^2 \tag{2.4}$$

is the ordinary least squares estimator in the linear model with regressors $\mathbf{c}_t(\omega_{j,n}) := (1, \cos(t\omega_{j,n}), \sin(t\omega_{j,n}))'$, corresponding to an L_2 -projection of the observed series onto the harmonic basis.

If, instead of a representation of Y_t itself, we are interested in a representation, in terms of the harmonic basis, of Y_t 's quantile of order τ , it may seem natural to replace that ordinary periodogram $I_n(\omega_{j,n})$ with

$$\hat{L}_{n,\tau}(\omega_{j,n}) := \frac{n}{4} \|\hat{\mathbf{b}}_{n,\tau}(\omega_{j,n})\|^2 := \frac{n}{4} \hat{\mathbf{b}}'_{n,\tau}(\omega_{j,n}) \begin{pmatrix} 1 & \mathbf{i} \\ -\mathbf{i} & 1 \end{pmatrix} \hat{\mathbf{b}}_{n,\tau}(\omega_{j,n}),$$

where

$$(\hat{a}_{n,\tau}(\omega_{j,n}), \hat{\mathbf{b}}_{n,\tau}(\omega_{j,n})) := \underset{(a, \mathbf{b}') \in \mathbb{R}^3}{\text{Argmin}} \sum_{t=1}^n \rho_\tau(Y_t - (a, \mathbf{b}')\mathbf{c}_t(\omega_{j,n})), \tag{2.5}$$

and

$$\rho_\tau(x) := x(\tau - I\{x \leq 0\}) = (1 - \tau)|x|I\{x \leq 0\} + \tau|x|I\{x > 0\}, \quad \tau \in (0, 1),$$

is the so-called *check function* (see Koenker [26]). In definition (2.5), the L_2 -loss function, which yields the classical definition (2.4), is thus replaced by Koenker and Bassett's weighted L_1 -loss which produces quantile regression estimates – see Koenker and Bassett [27]. That this indeed is a sensible definition will follow from the asymptotic results of Section 3.

This L_1 -approach has been taken by Li [30] for the particular case $\tau = 1/2$, leading to a least absolute deviations (LAD) regression coefficient $\hat{\mathbf{b}}_{n,0.5}$ and later by Li [31] for arbitrary $\tau \in (0, 1)$. More generally, for a given series Y_1, \dots, Y_n , define the *Laplace periodogram kernel* as the collection

$$\hat{L}_{n,\tau_1,\tau_2}(\omega_{j,n}) := \frac{n}{4} \hat{\mathbf{b}}'_{n,\tau_1}(\omega_{j,n}) \begin{pmatrix} 1 & \mathbf{i} \\ -\mathbf{i} & 1 \end{pmatrix} \hat{\mathbf{b}}_{n,\tau_2}(\omega_{j,n}), \quad \omega_{j,n} \in \mathcal{F}_n, (\tau_1, \tau_2) \in (0, 1)^2. \tag{2.6}$$

For any $(\tau_1, \tau_2, \omega_{j,n})$, computation of $\hat{L}_{n,\tau_1,\tau_2}(\omega_{j,n})$ is immediate via the simplex algorithm (as in classical Koenker–Bassett quantile regression, see Koenker [26]).

Similarly, define the *copula periodogram kernel* as the Laplace periodogram kernel $\hat{L}_{n,\tau_1,\tau_2}^U(\omega_{j,n})$ associated with the series U_1, \dots, U_n . This means that $\hat{L}_{n,\tau_1,\tau_2}^U(\omega_{j,n})$ is obtained from (2.6) by replacing the estimate $\hat{\mathbf{b}}_{n,\tau}$ by the second and third components of the vector

$$(\hat{a}, (\hat{\mathbf{b}}^U)') := \underset{(a, \mathbf{b}') \in \mathbb{R}^3}{\text{Argmin}} \sum_{t=1}^n \rho_\tau(U_t - (a, \mathbf{b}')\mathbf{c}_t(\omega_{j,n})).$$

Finally, because the distribution function F required for the calculation of $U_t = F(Y_t)$ is not known, we introduce the *empirical* or *rank-based Laplace periodogram kernel* as the Laplace periodogram kernel $\hat{\tilde{L}}_{n,\tau_1,\tau_2}(\omega_{j,n})$ associated with the series $n^{-1}R_1^{(n)}, \dots, n^{-1}R_n^{(n)}$, where $R_t^{(n)}$ denotes the rank of Y_t among Y_1, \dots, Y_n . In other words, $\hat{\tilde{L}}_{n,\tau_1,\tau_2}(\omega_{j,n})$ is obtained from (2.6)

by replacing the estimate $\hat{\mathbf{b}}_{n,\tau}$ by the second and third components of the vector

$$(\hat{a}, \hat{\mathbf{b}}') := \underset{(a, \mathbf{b}') \in \mathbb{R}^3}{\text{Argmin}} \sum_{t=1}^n \rho_\tau(n^{-1} R_t^{(n)} - (a, \mathbf{b}') \mathbf{c}_t(\omega_{j,n})).$$

A few remarks about the notation used in this paper are in order. With \hat{T} , we usually denote a statistic obtained from the original series Y_1, \dots, Y_n , such as $\hat{L}_{n,\tau_1,\tau_2}$. The notation \hat{T}^U means that \hat{T} has been computed from the probability integral transform U_1, \dots, U_n of the data – a typical example is $\hat{L}_{n,\tau_1,\tau_2}^U$. Finally, the notation \hat{T}^\sim reflects the fact that \hat{T} has been computed from the normalized ranks $n^{-1} R_1^{(n)}, \dots, n^{-1} R_n^{(n)}$ (see, for instance, the rank-based Laplace periodogram kernel $\hat{L}_{n,\tau_1,\tau_2}^\sim$).

3. Asymptotic properties

3.1. Asymptotics of Laplace and copula periodogram kernels

We now proceed to deriving the asymptotic distributions of the Laplace and rank-based Laplace periodogram kernels, which, as we shall see, establishes their relation to the spectral density kernels defined in (2.1) and (2.3). Throughout the rest of the paper, we make the following basic assumptions.

Assumption (A1). *The process $\{Y_t\}_{t \in \mathbb{Z}}$ is strictly stationary and β -mixing, such that*

$$\beta(n) := \sup_{k \geq 1} \mathbb{E} \sup_{B \in \mathcal{F}_{n+k}^\infty} |\mathbb{P}(B | \mathcal{F}_{-\infty}^k) - \mathbb{P}(B)| = O(n^{-\delta}), \quad \delta > 1, \text{ as } n \rightarrow \infty,$$

where $\mathcal{F}_l^m := \sigma(Y_l, \dots, Y_m)$ denotes the σ -field generated by Y_l, \dots, Y_m .

The class of β -mixing processes is well studied, and contains a wide range of linear and non-linear processes, including (possibly, under mild additional assumptions) ARMA, general non-linear scalar ARCH, threshold ARCH, and exponential ARCH processes (see Liebscher [32]), GARCH(p, q) processes with moments (see Boussama [5]) and GARCH(1, 1) processes with no assumptions regarding the moments (see Francq and Zakoian [13]), generalized polynomial random coefficient vector autoregressive processes, and a family of generalized hidden Markov processes (Carrasco and Chen [9]) which includes stochastic volatility ones.

Assumption (A2). *The distribution function F of Y_t and the joint distribution functions F_k of (Y_t, Y_{t+k}) are twice continuously differentiable, with uniformly (with respect to their arguments, and also with respect to k) bounded derivatives. Moreover, there exists a subset T of $[0, 1]$ and, for every $\tau \in T$, a positive real d_τ , such that $\inf_{|x - q_\tau| \leq d_\tau} f(x) > 0$, where f and $q_\tau := F^{-1}(\tau)$ denote the density and τ -quantile corresponding to the distribution function F .*

Denote by $\hat{L}_{n,\tau_1,\tau_2}$ and $\hat{L}_{n,\tau_1,\tau_2}^U$, respectively, the Laplace and copula periodogram kernels associated with a realization of length n . For each $(\tau_1, \tau_2) \in (0, 1)^2$ and $\omega \in (0, \pi)$, write

$$\overset{\circ}{f}_{\tau_1,\tau_2}(\omega) := f_{q_{\tau_1},q_{\tau_2}}(\omega)/(f(q_{\tau_1})f(q_{\tau_2})) \tag{3.1}$$

for the *scaled* version of the spectral density kernel $f_{q_{\tau_1},q_{\tau_2}}(\omega)$ defined in (2.3). In the following two statements, $\xrightarrow{\mathcal{L}}$ stands for convergence in distribution, and χ_k^2 denotes a chi-square distribution with k degrees of freedom. We also introduce the piecewise constant function (defined on the interval $(0, \pi)$)

$$g_n(\omega) := \omega_{j,n}, \tag{3.2}$$

where $\omega_{j,n}$ is the Fourier frequency closest to ω – more precisely, $\omega_{j,n}$ is such that ω belongs to $(\omega_{j,n} - \frac{2\pi}{n}, \omega_{j,n} + \frac{2\pi}{n}]$. The following result is the key for understanding the asymptotic properties of the Laplace periodogram kernel.

Theorem 3.1. *Let $\Omega := \{\omega_1, \dots, \omega_v\} \subset (0, \pi)$ and $T := \{\tau_1, \dots, \tau_p\} \subset (0, 1)$ denote distinct frequencies and distinct quantile orders, respectively. Let Assumptions (A1) and (A2) be satisfied with (A2) holding for every $\tau \in T$. Then*

$$\sqrt{n}(\hat{\mathbf{b}}_{n,\tau}(g_n(\omega)))_{\tau \in T, \omega \in \Omega} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (N_{\tau}(\omega))_{\tau \in T, \omega \in \Omega},$$

where $(N_{\tau}(\omega))_{\tau \in T, \omega \in \Omega}$ denotes a Gaussian random vector with mean zero and covariance

$$\begin{aligned} M_{\tau_1,\tau_2}^{\omega_1,\omega_2} &:= \text{Cov}(N_{\tau_1}(\omega_1), N_{\tau_2}(\omega_2)) \\ &= \begin{cases} 4\pi \begin{pmatrix} \overset{\circ}{\mathfrak{H}}_{\tau_1,\tau_2}(\omega) & \overset{\circ}{\mathfrak{F}}_{\tau_1,\tau_2}(\omega) \\ -\overset{\circ}{\mathfrak{F}}_{\tau_1,\tau_2}(\omega) & \overset{\circ}{\mathfrak{H}}_{\tau_1,\tau_2}(\omega) \end{pmatrix} & \text{if } \omega_1 = \omega_2 =: \omega, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \omega_1 \neq \omega_2. \end{cases} \end{aligned} \tag{3.3}$$

Proof. The proof consists of two basic steps which we only sketch here. Details are provided in Appendix A.

Step 1. The first step consists of a linearization of the estimate $\hat{\mathbf{b}}_{n,\tau}(\omega_{j,n})$ defined in (2.5). To be precise, for any $\tau \in (0, 1)$, $\omega \in (0, \pi)$, and $\delta \in \mathbb{R}^3$, let

$$\hat{Z}_{n,\tau,\omega}(\delta) := \sum_{t=1}^n (\rho_{\tau}(Y_t - q_{\tau} - n^{-1/2}\mathbf{c}'_t(\omega)\delta) - \rho_{\tau}(Y_t - q_{\tau})), \tag{3.4}$$

where $\mathbf{c}_t(\omega) := (1, \cos(\omega t), \sin(\omega t))'$, and q_{τ} denotes the τ -quantile of F . Further define

$$Z_{n,\tau,\omega}(\delta) := -\delta' \zeta_{n,\tau,\omega} + \frac{1}{2} \delta' \mathbf{Q}_{n,\tau,\omega} \delta,$$

where

$$\zeta_{n,\tau,\omega} := n^{-1/2} \sum_{t=1}^n \mathbf{c}_t(\omega) (\tau - I\{Y_t \leq q_\tau\}), \quad (3.5)$$

and

$$\mathbf{Q}_{n,\tau,\omega} := f(q_\tau) n^{-1} \sum_{t=1}^n \mathbf{c}_t(\omega) \mathbf{c}_t'(\omega). \quad (3.6)$$

We first show that the minimizers

$$\hat{\delta}_{n,\tau,\omega} := \arg \min_{\delta} \hat{Z}_{n,\tau,\omega}(\delta) \quad \text{and} \quad \delta_{n,\tau,\omega} := \arg \min_{\delta} Z_{n,\tau,\omega}(\delta) = (\mathbf{Q}_{n,\tau,\omega})^{-1} \zeta_{n,\tau,\omega} \quad (3.7)$$

are close in probability (uniformly with respect to $\omega \in \mathcal{F}_n$). Note that, from the definition in (2.5), it follows that the random variable $\sqrt{n} \hat{\mathbf{b}}_{n,\tau}(\omega_{j,n})$ coincides with the second and third components of the vector $\hat{\delta}_{n,\tau,\omega}$. Moreover, for $\omega_{j,n} = 2\pi j/n$, we have

$$\mathbf{Q}_{n,\tau,\omega_{j,n}} = f(q_\tau) \text{diag}(1, 1/2, 1/2), \quad (3.8)$$

where $\text{diag}(a_1, \dots, a_k)$ denotes the diagonal matrix with diagonal elements a_1, \dots, a_k . More precisely, we establish the following bound

$$\sup_{\omega \in \mathcal{F}_n} \|\hat{\delta}_{n,\tau,\omega} - \delta_{n,\tau,\omega}\| = \text{O}_P(r_n(\delta)), \quad (3.9)$$

$$r_n(\delta) := (n^{-1/8} \log n) \vee (n^{(1/4)(1-\delta)/(1+\delta)} (\log n)^{3/2}).$$

This result is obtained from the following arguments, for which the details are provided in Section A.1. Roughly speaking, bounds of the type (3.9) can be obtained by showing that the corresponding functions $\hat{Z}_{n,\tau,\omega}$ and $Z_{n,\tau,\omega}$ are uniformly close in probability. A precise statement is given in Lemma A.1 (see Section A.1.2), where we show that (3.9) follows if the bound

$$\sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta - \delta_{n,\tau,\omega}\| \leq \epsilon} |\hat{Z}_{n,\tau,\omega}(\delta) - Z_{n,\tau,\omega}(\delta)| = \text{O}_P(r_n(\delta)^2) \quad (3.10)$$

can be established for some $\epsilon > 0$.

Note that

$$\begin{aligned} & \mathbb{P}\left(\sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta - \delta_{n,\tau,\omega}\| \leq \epsilon} |\hat{Z}_{n,\tau,\omega}(\delta) - Z_{n,\tau,\omega}(\delta)| > r_n(\delta)^2\right) \\ & \leq \mathbb{P}\left(\sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta\| \leq \epsilon + \|\delta_{n,\tau,\omega}\|} |\hat{Z}_{n,\tau,\omega}(\delta) - Z_{n,\tau,\omega}(\delta)| > r_n(\delta)^2, \sup_{\omega \in \mathcal{F}_n} \|\delta_{n,\tau,\omega}\| \leq A\sqrt{\log n}\right) \\ & \quad + \mathbb{P}\left(\sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta\| \leq \epsilon + \|\delta_{n,\tau,\omega}\|} |\hat{Z}_{n,\tau,\omega}(\delta) - Z_{n,\tau,\omega}(\delta)| > r_n(\delta)^2, \sup_{\omega \in \mathcal{F}_n} \|\delta_{n,\tau,\omega}\| > A\sqrt{\log n}\right) \\ & \leq \mathbb{P}\left(\sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta\| \leq \epsilon + A\sqrt{\log n}} |\hat{Z}_{n,\tau,\omega}(\delta) - Z_{n,\tau,\omega}(\delta)| > r_n(\delta)^2\right) + \mathbb{P}\left(\sup_{\omega \in \mathcal{F}_n} \|\delta_{n,\tau,\omega}\| > A\sqrt{\log n}\right). \end{aligned}$$

By application of Lemma A.2, it is therefore sufficient to show that, for an enlarged A ,

$$\sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta\| \leq A\sqrt{\log n}} |\hat{Z}_{n,\tau,\omega}(\delta) - Z_{n,\tau,\omega}(\delta)| = O_P(r_n(\delta)^2) \tag{3.11}$$

and (3.10), hence also, in view of Lemma A.1, (3.9) is proved. The proof of (3.11) is deferred to Section A.1.1.

Step 2. As we have discussed at the beginning of the first step, the asymptotic properties of $\sqrt{n}\hat{\mathbf{b}}_{n,\tau}(\omega_{j,n})$ can be obtained from those of the random variables $\delta_{n,\tau,\omega}$ for which an explicit expression is available. More precisely, for given sets $\Omega := \{\omega_1, \dots, \omega_\nu\} \subset (0, \pi)$ of frequencies and $T := \{\tau_1, \dots, \tau_p\} \subset (0, 1)$, consider the linear combination with coefficients $\lambda_{ik} \in \mathbb{R}^2$, $i = 1, \dots, \nu$, $k = 1, \dots, p$

$$\begin{aligned} & \sum_{k=1}^p \sum_{i=1}^\nu \lambda'_{ik} \sqrt{n} \hat{\mathbf{b}}_{n,\tau_k}(g_n(\omega_i)) \\ &= \sum_{k=1}^p \sum_{i=1}^\nu \lambda'_{ik} \sum_{t=1}^n \frac{2}{f(q_{\tau_k})} \frac{\mathbf{v}_{tn}(\omega_i)}{\sqrt{n}} (\tau_k - I\{Y_t \leq q_{\tau_k}\}) + o_P(1), \end{aligned} \tag{3.12}$$

where $\mathbf{v}_{tn}(\omega) := (\cos(g_n(\omega)t), \sin(g_n(\omega)t))'$. The first equality is a consequence of (3.7), (3.8) and (3.9). Along the same lines as in the proof of Theorem 2 of Li [30], and using the fact that (A1) implies $\sum_{k=-\infty}^\infty |\gamma_k(q_{\tau_1}, q_{\tau_2})| \leq C \sum_{k=-\infty}^\infty |k|^{-\delta} < \infty$, we obtain that

$$\text{Cov} \left(\sum_{t=1}^n \frac{2}{f(q_{\tau_{k_1}})} \frac{\mathbf{v}_{tn}(\omega_{i_1})}{\sqrt{n}} (\tau_{k_1} - I\{Y_t \leq q_{\tau_{k_1}}\}), \sum_{t=1}^n \frac{2}{f(q_{\tau_{k_2}})} \frac{\mathbf{v}_{tn}(\omega_{i_2})}{\sqrt{n}} (\tau_{k_2} - I\{Y_t \leq q_{\tau_{k_2}}\}) \right)$$

converges to $M_{\tau_{k_1}, \tau_{k_2}}^{\omega_{i_1}, \omega_{i_2}}$ defined in (3.3). Hence, we have

$$\text{Var} \left(\sum_{t=1}^n \sum_{k=1}^p \sum_{i=1}^\nu \lambda'_{ik} \frac{2}{f(q_{\tau_k})} \frac{\mathbf{v}_{tn}(\omega_i)}{\sqrt{n}} (\tau_k - I\{Y_t \leq q_{\tau_k}\}) \right) \rightarrow \text{Var} \left(\sum_{k=1}^p \sum_{i=1}^\nu \lambda'_{ik} N_{\tau_k}(\omega_i) \right).$$

By an application of the central limit theorem for triangular arrays of strongly mixing random variables in Francq and Zakoian [12], with $\kappa = 0$, $T_n = 0$, $r^* = (\delta - 1)/(2 + 4\delta)$ and $\nu^* = 3/(\delta - 1)$, we deduce that

$$\sum_{t=1}^n \sum_{k=1}^p \sum_{i=1}^\nu \lambda'_{ik} \frac{2}{f(q_{\tau_k})} \frac{\mathbf{v}_{tn}(\omega_i)}{\sqrt{n}} (\tau_k - I\{Y_t \leq q_{\tau_k}\}) \xrightarrow{\mathcal{L}} \mathcal{N} \left(0, \text{Var} \left(\sum_{k=1}^p \sum_{i=1}^\nu \lambda'_{ik} N_{\tau_k}(\omega_i) \right) \right),$$

where $(N_\tau(\omega))_{\tau \in T, \omega \in \Omega}$ denotes a Gaussian random vector with mean zero and covariance matrix $\text{Cov}(N_{\tau_1}(\omega_1), N_{\tau_2}(\omega_2)) = M_{\tau_{k_1}, \tau_{k_2}}^{\omega_{i_1}, \omega_{i_2}}$. Because of (3.12), the quantity

$$\sqrt{n} \sum_{k=1}^p \sum_{i=1}^\nu \lambda'_{ik} \hat{\mathbf{b}}_{\tau_k}(g_n(\omega_i))$$

converges in distribution to the same normal limit. Thus, it follows from the traditional Cramér–Wold device that

$$(\sqrt{n}\hat{\mathbf{b}}_{n,\tau}(g_n(\omega)))_{\tau \in T, \omega \in \Omega} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (N_\tau(\omega))_{\tau \in T, \omega \in \Omega}. \quad \square$$

As an immediate consequence of the above result, the continuous mapping theorem yields the asymptotic distribution of a collection of Laplace periodogram kernels.

Theorem 3.2. *Under the assumptions of Theorem 3.1,*

$$(\hat{L}_{n,\tau_1,\tau_2}(g_n(\omega_1)), \dots, \hat{L}_{n,\tau_1,\tau_2}(g_n(\omega_\nu))) \xrightarrow{\mathcal{L}} (L_{\tau_1,\tau_2}(\omega_1), \dots, L_{\tau_1,\tau_2}(\omega_\nu)), \quad (3.13)$$

where the random variables L_{τ_1,τ_2} associated with distinct frequencies are mutually independent. Moreover,

$$L_{\tau_1,\tau_2}(\omega) \sim \pi \overset{\circ}{f}_{\tau_1,\tau_2}(\omega) \chi_2^2 \quad \text{if } \tau_1 = \tau_2, \quad (3.14)$$

and

$$L_{\tau_1,\tau_2}(\omega) \stackrel{d}{=} \frac{1}{4} (Z_{11}, Z_{12}) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} Z_{21} \\ Z_{22} \end{pmatrix} \quad \text{if } \tau_1 \neq \tau_2,$$

where $(Z_{11}, Z_{12}, Z_{21}, Z_{22})'$ is a Gaussian vector with mean zero and covariance matrix

$$\Sigma_4(\omega) := 4\pi \begin{pmatrix} \overset{\circ}{f}_{\tau_1,\tau_1}(\omega) & 0 & \overset{\circ}{\mathfrak{H}}_{\tau_1,\tau_2}(\omega) & \overset{\circ}{\mathfrak{S}}_{\tau_1,\tau_2}(\omega) \\ 0 & \overset{\circ}{f}_{\tau_1,\tau_1}(\omega) & -\overset{\circ}{\mathfrak{S}}_{\tau_1,\tau_2}(\omega) & \overset{\circ}{\mathfrak{H}}_{\tau_1,\tau_2}(\omega) \\ \overset{\circ}{\mathfrak{H}}_{\tau_1,\tau_2}(\omega) & -\overset{\circ}{\mathfrak{S}}_{\tau_1,\tau_2}(\omega) & \overset{\circ}{f}_{\tau_2,\tau_2}(\omega) & 0 \\ \overset{\circ}{\mathfrak{S}}_{\tau_1,\tau_2}(\omega) & \overset{\circ}{\mathfrak{H}}_{\tau_1,\tau_2}(\omega) & 0 & \overset{\circ}{f}_{\tau_2,\tau_2}(\omega) \end{pmatrix}. \quad (3.15)$$

It follows from Theorem 3.2 that $E[L_{\tau_1,\tau_2}(\omega)] = 2\pi \overset{\circ}{f}_{\tau_1,\tau_2}(\omega)$ for all $(\tau_1, \tau_2) \in (0, 1)^2$ and $\omega \in (0, \pi)$, which indicates that an estimator of the scaled spectral density $2\pi \overset{\circ}{f}_{\tau_1,\tau_2}(\omega)$ defined in (3.1) could be based on an average of quantities of the form $\hat{L}_{n,\tau_1,\tau_2}(\omega)$. Moreover, the following result, which is an immediate consequence of Theorem 3.2, yields the asymptotic distribution of the copula periodogram kernel.

Corollary 3.1. *Let $\Omega := \{\omega_1, \dots, \omega_\nu\} \subset (0, \pi)$ denote distinct frequencies and $(\tau_1, \tau_2) \in (0, 1)^2$. If Assumptions (A1)–(A2) hold for every $\tau \in \{\tau_1, \tau_2\}$, then*

$$(\hat{L}_{n,\tau_1,\tau_2}^U(g_n(\omega_1)), \dots, \hat{L}_{n,\tau_1,\tau_2}^U(g_n(\omega_\nu))) \xrightarrow{\mathcal{L}} (L_{\tau_1,\tau_2}^U(\omega_1), \dots, L_{\tau_1,\tau_2}^U(\omega_\nu)), \quad (3.16)$$

where $g_n(\omega)$ is defined in (3.2). The random variables L_{τ_1,τ_2}^U in (3.19) associated with distinct frequencies are mutually independent,

$$L_{\tau_1,\tau_2}^U(\omega) \sim \pi f_{q_{\tau_1},q_{\tau_2}}(\omega) \chi_2^2 \quad \text{if } \tau_1 = \tau_2, \quad (3.17)$$

and

$$L_{\tau_1, \tau_2}^U(\omega) \stackrel{d}{=} \frac{1}{4} (Z_{11}, Z_{12}) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} Z_{21} \\ Z_{22} \end{pmatrix} \quad \text{if } \tau_1 \neq \tau_2,$$

where $(Z_{11}, Z_{12}, Z_{21}, Z_{22})' \sim \mathcal{N}(0, \Sigma_4(\omega))$ with covariance matrix

$$\Sigma_4(\omega) := 4\pi \begin{pmatrix} \mathfrak{f}_{q_{\tau_1}, q_{\tau_1}}(\omega) & 0 & \Re \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) & \Im \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) \\ 0 & \mathfrak{f}_{q_{\tau_1}, q_{\tau_1}}(\omega) & -\Im \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) & \Re \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) \\ \Re \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) & -\Im \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) & \mathfrak{f}_{q_{\tau_2}, q_{\tau_2}}(\omega) & 0 \\ \Im \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) & \Re \mathfrak{f}_{q_{\tau_2}, q_{\tau_2}}(\omega) & 0 & \mathfrak{f}_{q_{\tau_2}, q_{\tau_2}}(\omega) \end{pmatrix}. \quad (3.18)$$

In particular, $E[L_{\tau_1, \tau_2}^U(\omega)] = 2\pi \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega)$. This indicates that the copula periodogram kernels $\hat{L}_{n, \tau_1, \tau_2}^U$, rather than the Laplace ones $\hat{L}_{n, \tau_1, \tau_2}$, are likely to be the appropriate tools for statistical inference about $\mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}$. Unfortunately, they are not statistics, since they involve the unknown marginal distribution F which in practice is unspecified. This problem is taken care of in the next section.

3.2. Asymptotics of rank-based Laplace periodogram kernels

The final result of this section establishes the asymptotic equivalence of the copula and rank-based Laplace periodogram kernels $\hat{L}_{n, \tau_1, \tau_2}^U(\omega)$ and $\tilde{L}_{n, \tau_1, \tau_2}(\omega)$, where the latter do not involve F , hence can be computed from the data. In particular, the following results show that $\hat{\mathbf{b}}_{n, \tau}$, and $\tilde{L}_{n, \tau_1, \tau_2}(\omega)$ are asymptotically distribution-free with respect to the marginal distribution of Y_t in the sense that their asymptotic distributions only depend on the process $\{U_t\}_{t \in \mathbb{Z}}$.

Theorem 3.3. *Let $\Omega := \{\omega_1, \dots, \omega_p\} \subset (0, \pi)$ and $T := \{\tau_1, \dots, \tau_p\} \subset (0, 1)$ denote distinct frequencies and quantile orders, respectively. Let Assumptions (A1)–(A2) be satisfied with (A2) holding for every $\tau \in T$. Then,*

$$\left(\hat{\mathbf{b}}_{n, \tau}(g_n(\omega)) \right)_{\tau \in T, \omega \in \Omega} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(N_{\tau, \omega}^U \right)_{\tau \in T, \omega \in \Omega},$$

where $(N_{\tau, \omega}^U)_{\tau \in T, \omega \in \Omega}$ is a Gaussian random vector with mean zero and covariance matrix

$$M_{\tau_1, \tau_2}^{\omega_1, \omega_2} := \text{Cov}(N_{\tau_1, \omega_1}^U, N_{\tau_2, \omega_2}^U) = \begin{cases} 4\pi \begin{pmatrix} \Re \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) & \Im \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) \\ -\Im \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) & \Re \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) \end{pmatrix} & \text{if } \omega_1 = \omega_2 =: \omega, \text{ and} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \omega_1 \neq \omega_2. \end{cases}$$

At first glance, the fact that replacing the U_t 's with their ranks does not have any impact on the asymptotic distribution of $\hat{\mathbf{b}}_{n, \tau}(g_n(\omega))$ seems quite surprising: a completely different

phenomenon indeed typically occurs when estimating a copula, see e.g. Genest and Segers [15]. The explanation for this is that the Bahadur representation for the vector $(\hat{a}, \hat{\mathbf{b}})$ is (see the proof of Theorem 3.3) of the very special form

$$\sqrt{n}((\hat{a}, \hat{\mathbf{b}})' - (q_\tau, 0, 0)') = (\mathbf{Q}_{n,\omega}^U)^{-1} n^{-1/2} \sum_{t=1}^n \mathbf{c}_t(\omega)(\tau - I\{U_t \leq \tau\} + F(\hat{F}_n^{-1}(\tau)) - \tau),$$

where the matrix $\mathbf{Q}_{n,\omega}^U := \frac{1}{n} \sum_{t=1}^n \mathbf{c}_t(\omega)\mathbf{c}_t'(\omega)$ is diagonal. The additional term $F(\hat{F}_n^{-1}(\tau)) - \tau$ comes into play because we are using ranks to estimate the unknown marginals. However, due to the fact that, for Fourier frequencies ω , $\sum_{t=1}^n \cos(\omega t) = \sum_{t=1}^n \sin(\omega t) = 0$, this effect is not present in the first-order expansion of $\hat{\mathbf{b}}$ and thus does not influence its asymptotic distribution.

Together with the above result, the continuous mapping theorem then yields the following result.

Theorem 3.4. *Under the assumptions of Theorem 3.3*

$$(\hat{\mathcal{L}}_{n,\tau_1,\tau_2}(g_n(\omega_1)), \dots, \hat{\mathcal{L}}_{n,\tau_1,\tau_2}(g_n(\omega_v))) \xrightarrow{\mathcal{L}} (L_{\tau_1,\tau_2}^U(\omega_1), \dots, L_{\tau_1,\tau_2}^U(\omega_v)), \tag{3.19}$$

where $g_n(\omega)$ and the distribution of the random variables L_{τ_1,τ_2}^U are defined in (3.2) and Corollary 3.1, respectively.

Proof of Theorem 3.3. Recall that \hat{F}_n denotes the empirical distribution function of Y_1, \dots, Y_n ; let $\mathbf{e}_1 := (1, 0, 0)'$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3)'$, and $U_t := F(Y_t)$. We introduce the functions

$$\begin{aligned} \hat{\mathcal{Z}}_{n,\tau,\omega}(\boldsymbol{\delta}) &:= \sum_{t=1}^n (\rho_\tau(\hat{F}_n(Y_t) - \tau - n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}) - \rho_\tau(\hat{F}_n(Y_t) - \tau)), \\ \hat{\mathcal{Z}}_{n,\tau,\omega}^U(\boldsymbol{\delta}) &:= \sum_{t=1}^n (\rho_\tau(U_t - \tau - n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}) - \rho_\tau(U_t - \tau)) - \delta_1\sqrt{n}(F(\hat{F}_n^{-1}(\tau)) - \tau), \\ \mathcal{Z}_{n,\tau,\omega}^U(\boldsymbol{\delta}) &:= -\boldsymbol{\delta}'(\boldsymbol{\zeta}_{n,\tau,\omega}^U + \mathbf{e}_1'\sqrt{n}(F(\hat{F}_n^{-1}(\tau)) - \tau)) + \frac{1}{2}\boldsymbol{\delta}'\mathbf{Q}_{n,\omega}^U\boldsymbol{\delta}, \end{aligned}$$

where $\mathbf{Q}_{n,\omega}^U := n^{-1} \sum_{t=1}^n \mathbf{c}_t(\omega)\mathbf{c}_t'(\omega)$ and $\boldsymbol{\zeta}_{n,\tau,\omega}^U := n^{-1/2} \sum_{t=1}^n \mathbf{c}_t(\omega)(\tau - I\{U_t \leq \tau\})$. If we can show that the difference $\hat{\mathcal{Z}}_{n,\tau,\omega}(\boldsymbol{\delta}) - \hat{\mathcal{Z}}_{n,\tau,\omega}^U(\boldsymbol{\delta})$ is uniformly small in probability, a slight modification of the arguments developed in the proof of Theorem 3.2 yields a uniform linearization of $\hat{\boldsymbol{\delta}}_{n,\tau,\omega} := \arg \min_{\boldsymbol{\delta}} \hat{\mathcal{Z}}_{n,\tau,\omega}(\boldsymbol{\delta})$. More precisely, we show that

$$\sup_{\omega \in \mathcal{F}_n} \|\hat{\boldsymbol{\delta}}_{n,\tau,\omega} - \boldsymbol{\delta}_{n,\tau,\omega}^U\| = \text{Op}(n^{(1/8)(1-\delta)/(1+\delta)} \log n), \tag{3.20}$$

where $\boldsymbol{\delta}_{n,\tau,\omega}^U := \arg \min_{\boldsymbol{\delta}} \mathcal{Z}_{n,\tau,\omega}^U(\boldsymbol{\delta}) = (\mathbf{Q}_{n,\omega}^U)^{-1}(\boldsymbol{\zeta}_{n,\tau,\omega}^U + \mathbf{e}_1'\sqrt{n}(F(\hat{F}_n^{-1}(\tau)) - \tau))$. The asymptotic normality of the linearization $\boldsymbol{\delta}_{n,\tau,\omega}^U$ then follows by the same arguments as in Step (2) of the proof of Theorem 3.1; details are omitted for the sake of brevity.

In order to prove (3.20), we note that Lemma A.1 in Appendix A also holds with $\hat{Z}_{n,\tau,\omega}(\delta)$, $Z_{n,\tau,\omega}^X(\delta)$, $\delta_{n,\tau,\omega}^X$ and $\hat{\delta}_{n,\tau,\omega}$ replaced by $\hat{\tilde{Z}}_{n,\tau,\omega}(\delta)$, $Z_{n,\tau,\omega}^U(\delta)$, $\delta_{n,\tau,\omega}^U$ and $\hat{\tilde{\delta}}_{n,\tau,\omega}$, respectively. Therefore, it suffices to establish that, for some $\epsilon > 0$,

$$\sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta - \delta_{n,\tau,\omega}^U\| \leq \epsilon} \left| \hat{\tilde{Z}}_{n,\tau,\omega}(\delta) - Z_{n,\tau,\omega}^U(\delta) \right| = O_P(n^{(1/4)(1-\delta)/(1+\delta)}(\log n)^2). \tag{3.21}$$

Note that $\delta_{n,\tau,\omega}^U$ decomposes into a term containing $\zeta_{n,\tau,\omega}^U$, to which Lemma A.2 applies, and a term involving $\sqrt{n}(F(\hat{F}_n^{-1}(\tau)) - \tau)$ which, for every τ , converges in distribution, so that $P(\sqrt{n}(F(\hat{F}_n^{-1}(\tau)) - \tau) > A\sqrt{\log n}) \rightarrow 0$ for any $A > 0$. Therefore, there exists a constant A such that $\lim_{n \rightarrow \infty} P(\sup_{\omega \in \mathcal{F}_n} \|\delta_{n,\tau,\omega}^U\| > A\sqrt{\log n}) = 0$. It follows that, in order to establish (3.21), we may restrict to a supremum with respect to the set $\|\delta\| \leq 2A\sqrt{\log n}$. Knight’s identity (Knight [25]; see page 121 of Koenker [26]) yields

$$\hat{\tilde{Z}}_{n,\tau,\omega}(\delta) = \hat{\tilde{Z}}_{n,\tau,\omega,1}(\delta) + \hat{\tilde{Z}}_{n,\tau,\omega,2}(\delta),$$

where

$$\hat{\tilde{Z}}_{n,\tau,\omega,1}(\delta) = -\delta' n^{-1/2} \sum_{t=1}^n \mathbf{c}_t(\omega) (\tau - I\{U_t \leq F(\hat{F}_n^{-1}(\tau))\}),$$

and

$$\hat{\tilde{Z}}_{n,\tau,\omega,2}(\delta) = \sum_{t=1}^n \int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\delta} (I\{U_t \leq F(\hat{F}_n^{-1}(s + \tau))\} - I\{U_t \leq F(\hat{F}_n^{-1}(\tau))\}) ds.$$

A similar representation holds for $\hat{Z}_{n,\tau,\omega}^U(\delta)$. Now the proof of (3.21) is a consequence of the following two auxiliary results, which are proved in Sections A.2.1–A.2.2:

$$\begin{aligned} \sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta\| \leq A\sqrt{\log n}} \left| \hat{\tilde{Z}}_{n,\tau,\omega,1}(\delta) - \delta' n^{-1/2} \sum_{t=1}^n \mathbf{c}_t(\omega) (\tau - I\{U_t \leq \tau\}) \right. \\ \left. - \delta_1 \sqrt{n} (F(\hat{F}_n^{-1}(\tau)) - \tau) \right| = O_P(n^{(1/4)(1-\delta)/(1+\delta)}(\log n)^2) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta\| \leq A\sqrt{\log n}} \left| \hat{\tilde{Z}}_{n,\tau,\omega,2}(\delta) - \sum_{t=1}^n \int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\delta} (I\{U_t \leq s + \tau\} - I\{U_t \leq \tau\}) ds \right| \\ = O_P(n^{(1/4)(1-\delta)/(1+\delta)}(\log n)^2). \end{aligned} \tag{3.23}$$

Note that the combination of (3.22) and (3.23) implies that $\hat{\tilde{Z}}_{n,\tau,\omega}$ and $\hat{Z}_{n,\tau,\omega}^U$ are uniformly close in probability. Finally, we obtain from (3.11) that

$$\sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta\| \leq A\sqrt{\log n}} \left| \hat{Z}_{n,\tau,\omega}^U(\delta) - Z_{n,\tau,\omega}^U(\delta) \right| = O_P(r_n(\delta)^2), \tag{3.24}$$

where we may replace $\hat{Z}_{n,\tau,\omega}(\delta)$ with $\hat{Z}_{n,\tau,\omega}^U(\delta)$ and $Z_{n,\tau,\omega}(\delta)$ with $Z_{n,\tau,\omega}^U(\delta)$, since U_1, \dots, U_n are β -mixing with the rate from (A1), as required, and the additional term $\delta_1 \sqrt{n}(F(\hat{F}_n^{-1}(\tau)) - \tau)$ appears in both $\hat{Z}_{n,\tau,\omega}^U(\delta)$ and $Z_{n,\tau,\omega}^U(\delta)$. Combining (3.22)–(3.24) yields (3.21), thus completing the proof of Theorem 3.3. \square

4. Smoothed periodograms

We have seen in Section 3.1 that the Laplace periodogram kernel, for all (τ_1, τ_2) , converges in distribution, and that the expectation of the limit is the *scaled* spectral density kernel (at (τ_1, τ_2))

$$2\pi \mathring{f}_{\tau_1, \tau_2}(\omega) := 2\pi \frac{\int_{q_{\tau_1}}^{q_{\tau_2}} f(q_{\tau_2}) \, dq_{\tau_1}}{f(q_{\tau_1})f(q_{\tau_2})} = \frac{1}{f(q_{\tau_1})f(q_{\tau_2})} \sum_{k=-\infty}^{\infty} \gamma_k(q_{\tau_1}, q_{\tau_2}) e^{-i\omega k}.$$

In practice, however, this is not enough, and consistent estimation is a minimal requirement. For this purpose, we consider, as in traditional spectral estimation, smoothed versions of our periodograms, of the form

$$\hat{f}_{n, \tau_1, \tau_2}(\omega_{j,n}) := \sum_{|k| \leq N_n} W_n(k) \hat{L}_{n, \tau_1, \tau_2}(\omega_{j+k,n}) \tag{4.1}$$

at the Fourier frequencies $\omega_{j,n} = 2\pi j/n$, where $N_n \rightarrow \infty$ as $n \rightarrow \infty$ is a sequence of positive integers, and $W_n = \{W_n(j) : |j| \leq N_n\}$ is a sequence of positive weights satisfying

$$W_n(k) = W_n(-k) \quad \text{for all } k \quad \text{and} \quad \sum_{|k| \leq N_n} W_n(k) = 1.$$

Extending the definition of $\hat{f}_{n, \tau_1, \tau_2}$ to the interval $(0, \pi)$, we introduce

$$\{(0, \pi) \ni \omega \mapsto \hat{f}_{n, \tau_1, \tau_2}(\omega) | (\tau_1, \tau_2) \in (0, 1)^2\}$$

as the *smoothed Laplace periodogram kernel*, where

$$\hat{f}_{n, \tau_1, \tau_2}(\omega) := \hat{f}_{n, \tau_1, \tau_2}^2(g_n(\omega)), \tag{4.2}$$

and the function g_n is defined in (3.2). In order to show that $\hat{f}_{n, \tau_1, \tau_2}(\omega)$ is a consistent estimator of the scaled spectral density $\mathring{f}_{\tau_1, \tau_2}(\omega)$, we make the following additional assumptions.

Assumption (A3). $N_n/n \rightarrow 0$, and $\sum_{|k| \leq N_n} W_n^2(k) = O(1/n)$ as $n \rightarrow \infty$.

Assumption (A4). For any $\tau_1, \tau_2, \tau_3, \tau_4 \in (0, 1)$,

$$\sum_{k_2, k_3, k_4 = -\infty}^{\infty} |\text{cum}(I\{Y_t \leq q_{\tau_1}\}, I\{Y_{t+k_2} \leq q_{\tau_2}\}, I\{Y_{t+k_3} \leq q_{\tau_3}\}, I\{Y_{t+k_4} \leq q_{\tau_4}\})| < \infty,$$

where $\text{cum}(\zeta_1, \dots, \zeta_r) := \sum (-1)^{p-1} (p-1)! (\mathbb{E} \prod_{j \in \nu_1} \zeta_j) \cdots (\mathbb{E} \prod_{j \in \nu_p} \zeta_j)$ (with summation extending over all partitions $\{\nu_1, \dots, \nu_p\}$, $p = 1, \dots, r$ of $\{1, \dots, r\}$) denotes the r th order joint cumulant of the random vector $(\zeta_1, \dots, \zeta_r)$ (cf. Brillinger [7], page 19).

Assumption (A5). The functions $\omega \mapsto \mathring{f}_{q_{\tau_1}, q_{\tau_2}}$ defined in (2.3) are continuously differentiable for all $(\tau_1, \tau_2) \in (0, 1)^2$.

Note that an assumption similar to (A4), but with the cumulant of Y_t 's instead of the cumulant of the indicators, is made when consistency of smoothed cross-periodograms is proved, and that (A5) follows if (A1) holds with $\delta > 2$, because this implies

$$\sum_{k \in \mathbb{Z}} |k| |\gamma_k(\tau_1, \tau_2)| < \infty.$$

Theorem 4.1. Let (A1)–(A5) hold. Then the smoothed Laplace periodogram defined in (4.1) and (4.2) is a consistent estimator for the scaled Laplace spectral density; more precisely,

$$\hat{f}_{n, \tau_1, \tau_2}(\omega) = 2\pi \mathring{f}_{\tau_1, \tau_2}(\omega) + O_P(R_n + n^{-1/2} + N_n/n) = 2\pi \mathring{f}_{\tau_1, \tau_2}(\omega) + o_P(1), \tag{4.3}$$

where $R_n = (n^{-1/8}(\log n)^{3/2}) \vee (n^{(1/4)(1-\delta)/(1+\delta)}(\log n)^{9/4})$.

Proof. The proof proceeds in several steps which are sketched here – technical details can be found in Appendix B. We first show (Section B.1) that

$$\hat{L}_{n, \tau_1, \tau_2}(\omega_{j,n}) = L_{n, \tau_1, \tau_2}(\omega_{j,n}) / (f(q_{\tau_1})f(q_{\tau_2})) + O_P(R_n), \tag{4.4}$$

uniformly in the Fourier frequencies $\omega_{j,n} := 2\pi j/n$, where

$$\begin{aligned} L_{n, \tau_1, \tau_2}(\omega_{j,n}) &:= n^{-1} d_n(\tau_1, \omega_{j,n}) d_n(\tau_2, -\omega_{j,n}), \\ d_n(\tau, \omega_{j,n}) &:= \sum_{t=1}^n e^{i\omega_{j,n}t} (\tau - I\{Y_t \leq q_\tau\}) = (1, i)n \mathbf{b}_{n, \tau, \omega_{j,n}} 2^{-1} f(q_\tau) \quad \text{and} \\ n^{1/2} \mathbf{b}_{n, \tau, \omega_{j,n}} &:= \frac{2}{f(q_\tau)} n^{-1/2} \sum_{t=1}^n \begin{pmatrix} \cos(\omega_{j,n}t) \\ \sin(\omega_{j,n}t) \end{pmatrix} (\tau - I\{Y_t \leq q_\tau\}). \end{aligned}$$

As an immediate consequence, we obtain

$$\hat{f}_{n, \tau_1, \tau_2}(\omega_{j,n}) = \sum_{|k| \leq N_n} W_n(k) L_{n, \tau_1, \tau_2}(\omega_{j+k,n}) / (f(q_{\tau_1})f(q_{\tau_2})) + O_P(R_n).$$

In Section B.2, we show that, for any $\omega_{j,n} = 2\pi j/n$,

$$K_n := \sum_{|k| \leq N_n} W_n(k) \left(\frac{L_{n, \tau_1, \tau_2}(\omega_{j+k,n})}{f(q_{\tau_1})f(q_{\tau_2})} - \mathring{f}_{\tau_1, \tau_2}(\omega_{j+k,n}) \right) = O_P(1/\sqrt{n}). \tag{4.5}$$

Now, let $\omega_{j_n n}$ be a sequence of Fourier frequencies such that $|\omega_{j_n n} - \omega| = O(N_n/n)$ for some $\omega \in (0, \pi)$: both for $f \equiv \mathfrak{N}\overset{\circ}{f}_{\tau_1, \tau_2}$ and $f \equiv \mathfrak{S}\overset{\circ}{f}_{\tau_1, \tau_2}$, we have

$$\begin{aligned} \left| \sum_{|k| \leq N_n} W_n(k) (f(\omega_{j_n+k, n}) - f(\omega)) \right| &\leq \sum_{|k| \leq N_n} W_n(k) |f'(\xi_{j_n+k, n})| |\omega_{j_n+k, n} - \omega| \\ &\leq C_n \sum_{|k| \leq N_n} W_n(k) |2\pi k/n + \omega_{j_n} - \omega| \\ &\leq C_n \sum_{|k| \leq N_n} W_n(k) |2\pi k/n| + C_n \sum_{|k| \leq N_n} W_n(k) |\omega_{j_n} - \omega| \\ &\leq C_n (2\pi N_n/n + |\omega_{j_n} - \omega|) \sum_{|k| \leq N_n} W_n(k) = O(N_n/n), \end{aligned}$$

where $|\xi_{j_n+k, n} - \omega| \leq |\omega - \omega_{j_n+k, n}|$ and $C_n := \sup_{\xi \in \Xi_n} |f'(\xi)|$ is the supremum over

$$\Xi_n = \{ \xi \mid \omega - |\omega - \omega_{j_n, n}| - \omega_{N_n, n} \leq \xi \leq \omega + |\omega - \omega_{j_n, n}| + \omega_{N_n, n} \}.$$

Note that, since $|\omega - \omega_{j_n, n}| \rightarrow 0$ and $\omega_{N_n, n} = 2\pi N_n/n \rightarrow 0$, $C_n \rightarrow f'(\omega)$, so that (C_n) is a bounded sequence. This yields

$$\left| \sum_{|k| \leq N_n} W_n(k) (\overset{\circ}{f}_{\tau_1, \tau_2}(\omega_{j_n+k}) - \overset{\circ}{f}_{\tau_1, \tau_2}(\omega)) \right| = O(N_n/n),$$

which completes the proof of Theorem 4.1. □

For a consistent estimation of the (unscaled) Laplace spectral density $f_{\tau_1, \tau_2}(\omega)$, we propose a smoothed version

$$\hat{f}_{n, \tau_1, \tau_2}(\omega) := \hat{f}_{n, \tau_1, \tau_2}(g_n(\omega)), \quad \hat{f}_{n, \tau_1, \tau_2}(\omega_{j, n}) := \sum_{|k| \leq N_n} W_n(k) \hat{L}_{n, \tau_1, \tau_2}(\omega_{j+k, n})$$

of the rank-based Laplace periodogram $\hat{L}_{n, \tau_1, \tau_2}(\omega)$. We then have the following result.

Theorem 4.2. *Let Assumptions (A1)–(A5) hold. Then the smoothed rank-based Laplace periodogram $\hat{f}_{n, \tau_1, \tau_2}$ is a consistent estimator of the (unscaled) Laplace spectral density $f_{q_{\tau_1}, q_{\tau_2}}$. More precisely,*

$$\hat{f}_{n, \tau_1, \tau_2}(\omega) = 2\pi f_{q_{\tau_1}, q_{\tau_2}}(\omega) + O_P(n^{(1/8)(1-\delta)/(1+\delta)} (\log n)^{3/2} + N_n/n) = 2\pi f_{q_{\tau_1}, q_{\tau_2}}(\omega) + o_P(1).$$

Proof. The proof is very similar to that of Theorem 4.1. The main difference lies in the asymptotic representation for the second and third coordinates $n^{1/2} \mathbf{b}_{n, \tau, \omega}^U$ of the quantity $\delta_{n, \tau, \omega}^U$ in

(3.20). Here we use (3.20), which implies that

$$\begin{aligned} & \sup_{\omega \in \mathcal{F}_n} \left\| n^{1/2} \mathbf{b}_{n,\tau,\omega}^U - 2n^{-1/2} \sum_{t=1}^n \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} (\tau - I\{F(Y_t) \leq \tau\}) \right\| \\ &= O_{\mathbb{P}}\left(n^{(1/8)(1-\delta)/(1+\delta)} (\log n)^{3/2}\right). \end{aligned}$$

The rest of the proof follows as in the proof of Theorem 4.1, yielding the estimate

$$\hat{\mathfrak{J}}_{n,\tau_1,\tau_2}(\omega) = 2\pi \mathfrak{f}_{\tau_1,\tau_2}(\omega) + O_{\mathbb{P}}\left(n^{(1/8)(1-\delta)/(1+\delta)} (\log n)^{3/2} + n^{-1/2} + N_n/n\right).$$

Finally, the assumptions imply that $n^{-1/2} = O(n^{(1/8)(1-\delta)/(1+\delta)} (\log n)^{3/2})$, which completes the proof of Theorem 4.2. □

Note that Theorem 4.1 solves an important open problem raised in Li [30,31], who considered the Laplace periodogram $\hat{L}_{n,\tau_1,\tau_2}$ for $\tau_1 = \tau_2$. This author established the asymptotic unbiasedness of a smoothed version of the Laplace periodogram, but not its consistency. Moreover, as pointed out in Theorem 3.1 the smoothed version of $\hat{L}_{n,\tau_1,\tau_2}$ is not consistent for the copula spectral density kernel, which is the main object of interest in this paper. Theorem 4.2 shows that the smoothed rank-based Laplace periodogram yields a consistent estimate of this quantity.

5. Finite-sample properties

5.1. Simulation results

In order to illustrate the finite-sample properties of the new estimates, we present a small simulation study, where we consider four models. In Models (1) and (2), the observations are AR(1) processes with $Y_t = -0.3Y_{t-1} + \varepsilon_t$, and $\mathcal{N}(0, 1)$ - and t_1 -distributed innovations ε_t . Note that in Model (2) no moments exist, hence the traditional spectral density is not defined. Model (3) is a QAR(1) model (cf. Koenker and Xiao [28]), that is, a model of the form $Y_t = \theta_0(U_t) + \theta_1(U_t)Y_{t-1}$, where (U_t) is a sequence of i.i.d. standard uniform random variables and θ_1 and θ_0 are functions from $[0, 1]$ to \mathbb{R} ; more specifically, we chose $\theta_1(u) = 1.9(u - 0.5)$ and $\theta_0(u) = 0.1\Phi^{-1}(u)$, with Φ the standard normal distribution function. Model (4) is the ARMA(1, 1) model $Y_t = -0.8Y_{t-1} + 1.25\varepsilon_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim t_3$. Note that this defines an *all-pass* ARMA(1, 1) process where the observations are uncorrelated, but not independent (cf. e.g., Breidt, Davis and Trindade [6]). All results presented in this section are based on 5000 independent replications.

For each of those four models, we generated pseudo-random time series of lengths $n = 100$, $n = 500$ and $n = 1000$, and computed the Laplace and rank-based Laplace periodogram for $\tau_1, \tau_2 \in \{0.05, 0.25, 0.5, 0.75, 0.95\}$. We also computed the smoothed estimates using Daniell kernels with parameters $(2, 1)$ for $n = 100$, $(10, 4)$ for $n = 500$, and $(10, 25)$ for $n = 1000$ – namely, the kernel $W_n^{(m_1, \dots, m_p)}(j)$ recursively defined, for parameters (m_1, \dots, m_p) , with $N_n =$

$\sum_{j=1}^p m_j < n/2$, as

$$\begin{aligned}
 W_n^{(m)}(j) &:= (2m - 1)^{-1} I\{|j| \leq m\}, \\
 W_n^{(m_1, \dots, m_p)}(j) &:= C (W_n^{(m_1, \dots, m_{p-1})} * W_n^{(m_p)})(j) \\
 &= C \sum_{|k| \leq m_p} (2m_p - 1)^{-1} W_n^{(m_1, \dots, m_{p-1})}(j - k),
 \end{aligned}$$

where $*$ denotes convolution of two kernels and the constant C is chosen such that $\sum_{|j| \leq N_n} W_n^{(m_1, \dots, m_p)}(j) = 1$; the parameters m_1 and m_2 , $N_n = m_1 + m_2$, were chosen by empirical considerations.

From all calculated periodograms, we determine the mean as an approximation to the expectation of the various estimates. Each of the following figures subdivides into nine subfigures. For any combination of τ_1 and τ_2 , the imaginary parts of periodograms and spectra are represented above the diagonal, and the real parts below; for $\tau_1 = \tau_2$, those quantities are real and we represent them on the diagonal. All curves are plotted against $\omega/(2\pi)$. In all figures, the dashed line represents the “true” spectrum (scaled for Figures 1–4; unscaled for Figures 5–8) and the

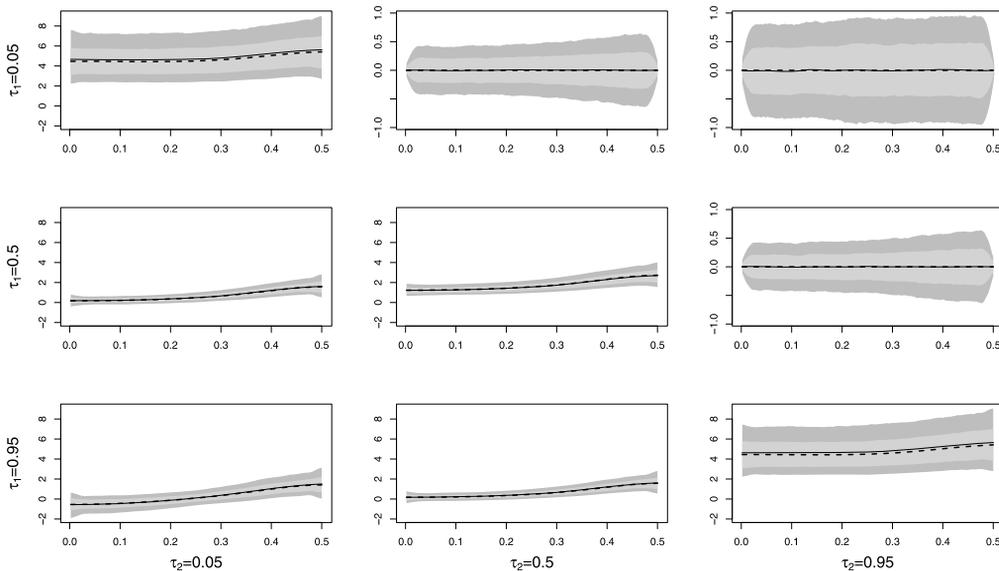


Figure 1. Smoothed Laplace periodograms and (scaled) spectral densities as defined in (3.1) from 5000 replications of length $n = 500$ of an AR(1) process with $\mathcal{N}(0, 1)$ innovations. All curves are plotted against $\omega/(2\pi)$. Real parts (Imaginary parts) of the periodogram and spectrum are presented in subfigures with $\tau_2 \leq \tau_1$ ($\tau_2 > \tau_1$): the dashed line represents the actual scaled spectrum [cf. (3.1)], the solid line the (pointwise) mean of the simulated smoothed Laplace periodograms. The gray areas represent the 0.1, 0.25, 0.75 and 0.9 (pointwise) sample quantiles of the smoothed periodograms over the 5000 replications.

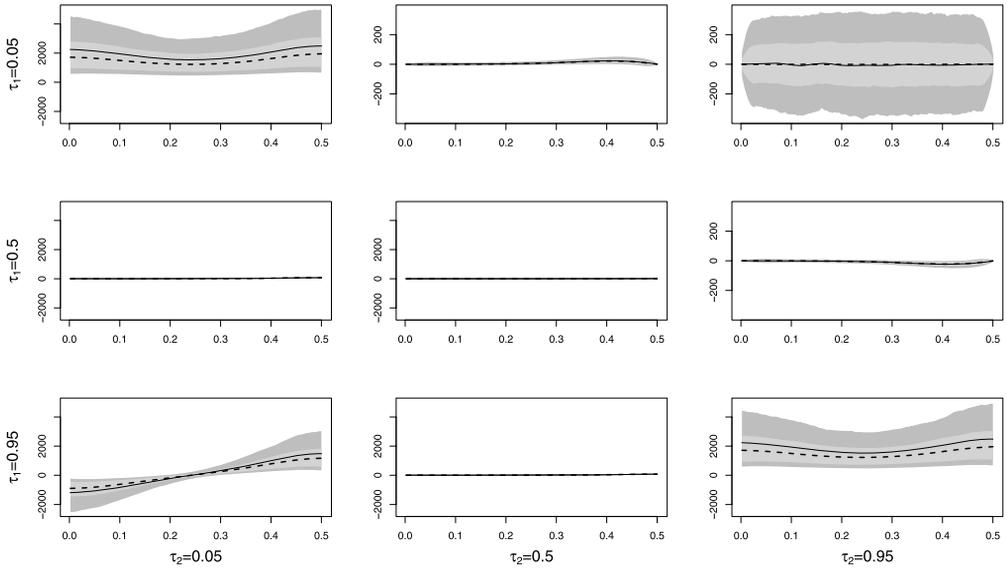


Figure 2. Smoothed Laplace periodograms and (scaled) spectral densities as defined in (3.1). The process is an AR(1) process with t_1 innovations and the sample size is 500. All curves are plotted against $\omega/(2\pi)$. Real parts (Imaginary parts) of the periodogram and spectrum are presented in subfigures with $\tau_2 \leq \tau_1$ ($\tau_2 > \tau_1$): the dashed line represents the actual scaled spectrum [cf. (3.1)], the solid line the (pointwise) mean of the simulated smoothed Laplace periodograms. The gray areas represent the 0.1, 0.25, 0.75 and 0.9 (pointwise) sample quantiles of the smoothed periodograms over the 5000 replications.

solid line the (pointwise) mean of the simulated smoothed Laplace periodograms. The gray areas represent the 0.1, 0.25, 0.75 and 0.9 (pointwise) sample quantiles of the smoothed periodograms from the 5000 simulation runs.

For the sake of brevity, only results for sample size $n = 500$ are presented here, but further results, which show a similar behavior, are available from the authors.

We first discuss the results for the smoothed Laplace periodogram in the case of an AR(1) process. Figure 1 is with Gaussian innovations, while the case of t_1 -distributed innovations is shown in Figure 2. Inspection of these figures reveals that the imaginary component of the spectrum is vanishing in the case of Gaussian innovations (see Figure 1). This observation reflects the fact that AR processes with Gaussian innovations are time-reversible. On the other hand, for t_1 -distributed innovations, this phenomenon only takes place for the extreme quantiles ($\tau_1 = 0.05, \tau_2 = 0.95$), meaning that $P(X_t \leq q_{0.05}, X_{t+k} \leq q_{0.95})$ is approximately equal to $P(X_t \leq q_{0.95}, X_{t+k} \leq q_{0.05})$. This, however, does not hold for $\tau_1 = 0.5$ and $\tau_2 = 0.05$ or 0.95 , which indicates a time-irreversible impact of extreme values on the central ones.

In Figure 3, the simulation results for the QAR(1) process are shown. We see that the (scaled) copula spectrum for $\tau_1 = \tau_2 = 0.25$ has the shape previously observed in the case of the AR(1) process, where the autoregressive parameter was negative. Note that the function $\theta_1(u)$ takes negative values for $u \in (0, 0.5)$. On the other hand, for $\tau_1 = \tau_2 = 0.75$, it

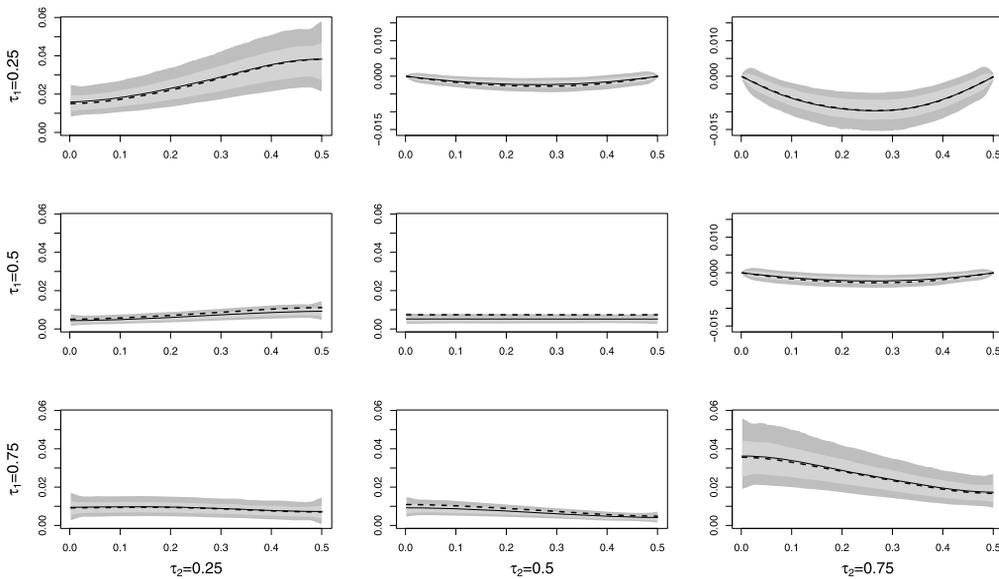


Figure 3. Smoothed Laplace periodograms and (scaled) spectral densities as defined in (3.1). The process is a QAR(1) process with $\theta_1(u) = 1.9(u - 0.5)$, $\theta_0(u) = 0.1\Phi^{-1}(u)$ and the sample size is 500. All curves are plotted against $\omega/(2\pi)$. Real parts (Imaginary parts) of the periodogram and spectrum are presented in subfigures with $\tau_2 \leq \tau_1$ ($\tau_2 > \tau_1$): the dashed line represents the actual scaled spectrum [cf. (3.1)], the solid line the (pointwise) mean of the simulated smoothed Laplace periodograms. The gray areas represent the 0.1, 0.25, 0.75 and 0.9 (pointwise) sample quantiles of the smoothed periodograms over the 5000 replications.

has the shape of the spectral density in the AR(1) case when the autoregressive parameter is positive, while $\theta_1(u)$ is positive for $u \in (0.5, 1)$. For $\tau_1 = \tau_2 = 0.5$ we observe a flat spectrum, indicating that the sequence $(I\{Y_t \leq q_{0.5}\})$ has zero autocorrelation, which would imply $P(X_t \leq q_{0.5}, X_{t+k} \leq q_{0.5}) = P(X_t \leq q_{0.5})P(X_{t+k} \leq q_{0.5})$. The imaginary part of the spectrum clearly indicates time-irreversibility, which implies that the QAR(1) process, irrespective of the choice of θ_0 , cannot be a Gaussian process.

The simulation results for the all-pass ARMA(1, 1) process are shown in Figure 3. We see here that the statistics proposed are very able of capturing the serial dependence which (due to uncorrelatedness) would completely escape the traditional analysis. Another finding is that, in most cases, the bias is larger for the estimation of the Laplace spectrum with $\tau_1 = \tau_2$: see, for instance, the diagonals of Figures 1–4.

The corresponding rank-based Laplace periodograms are shown in Figures 5–8, respectively. The results indicate the same type of time-reversibility features as observed with the Laplace periodogram. It is interesting to note that, for the rank-based Laplace periodograms, the bias appears to be much smaller, and smoothing seems to be more effective.

Finally, we investigate the quality of the estimates by their mean squared properties. In Table 1, we provide the square roots of the integrated mean squared errors (MSE). We consider the

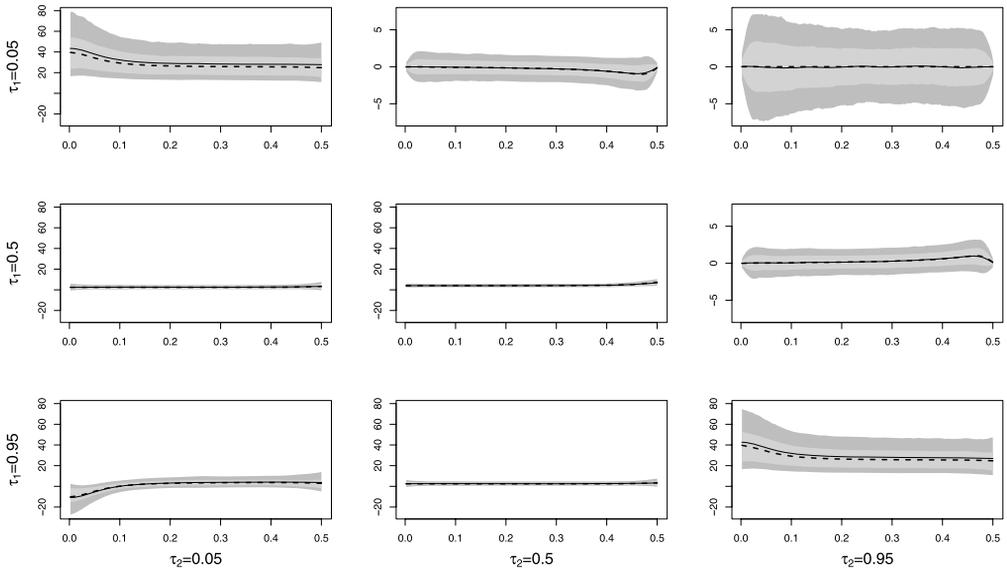


Figure 4. Smoothed Laplace periodograms and (scaled) spectral densities as defined in (3.1). The process is an ARMA(1, 1) process with t_3 innovations and the sample size is 500. All curves are plotted against $\omega/(2\pi)$. Real parts (Imaginary parts) of the periodogram and spectrum are presented in subfigures with $\tau_2 \leq \tau_1$ ($\tau_2 > \tau_1$): the dashed line represents the actual scaled spectrum [cf. (3.1)], the solid line the (pointwise) mean of the simulated smoothed Laplace periodograms. The gray areas represent the 0.1, 0.25, 0.75 and 0.9 (pointwise) sample quantiles of the smoothed periodograms over the 5000 replications.

smoothed rank-based Laplace periodograms for sample sizes $n = 100, 500,$ and 1000 . Note that, because of symmetry, we do not display all combinations. For example, the spectra corresponding to the quantiles (0.05, 0.05) and (0.95, 0.95) coincide in the scenario under consideration. In all cases, we observe, from the point of view of MSE, a reasonable behavior of the rank-based Laplace periodograms. It is interesting to note that the integrated MSE is larger when quantiles in the neighborhood of $\tau = 0.5$ are involved. For example, the integrated MSE is increasing from (0.05, 0.05) to (0.05, 0.25) and (0.05, 0.50), then decreasing from (0.05, 0.75) to (0.05, 0.95). This phenomenon is closely related to the fact that the empirical copula has variance zero at the boundaries of the unit cube, see Genest and Segers [15] for more details on this fact.

5.2. An empirical application: S&P 500 returns

The smoothed rank-based Laplace periodogram was computed from the series of daily return values of the S&P 500 index (Jan/2/1963–Dec/31/2009, $n = 11\,844$), based on a Daniell kernel with parameters (200, 100), for the same quantile orders as in the previous section. The results for the smoothed traditional periodogram are shown in Figure 9, and those for the rank-based Laplace periodogram in Figure 10, with the same convention as in Section 5.1.

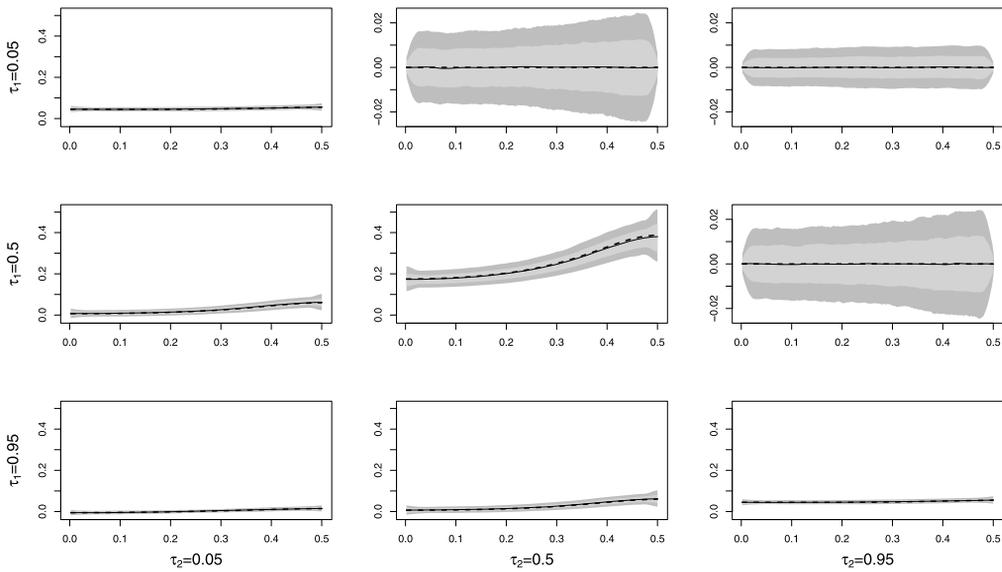


Figure 5. Smoothed rank-based Laplace periodograms and (unscaled) spectral densities as defined in (2.3). The process is an AR(1) process with $\mathcal{N}(0, 1)$ innovations and the sample size is 500. All curves are plotted against $\omega/(2\pi)$. Real parts (Imaginary parts) of the periodogram and spectrum are presented in subfigures with $\tau_2 \leq \tau_1$ ($\tau_2 > \tau_1$): the dashed line represents the actual scaled spectrum [cf. (2.3)], the solid line the (pointwise) mean of the simulated smoothed Laplace periodograms. The gray areas represent the 0.1, 0.25, 0.75 and 0.9 (pointwise) sample quantiles of the smoothed periodograms over the 5000 replications.

The nonlinear features of that series have been stressed by many authors (see, e.g., Abhyankar, Copeland and Wong [1], Berg, Paparoditis and Politis [3], Brock, Hsieh and LeBaron [8], Hinich and Patterson [18,19], Hsieh [22], and Vaidyanathan and Krehbiel [40]). Those nonlinear features cannot be detected by classical correlogram-based spectral methods, and hence do not appear in Figure 9, where the traditional smoothed periodogram is depicted. They do appear, however, in the plots of Figure 10. Whereas the picture for the central quantiles $\tau_1 = \tau_2 = 0.5$ looks quite similar to that in Figure 9, the remaining ones, which involve at least one extreme quantile, are drastically different, indicating a marked discrepancy between tail and central dependence structures. All plots involving at least one extremal quantile yield a peak at the origin, which possibly corresponds to a long-range memory for extremal events. Imaginary parts are not zero, suggesting again time-irreversibility. Such features entirely escape a traditional spectral analysis.

Appendix A: Technical details for the proofs in Section 3

In this section, we give the technical details for the proofs of Theorems 3.1 and 3.3. Those proofs rely on a series of lemmas. Two of them (Lemmas A.6 and A.7) are general results, to be used at several places in both proofs; their statements and proofs are postponed to Section A.3. Two

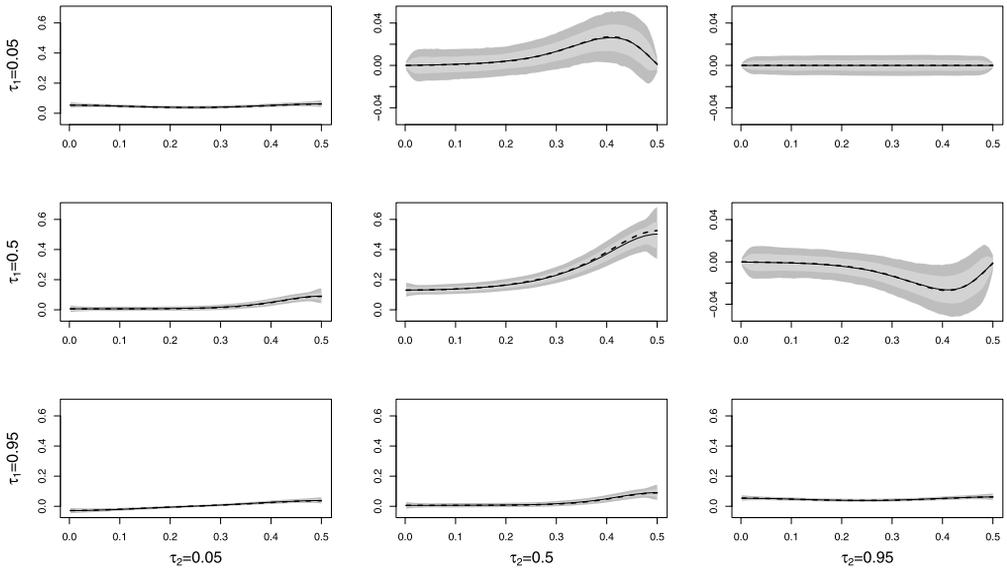


Figure 6. Smoothed rank-based Laplace periodograms and (unscaled) spectral densities as defined in (2.3). The process is an AR(1) process with t_1 innovations and the sample size is 500. All curves are plotted against $\omega/(2\pi)$. Real parts (Imaginary parts) of the periodogram and spectrum are presented in subfigures with $\tau_2 \leq \tau_1$ ($\tau_2 > \tau_1$): the dashed line represents the actual scaled spectrum [cf. (2.3)], the solid line the (pointwise) mean of the simulated smoothed Laplace periodograms. The gray areas represent the 0.1, 0.25, 0.75 and 0.9 (pointwise) sample quantiles of the smoothed periodograms over the 5000 replications.

further ones (Lemmas A.4 and A.5) are specific to the proof of (3.20) and Theorem 3.3: they are presented in Section A.2.3. Finally, Lemmas A.1 and A.2 are auxiliary results used in the proofs of (3.9) and (3.20); they are regrouped in Section A.1.2, along with Lemma A.3, which is specific to the proof of (3.9).

A.1. Details for the proof of (3.9)

Recall that (3.9) was obtained by combining Lemmas A.1 and A.2 with Equation (3.11). In Section A.1.1, we establish (3.11), thus completing (but for Lemmas A.1–A.3) the proof of Theorem 3.1. In Section A.1.2, we state and prove Lemmas A.1–A.3, which completes the proof of (3.9). The notation of Theorem 3.1 is used throughout this section.

A.1.1. Proof of (3.11)

In this proof, we use a blocking argument by Yu [42] – call it the *independent blocks argument*. Let $m_n := \lceil n^{1/(1+\delta)} \log n \rceil$, $\mu_n := \lfloor n/(2m_n) \rfloor$, and split the set $\{1, \dots, n\}$ into $2\mu_n$ subsets of size

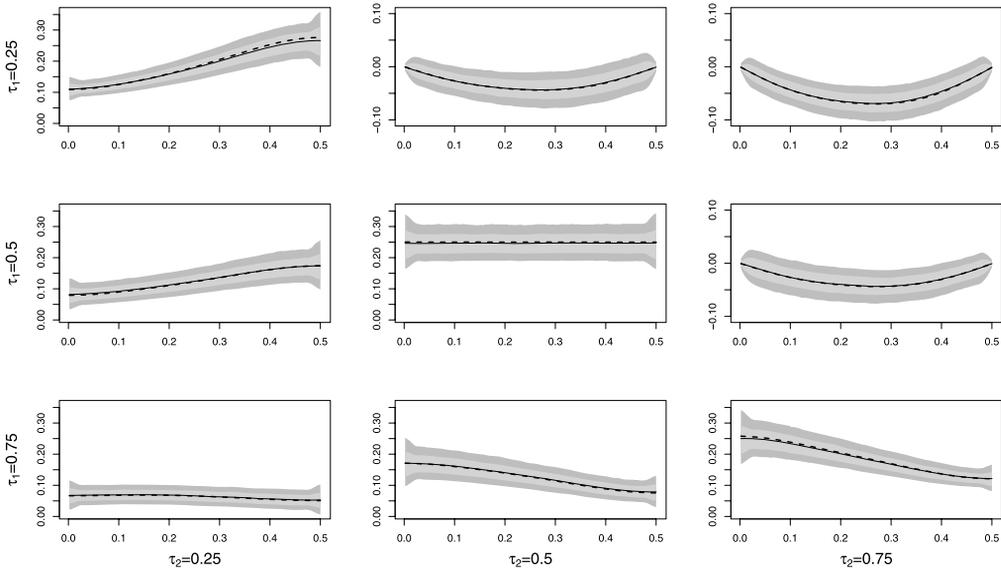


Figure 7. Smoothed rank-based Laplace periodograms and (unscaled) spectral densities as defined in (2.3). The process is a QAR(1) process with $\theta_1(u) = 1.9(u - 0.5)$, $\theta_0(u) = 0.1\Phi^{-1}(u)$ and the sample size is 500. All curves are plotted against $\omega/(2\pi)$. Real parts (Imaginary parts) of the periodogram and spectrum are presented in subfigures with $\tau_2 \leq \tau_1$ ($\tau_2 > \tau_1$): the dashed line represents the actual scaled spectrum [cf. (2.3)], the solid line the (pointwise) mean of the simulated smoothed Laplace periodograms. The gray areas represent the 0.1, 0.25, 0.75 and 0.9 (pointwise) sample quantiles of the smoothed periodograms over the 5000 replications.

m_n and a “residual” subset of size $n - 2m_n\mu_n$:

$$\begin{aligned}
 S_i &:= \{k \in \mathbb{N}: 2(i - 1)m_n + 1 \leq k \leq (2i - 1)m_n\}, & i = 1, \dots, \mu_n, \\
 T_i &:= \{k \in \mathbb{N}: (2i - 1)m_n + 1 \leq k \leq 2im_n\}, & i = 1, \dots, \mu_n, \\
 R_n &:= \{2m_n\mu_n + 1, \dots, n\}.
 \end{aligned}
 \tag{A.1}$$

Associated with this partition of $\{1, \dots, n\}$, consider the partition

$$(Y_t)_{t \in S_1}, (Y_t)_{t \in T_1}; (Y_t)_{t \in S_2}, \dots, (Y_t)_{t \in T_{\mu_n-1}}; (Y_t)_{t \in S_{\mu_n}}, (Y_t)_{t \in T_{\mu_n}}; (Y_t)_{t \in R_n}$$

of $\{Y_1, \dots, Y_n\}$ into $2\mu_n$ blocks of length m_n and a “residual” block of length $n - 2m_n\mu_n$. The independent block m_n -sequence then is defined as a collection of $2\mu_n$ mutually independent m_n -dimensional random variables $(X_t)_{t \in S_i}, (X_t)_{t \in T_i}, i = 1, \dots, \mu_n$, such that $(X_t)_{t \in S_i} \stackrel{d}{=} (Y_t)_{t \in S_i}$ and $(X_t)_{t \in T_i} \stackrel{d}{=} (Y_t)_{t \in T_i}$, along with an $(n - 2m_n\mu_n)$ -dimensional variable $(X_t)_{t \in R_n}$ independent of all other (X_t) 's.

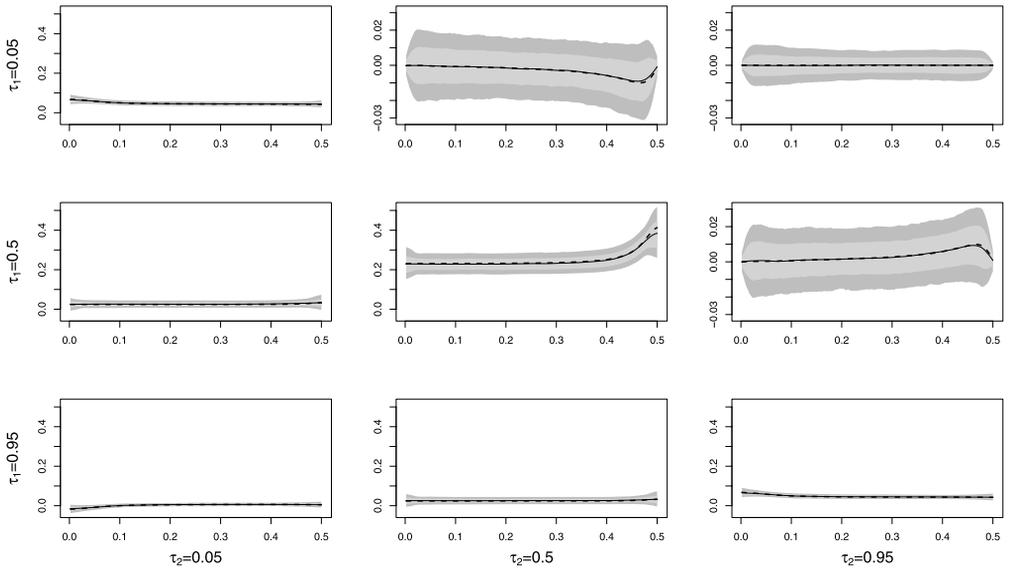


Figure 8. Smoothed rank-based Laplace periodograms and (unscaled) spectral densities as defined in (2.3). The process is an ARMA(1, 1) process with t_3 innovations and the sample size is 500. All curves are plotted against $\omega/(2\pi)$. Real parts (Imaginary parts) of the periodogram and spectrum are presented in subfigures with $\tau_2 \leq \tau_1$ ($\tau_2 > \tau_1$): the dashed line represents the actual scaled spectrum [cf. (2.3)], the solid line the (pointwise) mean of the simulated smoothed Laplace periodograms. The gray areas represent the 0.1, 0.25, 0.75 and 0.9 (pointwise) sample quantiles of the smoothed periodograms over the 5000 replications.

The independent blocks argument will be used to establish results of the form

$$P\left(\sup_{\theta \in \Theta_n} \left| \sum_{t=1}^n \theta(t, Y_t) \right| > \eta_n\right) = o(1),$$

where Θ_n are sets of measurable functions $\theta: \mathbb{R}^2 \rightarrow \mathbb{R}$. For the argument, consider the decomposition

$$\begin{aligned} & P\left(\sup_{\theta \in \Theta_n} \sum_{t=1}^n \theta(t, Y_t) > \eta_n\right) \\ & \leq P\left(\sup_{\theta \in \Theta_n} \left| \sum_{i=1}^{\mu_n} \sum_{t \in S_i} \theta(t, Y_t) \right| > \eta_n/3\right) \\ & \quad + P\left(\sup_{\theta \in \Theta_n} \left| \sum_{i=1}^{\mu_n} \sum_{t \in T_i} \theta(t, Y_t) \right| > \eta_n/3\right) + P\left(\sup_{\theta \in \Theta_n} \left| \sum_{t \in R_n} \theta(t, Y_t) \right| > \eta_n/3\right) \\ & =: P_n^{(1)} + P_n^{(2)} + P_n^{(3)}. \end{aligned}$$

Table 1. Root Integrated Mean Square Errors of smoothed, rank-based Laplace periodograms, for the four models described in Section 5.1, and various series lengths

Y_t	n	(τ_1, τ_2)							
		(0.05, 0.05)	(0.05, 0.25)	(0.05, 0.5)	(0.05, 0.75)	(0.05, 0.95)	(0.25, 0.25)	(0.25, 0.5)	(0.5, 0.5)
Model (1)	100	0.0212	0.0408	0.0459	0.0401	0.0219	0.0651	0.0837	0.0876
	500	0.0085	0.0185	0.0215	0.0189	0.0099	0.0347	0.0429	0.0474
	1000	0.0054	0.0117	0.0137	0.0121	0.0064	0.0225	0.0275	0.0310
Model (2)	100	0.0223	0.0418	0.0462	0.0405	0.0234	0.0672	0.0852	0.0929
	500	0.0091	0.0188	0.0213	0.0188	0.0110	0.0353	0.0441	0.0506
	1000	0.0059	0.0120	0.0135	0.0120	0.0072	0.0228	0.0282	0.0330
Model (3)	100	0.0207	0.0398	0.0452	0.0386	0.0214	0.0652	0.0830	0.0873
	500	0.0084	0.0184	0.0213	0.0186	0.0098	0.0349	0.0428	0.0471
	1000	0.0053	0.0115	0.0135	0.0119	0.0064	0.0227	0.0277	0.0309
Model (4)	100	0.0220	0.0412	0.0453	0.0398	0.0226	0.0654	0.0834	0.0873
	500	0.0097	0.0191	0.0214	0.0190	0.0108	0.0344	0.0422	0.0465
	1000	0.0064	0.0122	0.0135	0.0121	0.0071	0.0226	0.0271	0.0306

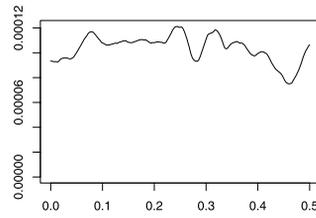


Figure 9. Smoothed traditional periodogram, S&P 500 returns curve is plotted against $\omega/(2\pi)$.

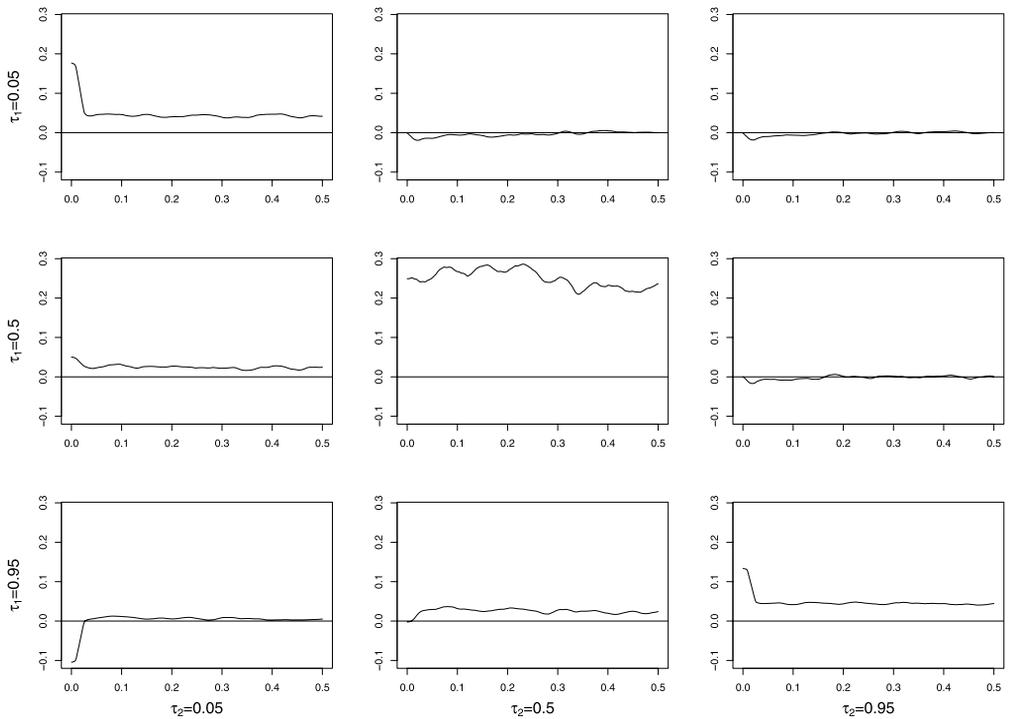


Figure 10. Smoothed rank-based Laplace periodograms, S&P 500 returns. All curves are plotted against $\omega/(2\pi)$. Real parts (Imaginary parts) of the periodogram and spectrum are presented in subfigures with $\tau_2 \leq \tau_1$ ($\tau_2 > \tau_1$).

The last probability $P_n^{(3)}$ is zero as soon as

$$(i) \quad \sup_{\theta \in \Theta_n} \sup_{t=1, \dots, n} |\theta(t, Y_t)| \leq C_n \quad \text{a.s. and} \quad m_n C_n < \eta_n/3,$$

which will be the case in all applications of the independent blocks argument. The first probability $P_n^{(1)}$ can be handled by applying Lemma 4.1 from Yu [42], by which we have

$$\mathbb{P} \left(\sup_{\theta \in \Theta_n} \left| \sum_{i=1}^{\mu_n} \sum_{t \in S_i} \theta(t, Y_t) \right| > \eta_n/3 \right) \leq \mathbb{P} \left(\sup_{\theta \in \Theta_n} \left| \sum_{i=1}^{\mu_n} \sum_{t \in S_i} \theta(t, X_t) \right| > \eta_n/3 \right) + o(1),$$

since by the choice of m_n we have $\mu_n \beta(m_n) = o(1)$. A similar argument applies to the second probability $P_n^{(2)}$. We assume that the set Θ_n consists of finitely many, say $|\Theta_n|$, elements to further obtain

$$\mathbb{P} \left(\sup_{\theta \in \Theta_n} \left| \sum_{i=1}^{\mu_n} \sum_{t \in S_i} \theta(t, X_t) \right| > \eta_n/3 \right) \leq |\Theta_n| \sup_{\theta \in \Theta_n} \mathbb{P} \left(\left| \sum_{i=1}^{\mu_n} \sum_{t \in S_i} \theta(t, X_t) \right| > \eta_n/3 \right),$$

where the summands $\sum_{t \in S_i} \theta_t(X_t)$, $i = 1, \dots, \mu_n$ are independent by construction. If we additionally show that

$$(ii) \quad \sup_{\theta \in \Theta_n} \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in S_j} \theta(t, X_t) \right) \leq V_n^2 \quad \text{and} \quad \sup_{\theta \in \Theta_n} \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in T_j} \theta(t, X_t) \right) \leq V_n^2,$$

the version of Bennett's inequality given in Lemma A.6 can be applied, so that, under (i) and (ii),

$$\begin{aligned} \mathbb{P} \left(\sup_{\theta \in \Theta_n} \left| \sum_{i=1}^{\mu_n} \sum_{t \in S_i} \theta(t, X_t) \right| > \eta_n/3 \right) &\leq \mathbb{P} \left(\sup_{\theta \in \Theta_n} \left| \sum_{i=1}^{\mu_n} \sum_{t \in S_i} (\theta(t, X_t) - \mathbb{E}[\theta(t, X_t)]) \right| > \lambda_n \right) \\ &\leq 2|\Theta_n| \exp \left(-\frac{\log 2}{4} \left(\frac{\lambda_n^2}{2V_n^2} \wedge \frac{\lambda_n}{m_n C_n} \right) \right), \end{aligned}$$

where $\lambda_n := \eta_n/3 - n \sup_{\theta \in \Theta_n} \sup_{t=1, \dots, n} |\mathbb{E}[\theta(t, X_t)]|$. Exactly the same argument can be used to handle the probability $P_n^{(2)}$. Hence, we obtain

$$\begin{aligned} \mathbb{P} \left(\sup_{\theta \in \Theta_n} \left| \sum_{t=1}^n \theta(t, Y_t) \right| > \eta_n \right) &\leq E_n + o(1), \\ E_n &:= 4|\Theta_n| \exp \left(-\frac{\log 2}{4} \left(\frac{\lambda_n^2}{2V_n^2} \wedge \frac{\lambda_n}{m_n C_n} \right) \right). \end{aligned} \tag{A.2}$$

An application of the independent block argument for finite Θ_n thus boils down to establishing (i)–(ii) discussed above and ensuring that $E_n = o(1)$.

Regarding the proof of (3.11) note that, it is obviously possible to construct $N = o(n^5)$ points d_1, \dots, d_N (dependence on n is not reflected in the notation) such that, for every δ with $\|\delta\| \leq$

$A\sqrt{\log n}$, there exists an index $j(\delta)$ for which $\|\delta - d_{j(\delta)}\| \leq n^{-3/2}$. Define

$$K_n(\delta; \tau, \omega) := \sum_{t=1}^n \left(\int_0^{n^{-1/2} \mathbf{c}'_t(\omega) \delta} (I\{Y_t \leq s + q_\tau\} - I\{Y_t \leq q_\tau\}) ds - f(q_\tau)(2n)^{-1} (\mathbf{c}'_t(\omega) \delta)^2 \right)$$

and note, by direct calculation, that, for $n \geq n_0$ with $n_0 \in \mathbb{N}$ independent of τ and ω ,

$$\sup_{\omega \in \mathcal{F}_n} |K_n(a; \tau, \omega) - K_n(b; \tau, \omega)| \leq 1.5\sqrt{n}\|a - b\|.$$

By applying Knight's identity, we therefore have

$$\sup_{\omega \in \mathcal{F}_n} \sup_{\|\delta\| \leq A\sqrt{\log n}} |\hat{Z}_{n,\tau,\omega}(\delta) - Z_{n,\tau,\omega}(\delta)| = \sup_{\theta \in \Theta_n} \left| \sum_{t=1}^n \theta(t, Y_t) \right| + \text{O}_P(n^{-1}),$$

where

$$\Theta_n := \left\{ \theta(t, y) := \int_0^{n^{-1/2} \mathbf{c}'_t(\omega) d_j} (I\{y \leq s + q_\tau\} - I\{y \leq q_\tau\}) ds - f(q_\tau)(2n)^{-1} (\mathbf{c}'_t(\omega) d_j)^2 \right. \\ \left. \omega \in \mathcal{F}_n, j = 1, \dots, N \right\}.$$

In order to show that $\sup_{\theta \in \Theta_n} |\sum_{t=1}^n \theta(t, Y_t)| = \text{O}_P(r_n(\delta)^2)$, we apply the independent blocks argument with Θ_n defined above and $\eta_n := Dr_n(\delta)^2$ for a suitable constant D .

Due to the fact that $n^{(1-\delta)/(2+2\delta)}(\log n)^{3/2} \ll r_n(\delta)^2$ and that, by Lemma A.3,

$$\sup_{\theta \in \Theta_n} \sup_{t=1, \dots, n} |\theta(t, Y_t)| \leq Cn^{-1/2}(\log n)^{1/2} =: C_n,$$

almost surely, (i) in the independent blocks argument follows.

Next, a direct calculation shows that (ii) in the independent blocks argument holds with $V_n^2 := Cn^{-1/2}(\log n)^2$.

Finally, let us complete the independent blocks argument by establishing that for E_n defined in (A.2) we have $E_n = o(1)$. Observe that the bounds in Lemma A.3 imply

$$\sup_{\theta \in \Theta_n} \sup_{t=1, \dots, n} \mathbb{E}[|\theta(t, X_t)|] \leq C \log(n)^3 n^{-3/2} = o(n^{-1} r_n(\delta)^2).$$

Thus, we find that for sufficiently large n

$$\lambda_n := D \left(r_n(\delta)^2 / 3 - n \sup_{\theta \in \Theta_n} \sup_{t=1, \dots, n} \mathbb{E}[|\theta(t, X_t)|] \right) \leq Dr_n(\delta)^2 / 6.$$

Noting that $|\Theta_n| = Nn = o(n^6)$ direct calculations yield $E_n = o(1)$ for D in the definition of η_n being large enough. This completes the application of the independent blocks argument and shows that $\sup_{\theta \in \Theta_n} |\sum_{t=1}^n \theta(t, Y_t)| = \text{O}_P(r_n(\delta)^2)$.

Summing up, except for Lemma A.3 which is taken care of in the next section, we have proven (3.11). If we now prove Lemmas A.1 and A.2, (3.10) and (3.9), hence Theorem 3.1, follow. The purpose of Section A.1.2 below is to complete the proof of Theorem 3.1 by establishing the missing Lemmas A.1–A.3.

A.1.2. *Three auxiliary lemmas*

We now state and prove the three lemmas that have been used in the proof of Theorem 3.1. Lemma A.1 generalizes ideas from Pollard [37].

Lemma A.1. *Let $B_{a_n}(\mathbf{x})$ denote the closed ball (in \mathbb{R}^3) with center \mathbf{x} and radius $a_n > 0$. Assume that, for some sequence of real numbers $a_n = o(1)$,*

$$\Delta_n := \sup_{\omega \in \mathcal{F}_n} \sup_{\delta \in B_{a_n}(\delta_{n,\tau,\omega})} |\hat{Z}_{n,\tau,\omega}(\delta) - Z_{n,\tau,\omega}(\delta)| = o_{\mathbb{P}}(a_n^2).$$

Then, $\sup_{\omega \in \mathcal{F}_n} |\hat{\delta}_{n,\tau,\omega} - \delta_{n,\tau,\omega}| = o_{\mathbb{P}}(a_n)$.

Proof. Let $r_{n,\tau,\omega}(\delta) := \hat{Z}_{n,\tau,\omega}(\delta) - Z_{n,\tau,\omega}(\delta)$. Simple algebra and the explicit form (3.7) of $\delta_{n,\tau,\omega}$ yield

$$\hat{Z}_{n,\tau,\omega}(\delta) = \frac{1}{2}(\delta - \delta_{n,\tau,\omega})' \mathbf{Q}_{n,\tau,\omega} (\delta - \delta_{n,\tau,\omega}) - \frac{1}{2}(\delta_{n,\tau,\omega})' \mathbf{Q}_{n,\tau,\omega} \delta_{n,\tau,\omega} + r_{n,\tau,\omega}(\delta). \quad (\text{A.3})$$

Any $\delta \in \mathbb{R}^3 \setminus B_{a_n}(\delta_{n,\tau,\omega})$ with distance $l_n := \|\delta - \delta_{n,\tau,\omega}\| > a_n$ to $\delta_{n,\tau,\omega}$ can be represented as $\delta = \delta_{n,\tau,\omega} + l_{n,\tau,\omega} \mathbf{d}_{n,\tau,\omega}$, where $\mathbf{d}_{n,\tau,\omega} := l_{n,\tau,\omega}^{-1}(\delta - \delta_{n,\tau,\omega})$.

The point $\delta_{n,\tau,\omega}^* = \delta_{n,\tau,\omega} + a_n \mathbf{d}_{n,\tau,\omega}$ lies on the boundary of the ball $B_{a_n}(\delta_{n,\tau,\omega})$. The convexity of $\hat{Z}_{n,\tau,\omega}(\delta)$ therefore implies

$$\begin{aligned} & a_n l_{n,\tau,\omega}^{-1} \hat{Z}_{n,\tau,\omega}(\delta) + (1 - a_n l_{n,\tau,\omega}^{-1}) \hat{Z}_{n,\tau,\omega}(\delta_{n,\tau,\omega}) \\ & \geq \hat{Z}_{n,\tau,\omega}(\delta_{n,\tau,\omega}^*) = Z_{n,\tau,\omega}(\delta_{n,\tau,\omega}^*) + r_{n,\tau,\omega}(\delta_{n,\tau,\omega}^*) \\ & \geq \frac{1}{2} \mathbf{d}'_{n,\tau,\omega} \mathbf{Q}_{n,\tau,\omega} \mathbf{d}_{n,\tau,\omega} a_n^2 - \frac{1}{2} (\delta_{n,\tau,\omega})' \mathbf{Q}_{n,\tau,\omega} \delta_{n,\tau,\omega} - \Delta_n \\ & \geq \frac{1}{2} \mathbf{d}'_{n,\tau,\omega} \mathbf{Q}_{n,\tau,\omega} \mathbf{d}_{n,\tau,\omega} a_n^2 + \hat{Z}_{n,\tau,\omega}(\delta_{n,\tau,\omega}) - 2\Delta_n. \end{aligned}$$

Rearranging and taking the infimum over ω and δ , we obtain

$$\begin{aligned} & \inf_{\omega \in \mathcal{F}_n} \inf_{\delta: |\delta - \delta_{n,\tau,\omega}^*| > a_n} (\hat{Z}_{n,\tau,\omega}(\delta) - \hat{Z}_{n,\tau,\omega}(\delta_{n,\tau,\omega})) \\ & \geq \inf_{\omega \in \mathcal{F}_n} \inf_{\delta: |\delta - \delta_{n,\tau,\omega}| > a_n} l_{n,\tau,\omega} a_n^{-1} \left(\frac{1}{2} \mathbf{d}'_{n,\tau,\omega} \mathbf{Q}_{n,\tau,\omega} \mathbf{d}_{n,\tau,\omega} a_n^2 - 2\Delta_n \right). \end{aligned} \quad (\text{A.4})$$

By assumption, the smallest eigenvalue of $\mathbf{Q}_{n,\tau,\omega}$ is bounded away from zero uniformly in $\omega \in \mathcal{F}_n$, for n sufficiently large. Hence, $2\Delta_n < \frac{1}{2} \mathbf{d}'_{n,\tau,\omega} \mathbf{Q}_{n,\tau,\omega} \mathbf{d}_{n,\tau,\omega} a_n^2$ with probability tending to one, the right-hand side in (A.4) is strictly positive, and the minimum of $\hat{Z}_{n,\tau,\omega}(\delta)$ cannot be attained at any δ with $|\delta - \delta_{n,\tau,\omega}| > a_n$. \square

Lemma A.2. *Let (A1) hold, and $\delta_{n,\tau,\omega}$ be defined as in (3.7). Then, for any $\tau \in (0, 1)$ for which $f(q_\tau) > 0$, there exists a constant A such that*

$$\lim_{n \rightarrow \infty} P\left(\sup_{\omega \in \mathcal{F}_n} \|\delta_{n,\tau,\omega}\| > A\sqrt{\log n}\right) = 0.$$

Proof. Denote by $\|\mathbf{x}\|_\infty$ the sup-norm of \mathbf{x} . Since, for $\mathbf{x} \in \mathbb{R}^3$, $\sqrt{3}\|\mathbf{x}\|_\infty \geq \|\mathbf{x}\|$, it suffices to prove that

$$\lim_{n \rightarrow \infty} P\left(\sup_{\omega \in \mathcal{F}_n} \|\delta_{n,\tau,\omega}\|_\infty > 3^{-1/2}A\sqrt{\log n}\right) = 0.$$

Next note that $\sqrt{n} \sup_{\omega \in \mathcal{F}_n} \|\delta_{n,\tau,\omega}\|_\infty = \sup_{\theta \in \Theta_n} |\sum_{t=1}^n \theta(t, Y_t)|$, where

$$\Theta_n := \{\theta(t, y) := f(q_\tau)^{-1} c_{t,k}(\omega)(\tau - I\{y \leq q_\tau\}) | k = 1, 2, 3, \omega \in \mathcal{F}_n\},$$

with $(c_{t,1}(\omega), c_{t,2}(\omega), c_{t,3}(\omega)) := (1, \cos(\omega t), \sin(\omega t))$.

We apply the independent blocks argument described in Section A.1.1, with Θ_n defined above and $\eta_n := 3^{-1/2}An^{1/2}(\log n)^{1/2}$ with a suitably chosen constant A . To this end, remark that (i) in the independent blocks argument holds for A large enough, because we have, almost everywhere,

$$\sup_{\theta \in \Theta_n} \sup_{t=1, \dots, \mu_n m_n} |\theta(t, Y_t)| \leq \frac{2}{f(q_\tau)} =: C_n$$

which implies,

$$\sup_{\theta \in \Theta_n} \left| \sum_{t \in R_n} \theta(t, Y_t) \right| \leq \frac{2m_n}{f(q_\tau)} \quad \text{a.e.}$$

Regarding (ii) from the independent blocks argument note that for all $\theta \in \Theta_n$

$$\begin{aligned} \text{Var}\left(\sum_{t \in S_i} \theta(t, X_t)\right) &= \sum_{s \in S_i} \sum_{t \in S_i} E[\theta(s, X_s)\theta(t, X_t)] \\ &= (\mathbf{Q}_{n,\tau,\omega})^{-2} \sum_{|l| < m_n} \gamma_l(\tau, \tau) \sum_{j=2(i-1)m_n+1+(0 \vee l)}^{(2i-1)m_n+(l \wedge 0)} c_{j+l,k}(\omega)c_{j,k}(\omega)'. \end{aligned}$$

Since $|c_{t,k}(\omega)| \leq 1$ and

$$\sum_{l=-\infty}^{\infty} |\gamma_l(\tau, \tau)| \leq 1 + C_1 \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} l^{-\delta} =: C < \infty,$$

we have

$$\sum_{i=1}^{\mu_n} \text{Var}\left(\sum_{t \in S_i} \theta(t, X_t)\right) \leq 4C(f(q_\tau))^{-2}n =: V_n^2.$$

Direct calculations show that E_n defined in (A.2) of the independent blocks argument satisfies $E_n = o(1)$. This completes the independent blocks argument and concludes the proof. \square

Lemma A.3. For the Fourier frequencies $\omega \in \mathcal{F}_n$, let

$$H_t(\boldsymbol{\delta}; \tau, \omega) := \int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\boldsymbol{\delta}} (I\{X_t \leq s + q_\tau\} - I\{X_t \leq q_\tau\}) ds \quad (\text{A.5})$$

and define

$$W_{t,n}(\omega, \boldsymbol{\delta}) := H_t(\boldsymbol{\delta}; \tau, \omega) - f(q_\tau)(2n)^{-1}(\mathbf{c}'_t(\omega)\boldsymbol{\delta})^2. \quad (\text{A.6})$$

Then, for some finite constant C (independent of t, t_1, t_2) and n large enough,

$$\begin{aligned} \sup_{\omega \in \mathcal{F}_n} \sup_t |E[W_{t,n}(\omega, \boldsymbol{\delta})]| &\leq C \|\boldsymbol{\delta}\|^3 n^{-3/2}, \\ \sup_{\omega \in \mathcal{F}_n} \sup_t |W_{t,n}(\omega, \boldsymbol{\delta})| &\leq C(n^{-1/2} \|\boldsymbol{\delta}\| + n^{-1} \|\boldsymbol{\delta}\|^2) \end{aligned} \quad (\text{A.7})$$

almost surely, and

$$\sup_{\omega \in \mathcal{F}_n} |E[W_{t_1,n}(\omega, \boldsymbol{\delta})W_{t_2,n}(\omega, \boldsymbol{\delta})]| \leq C(\|\boldsymbol{\delta}\|^4 \vee 1)(n^{-3/2}I\{t_1 = t_2\} + n^{-2}I\{t_1 \neq t_2\}). \quad (\text{A.8})$$

Proof. First, note that

$$\begin{aligned} E[H_t(\boldsymbol{\delta}; \tau, \omega)] &= E\left[\int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\boldsymbol{\delta}} (I\{X_t \leq u + q_\tau\} - I\{X_t \leq q_\tau\}) du\right] \\ &= \int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\boldsymbol{\delta}} (f(q_\tau)u + r_4(u, \tau)) du = \frac{f(q_\tau)}{2n}(\mathbf{c}'_t(\omega)\boldsymbol{\delta})^2 + r_1(\tau, \omega), \end{aligned} \quad (\text{A.9})$$

where $|r_4(u, \tau)| \leq C_3u^2$, hence $|r_1(\omega, \tau)| \leq C_4\|\boldsymbol{\delta}\|^3n^{-3/2}$. Next, observe that

$$\begin{aligned} &E[H_t(\boldsymbol{\delta}; \tau, \omega)^2] \\ &= E\left[\int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\boldsymbol{\delta}} \int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\boldsymbol{\delta}} (I\{X_t \leq u + q_\tau\} - I\{X_t \leq q_\tau\}) \right. \\ &\quad \left. \times (I\{X_t \leq v + q_\tau\} - I\{X_t \leq q_\tau\}) du dv\right] \\ &= E\left[\int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\boldsymbol{\delta}} \int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\boldsymbol{\delta}} (I\{X_t \leq (u \wedge v) + q_\tau\} - I\{X_t \leq (u \wedge 0) + q_\tau\} \right. \\ &\quad \left. - I\{X_t \leq (v \wedge 0) + q_\tau\} + I\{X_t \leq q_\tau\}) du dv\right] \end{aligned}$$

$$= \int_0^{n^{-1/2}c'_t(\omega)\delta} \int_0^{n^{-1/2}c'_t(\omega)\delta} (u \wedge v - u \wedge 0 - v \wedge 0) f(q_\tau) + r_2(u, v, \tau) \, du \, dv \quad (\text{A.10})$$

$$= 3^{-1}n^{-3/2} f(q_\tau) |c'_t(\omega)\delta|^3 + r_3(\omega, \tau), \quad (\text{A.11})$$

where $|r_2(u, v, \tau)| \leq C_1(u^2 + v^2)$, hence $|r_3(\omega, \tau)| \leq C_2\|\delta\|^4 n^{-2}$. Equality (A.10) follows via a Taylor expansion, (A.11) from the fact that $\int_0^x \int_0^x (u \wedge v - u \wedge 0 - v \wedge 0) \, du \, dv = \frac{1}{3}|x|^3$. Similarly, for $t_1 \neq t_2$, but from the same block (otherwise H_{t_1} and H_{t_2} are independent and the previously derived approximation of their expectations can be used for the proof),

$$\begin{aligned} & \mathbb{E}[H_{t_1}(\delta; \tau, \omega) H_{t_2}(\delta; \tau, \omega)] \\ &= \mathbb{E} \left[\int_0^{n^{-1/2}c'_{t_1}(\omega)\delta} \int_0^{n^{-1/2}c'_{t_2}(\omega)\delta} (I\{X_{t_1} \leq u + q_\tau\} - I\{X_{t_1} \leq q_\tau\}) \right. \\ & \qquad \qquad \qquad \left. \times (I\{X_{t_2} \leq v + q_\tau\} - I\{X_{t_2} \leq q_\tau\}) \, du \, dv \right] \\ &= \int_0^{n^{-1/2}c'_{t_1}(\omega)\delta} \int_0^{n^{-1/2}c'_{t_2}(\omega)\delta} F_{t_2-t_1}(u + q_{n,\tau}, v + q_\tau) - F_{t_2-t_1}(q_\tau, v + q_\tau) \\ & \qquad \qquad \qquad - F_{t_2-t_1}(u + q_\tau, q_\tau) + F_{t_2-t_1}(q_{n,\tau}, q_\tau) \, du \, dv \\ &= \int_0^{n^{-1/2}c'_{t_1}(\omega)\delta} \int_0^{n^{-1/2}c'_{t_1}(\omega)\delta} r_6(u, v, \tau) \, du \, dv = r_7(\omega, \tau), \end{aligned} \quad (\text{A.12})$$

where $|r_6(u, v, \tau)| \leq C_6(u^2 + v^2)$, hence $|r_7(u, v, \tau)| \leq C_7\|\delta\|^4 n^{-2}$; equality (A.12) follows via a Taylor expansion and some straightforward algebra. This completes the proof. \square

A.2. Details for the proof of (3.20)

We now turn to the proof of Theorem 3.3. Sections A.2.1–A.2.2 contain the proofs of (3.22) and (3.23), which are basic in establishing that theorem. Some auxiliary results used in the proofs are collected in Section A.2.3 under the form of Lemmas A.4 and A.5. Denote by \hat{F}_n the empirical distribution function of Y_1, \dots, Y_n . Throughout this section, the notation from Section 3.2 is used.

A.2.1. Proof of (3.22)

Plugging into (3.22) the definition of $\hat{Z}_{n,\tau,\omega,1}(\delta)$, it remains to show that [recall that $c_{t,1}(\omega) = 1$]

$$\begin{aligned} & \max_{k=2,3} \sup_{\omega \in \mathcal{F}_n} \left| n^{-1/2} \sum_{t=1}^n c_{t,k}(\omega) (I\{U_t \leq F(\hat{F}_n^{-1}(\tau))\} - I\{U_t \leq \tau\}) \right| \\ &= \text{Op}(n^{-1/4} m_n^{1/2} \log n) \end{aligned} \quad (\text{A.13})$$

and

$$\begin{aligned} & \left| n^{-1/2} \sum_{t=1}^n (I\{U_t \leq F(\hat{F}_n^{-1}(\tau))\} - I\{U_t \leq \tau\}) - \sqrt{n}(F(\hat{F}_n^{-1}(\tau)) - \tau) \right| \\ & = \text{Op}(n^{-1/4} m_n^{1/2} \log n). \end{aligned} \quad (\text{A.14})$$

First, consider (A.13). Since, by Lemma A.4, $|F(\hat{F}_n^{-1}(\tau)) - \tau| = \text{Op}(n^{-1/2} \sqrt{\log n})$, we obtain

$$\begin{aligned} & \sup_{\omega \in \mathcal{F}_n} \left| n^{-1/2} \sum_{t=1}^n c_{t,k}(\omega) (I\{U_t \leq F(\hat{F}_n^{-1}(\tau))\} - I\{U_t \leq \tau\}) \right| \\ & \leq \sup_{\omega \in \mathcal{F}_n} n^{-1/2} \sup_{|x-\tau| \leq n^{-1/2} \log n} \left| \sum_{t=1}^n c_{t,k}(\omega) (I\{U_t \leq x\} - I\{U_t \leq \tau\} - (x - \tau)) \right| \\ & \quad + \sup_{\omega \in \mathcal{F}_n} n^{-1} \log n \left| \sum_{t=1}^n c_{t,k}(\omega) \right| \end{aligned} \quad (\text{A.15})$$

for $k = 2, 3$, with probability tending to one. The second term in (A.15) vanishes, because, for all $\omega \in \mathcal{F}_n$, $\sum_{t=1}^n \cos(\omega t) = \sum_{t=1}^n \sin(\omega t) = 0$. In order to bound the first term, cover the set $\mathcal{Z} := \{u: |u - \tau| \leq n^{-1/2} \log n\}$ with $N < n$ balls of radius $1/n$ and centers $u_1, \dots, u_N \in \mathcal{Z}$, and define $\mathbb{G}_{n,\omega,k}(u) := n^{-1/2} \sum_{t=1}^n c_{t,k}(\omega) (I\{U_t \leq u\} - u)$. Then, almost surely,

$$\begin{aligned} & \sup_j \sup_{\omega \in \mathcal{F}_n} \sup_{|u-u_j| \leq n^{-1}} |\mathbb{G}_{n,\omega,k}(u) - \mathbb{G}_{n,\omega,k}(u_j)| \\ & \leq \sup_{u \in \mathcal{Z}} n^{-1/2} \sum_{t=1}^n (I\{U_t \leq u + 2n^{-1}\} - I\{U_t \leq u - 2n^{-1}\} + 4n^{-1}) + \text{O}(n^{-1/2}) \\ & \leq \sqrt{n} \sup_{j=1, \dots, N} |\hat{F}_{n,U}(u_j + 2n^{-1}) - \hat{F}_{n,U}(u_j - 2n^{-1}) - 4n^{-1}| + \text{O}(n^{-1/2}), \end{aligned}$$

where the latter bound, in view of Lemma A.7, is $\text{Op}(n^{(1-\delta)/(2+2\delta)} \log n)$. Thus,

$$\sup_j \sup_{\omega \in \mathcal{F}_n} \sup_{|u-u_j| \leq n^{-1}} |\mathbb{G}_{n,\omega,k}(u) - \mathbb{G}_{n,\omega,k}(u_j)| = \text{Op}(n^{(1-\delta)/(2+2\delta)} \log n), \quad k = 2, 3,$$

and therefore

$$\begin{aligned} & \max_{k=2,3} \sup_{\omega \in \mathcal{F}_n} \left| n^{-1/2} \sum_{t=1}^n c_{t,k}(\omega) (I\{U_t \leq F(F_n^{-1}(\tau))\} - I\{U_t \leq \tau\}) \right| \\ & \leq \max_{k=2,3} \sup_{j=1, \dots, N} \sup_{\omega \in \mathcal{F}_n} |\mathbb{G}_{n,\omega,k}(u_j) - \mathbb{G}_{n,\omega,k}(\tau)| + \text{Op}(n^{(1-\delta)/(2+2\delta)} \log n). \end{aligned} \quad (\text{A.16})$$

Now, by construction, $\max_j |u_j - \tau| \leq n^{-1/2} \log n$.

Moreover,

$$\max_{k=2,3} \sup_{j=1,\dots,N} \sup_{\omega \in \mathcal{F}_n} |\mathbb{G}_{n,\omega,k}(u_j) - \mathbb{G}_{n,\omega,k}(\tau)| = \sup_{\theta \in \Theta_n} \left| \sum_{t=1}^n \theta(t, U_t) \right|,$$

where

$$\Theta_n := \{ \theta(t, u) := n^{-1/2} c_{t,k}(\omega) (I\{u \leq u_j\} - I\{u \leq \tau\} - (u_j - \tau)) \mid \omega \in \mathcal{F}_n, j = 1, \dots, N, k = 2, 3 \}.$$

Apply the independent blocks argument with $\eta_n := \tilde{C} n^{-1/2} \sqrt{\log n} (n^{1/2} m_n \log n)^{1/2}$, where \tilde{C} is a large enough constant, and Θ_n defined above. Direct calculations show that

$$\sup_{\theta \in \Theta_n} |\theta(t, U_t)| \leq 2n^{-1/2} =: C_n \quad \text{a.s.},$$

which yields (i) from the independent blocks argument, since $m_n C_n \sim m_n n^{-1/2} \log n \ll \eta_n$. Additionally, for some finite constant C independent of $\theta \in \Theta_n$ $E|\theta(t, U_t)|^2 \leq C n^{-3/2} \log n$, and $E[\theta(t_1, U_{t_1})\theta(t_2, U_{t_2})] \leq C n^{-2} (\log n)^2$, and thus

$$\begin{aligned} \sup_{\theta \in \Theta_n} \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in \mathcal{S}_j} \theta(t, U_t) \right) &\leq \tilde{C} n^{-1/2} \log n =: V_n^2, \\ \sup_{\theta \in \Theta_n} \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in \mathcal{T}_j} \theta(t, U_t) \right) &\leq V_n^2. \end{aligned}$$

Hence, (ii) from the independent blocks argument holds and the fact that $E_n = o(1)$ with E_n defined in (A.2) follows by a simple calculation. The independent blocks argument thus yields

$$\max_{k=2,3} \sup_{j=1,\dots,N} \sup_{\omega \in \mathcal{F}_n} |\mathbb{G}_{n,\omega,k}(u_j) - \mathbb{G}_{n,\omega,k}(\tau)| = O_P(n^{-1/4} m_n \log n) = O_P(n^{(1-\delta)/(4+4\delta)} \log n).$$

Together with (A.16), this establishes (A.13). Turning to (A.14), Lemmas A.4 and A.7 yield

$$\begin{aligned} &\left| n^{-1/2} \sum_{t=1}^n (I\{U_t \leq F(\hat{F}_n^{-1}(\tau))\} - I\{U_t \leq \tau\} - (F(\hat{F}_n^{-1}(\tau)) - \tau)) \right| \\ &\leq \sup_{|u-\tau| \leq n^{-1/2} \log n} \left| n^{-1/2} \sum_{t=1}^n (I\{U_t \leq u\} - I\{U_t \leq \tau\} - (u - \tau)) \right| \\ &= n^{1/2} \sup_{|u-\tau| \leq n^{-1/2} \log n} \left| \hat{F}_{n,U}(u) - \hat{F}_{n,U}(\tau) - (u - \tau) \right| \\ &= O_P(n^{-1/2} (m_n \vee n^{1/4}) \log n) \leq O_P(n^{-1/4} m_n^{1/2} \log n). \end{aligned}$$

A.2.2. Proof of (3.23)

Observe the decomposition

$$\begin{aligned}
 & \hat{Z}_{n,\tau,\omega,2}(\delta) - \sum_{t=1}^n \int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\delta} (I\{U_t \leq s + \tau\} - I\{U_t \leq \tau\}) \, ds \\
 &= \sum_{t=1}^n \int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\delta} (I\{U_t \leq F(\hat{F}_n^{-1}(s + \tau))\} - I\{U_t \leq F(\hat{F}_n^{-1}(\tau))\} \\
 &\quad - I\{U_t \leq s + \tau\} + I\{U_t \leq \tau\}) \, ds \\
 &= \int_{-2\|\delta\|}^{2\|\delta\|} n^{-1/2} \sum_{t=1}^n (I\{U_t \leq F(\hat{F}_n^{-1}(n^{-1/2}s + \tau))\} - I\{U_t \leq F(\hat{F}_n^{-1}(\tau))\} \\
 &\quad - I\{U_t \leq n^{-1/2}s + \tau\} + I\{U_t \leq \tau\}) \\
 &\quad \times (I\{0 \leq s \leq \mathbf{c}'_t(\omega)\delta\} - I\{0 \geq s \geq \mathbf{c}'_t(\omega)\delta\}) \, ds \\
 &= A_n^{(1)} - A_n^{(2)} - A_n^{(3)} + A_n^{(4)}, \quad \text{say,}
 \end{aligned}$$

where

$$A_n^{(1)} := \int_{-2\|\delta\|}^{2\|\delta\|} (\mathbb{S}_{n,\omega,\delta}^{(+)}(F(\hat{F}_n^{-1}(n^{-1/2}s + \tau)), n^{-1/2}s + \tau; s) - \mathbb{S}_{n,\omega,\delta}^{(+)}(F(\hat{F}_n^{-1}(\tau)), \tau; s)) \, ds,$$

$$\begin{aligned}
 A_n^{(2)} &:= \int_{-2\|\delta\|}^{2\|\delta\|} n^{-1/2} \sum_{t=1}^n [(F(\hat{F}_n^{-1}(n^{-1/2}s + \tau)) - (n^{-1/2}s + \tau)) - (F(\hat{F}_n^{-1}(\tau)) - \tau)] \\
 &\quad \times I\{0 \leq s \leq \mathbf{c}'_t(\omega)\delta\} \, ds,
 \end{aligned}$$

$$A_n^{(3)} := \int_{-2\|\delta\|}^{2\|\delta\|} (\mathbb{S}_{n,\omega,\delta}^{(-)}(F(\hat{F}_n^{-1}(n^{-1/2}s + \tau)), n^{-1/2}s + \tau; s) - \mathbb{S}_{n,\omega,\delta}^{(-)}(F(\hat{F}_n^{-1}(\tau)), \tau; s)) \, ds,$$

$$\begin{aligned}
 A_n^{(4)} &:= \int_{-2\|\delta\|}^{2\|\delta\|} n^{-1/2} \sum_{t=1}^n [(F(\hat{F}_n^{-1}(n^{-1/2}s + \tau)) - (n^{-1/2}s + \tau)) - (F(\hat{F}_n^{-1}(\tau)) - \tau)] \\
 &\quad \times I\{0 \geq s \geq \mathbf{c}'_t(\omega)\delta\} \, ds,
 \end{aligned}$$

and

$$\mathbb{S}_{n,\omega,\delta}^{(+)}(u, v; s) := n^{-1/2} \sum_{t=1}^n (I\{U_t \leq u\} - I\{U_t \leq v\} - (u - v)) I\{0 \leq s \leq \mathbf{c}'_t(\omega)\delta\},$$

$$\mathbb{S}_{n,\omega,\delta}^{(-)}(u, v; s) := n^{-1/2} \sum_{t=1}^n (I\{U_t \leq u\} - I\{U_t \leq v\} - (u - v)) I\{0 \geq s \geq \mathbf{c}'_t(\omega)\delta\}.$$

First note that, in view of Lemma A.4,

$$\begin{aligned}
 |A_n^{(2)}| &\leq 4\|\delta\|\sqrt{n} \sup_{|u-\tau|\leq 2\|\delta\|/\sqrt{n}} |F(\hat{F}_n^{-1}(u)) - u - (F(\hat{F}_n^{-1}(\tau)) - \tau)| \\
 &= O_P(\rho_n(2(\log n)^{1/2}n^{-1/2}, \delta)\sqrt{n \log n}) \\
 &= O_P((n^{-1/4}(\log n)^{5/4}) \vee (n^{(1-\delta)/(2+2\delta)}(\log n)^{3/2})) \\
 &= O_P(n^{-1/4}m_n^{1/2} \log n).
 \end{aligned}$$

A similar bound can be obtained for $A_n^{(4)}$. Next, for sufficiently large n , still in view of Lemma A.4,

$$\begin{aligned}
 &\int_{-2\|\delta\|}^{2\|\delta\|} |\mathbb{S}_{n,\omega,\delta}^{(+)}(F(\hat{F}_n^{-1}(n^{-1/2}s + \tau)), n^{-1/2}s + \tau; s)| ds \\
 &\leq \int_{-2\|\delta\|}^{2\|\delta\|} \sup_{v:|v-\tau|\leq 2\|\delta\|/\sqrt{n}} |\mathbb{S}_{n,\omega,\delta}^{(+)}(F(\hat{F}_n^{-1}(v)), v; s)| ds \\
 &\leq \int_{-2\|\delta\|}^{2\|\delta\|} \sup_{v:|v-\tau|\leq 2\|\delta\|/\sqrt{n}} \sup_{u:|u-v|\leq n^{-1/2} \log n} |\mathbb{S}_{n,\omega,\delta}^{(+)}(u, v; s)| ds \\
 &\leq 4\|\delta\| \sup_{s:|s|\leq 2\|\delta\|} \sup_{v:|v-\tau|\leq 2\|\delta\|/\sqrt{n}} \sup_{u:|u-v|\leq n^{-1/2} \log n} |\mathbb{S}_{n,\omega,\delta}^{(+)}(u, v; s)|.
 \end{aligned}$$

Similar inequalities hold for $\int_{-2\|\delta\|}^{2\|\delta\|} |\mathbb{S}_{n,\omega,\delta}^{(+)}(F(\hat{F}_n^{-1}(\tau)), \tau; s)| ds$. Let us show that

$$\begin{aligned}
 &\sup_{\omega \in \mathcal{F}_n} \sup_{\delta: \|\delta\| \leq A\sqrt{\log n}} \sup_{s: |s| \leq 2\|\delta\|} \sup_{\substack{(u,v): |v-\tau| \leq 2\|\delta\|/\sqrt{n} \\ |u-v| \leq n^{-1/2} \log n}} |\mathbb{S}_{n,\omega,\delta}^{(+)}(u, v; s)| \\
 &= O_P(n^{-1/4}m_n^{1/2} \log n).
 \end{aligned} \tag{A.17}$$

For any $C > 0$, we have $I\{0 \leq s \leq \mathbf{c}'_t \delta\} = I\{0 \leq Cs \leq C\mathbf{c}'_t \delta\}$. Thus, it is sufficient to consider vectors δ satisfying $\|\delta\| = 1$. Since, by definition, $\|\mathbf{c}_t(\omega)\| = \sqrt{2}$, it also is sufficient to consider values of s in the interval $[0, \sqrt{2}]$. Finally, note that if

$$I\{0 \leq s_1 \leq \mathbf{c}'_t \delta_1\} = I\{0 \leq s_2 \leq \mathbf{c}'_t \delta_2\} \quad \text{for all } t = 1, \dots, n,$$

then also $\mathbb{S}_{n,\omega,\delta_1}^{(+)}(u, v; s_1) = \mathbb{S}_{n,\omega,\delta_2}^{(+)}(u, v; s_2)$. We thus can rewrite (A.17) as

$$G_n := \sup_{T \in \mathcal{M}_n} \sup_{\substack{(u,v): |v-\tau| \leq 2\|\delta\|/\sqrt{n} \\ |u-v| \leq n^{-1/2} \log n}} |\tilde{\mathbb{S}}_n^{(+)}(u, v; T)| = O_P(n^{-1/4}m_n^{1/2} \log n), \tag{A.18}$$

where

$$\mathcal{M}_n := \{T = \{t \in \{1, \dots, n\}: 0 \leq s \leq c'_t \delta\} | \omega \in \mathcal{F}_n, s \in [0, \sqrt{2}], \|\delta\| = 1\} \quad (\text{A.19})$$

and

$$\bar{\mathbb{S}}_n^{(+)}(u, v; T) := n^{-1/2} \sum_{t \in T} (I\{U_t \leq u\} - u - (I\{U_t \leq v\} - v)).$$

In order to prove (A.17) (equivalently, (A.18)), define the set

$$\mathcal{Z}_n := \{(u, v) \in \mathbb{R}^2: |u - v| \leq n^{-1/2} \log n, |v - \tau| \leq 2An^{-1/2} \sqrt{\log n}\}$$

and cover it with $N < n^2$ balls of radius $1/n$ with centers $z_1, \dots, z_N \in \mathcal{Z}_n$. For any (u, v) in \mathcal{Z}_n there exists a j such that $\|(u, v) - (z_{1j}, z_{2j})\| \leq 1/n$ and, letting $z_j := (z_{1j}, z_{2j})$, we have, almost surely,

$$\begin{aligned} \rho(u, v, z_j) &:= |\bar{\mathbb{S}}_n^{(+)}(u, v; T) - \bar{\mathbb{S}}_n^{(+)}(z_{1j}, z_{2j}; T)| \\ &\leq n^{-1/2} \sum_{t=1}^n (I\{|U_t - z_{1j}| \leq n^{-1}\} + I\{|U_t - z_{2j}| \leq n^{-1}\} + |u - z_{1j}| + |v - z_{2j}|) \\ &\leq 2n^{-1/2} + n^{-1/2} \sum_{t=1}^n (I\{|U_t - z_{1j}| \leq n^{-1}\} + I\{|U_t - z_{2j}| \leq n^{-1}\}) \\ &= 2n^{-1/2} + n^{-1/2} \sum_{t=1}^n (I\{U_t \leq z_{1j} + n^{-1}\} - I\{U_t < z_{1j} - n^{-1}\} \\ &\quad + I\{U_t \leq z_{2j} + n^{-1}\} - I\{U_t < z_{2j} - n^{-1}\}) \\ &\leq n^{1/2} (\hat{F}_{n,U}(z_{1j} + 2n^{-1}) - (z_{1j} + 2n^{-1}) - (\hat{F}_{n,U}(z_{1j} - 2n^{-1}) - (z_{1j} - 2n^{-1}))) \\ &\quad + \hat{F}_{n,U}(z_{2j} + 2n^{-1}) - (z_{2j} + 2n^{-1}) \\ &\quad - (\hat{F}_{n,U}(z_{2j} - 2n^{-1}) - (z_{2j} - 2n^{-1}))) + O(n^{-1/2}), \end{aligned}$$

where $\hat{F}_{n,U}$ denotes the empirical distribution function of U_1, \dots, U_n . From Lemma A.7,

$$\begin{aligned} &\sup_{z_1, \dots, z_N} \sup_{\substack{(u,v) \in [0,1]^2 \\ \|z_j - (u,v)\| < n^{-1}}} |\rho(u, v, z_j)| \\ &\leq n^{1/2} \sup_{z_j \in \mathcal{Z}} |\hat{F}_{n,U}(z_{1j} + 2n^{-1}) - \hat{F}_{n,U}(z_{1j} - 2n^{-1}) - 4n^{-1}| \\ &\quad + n^{1/2} \sup_{z_j \in \mathcal{Z}} |\hat{F}_{n,U}(z_{2j} + 2n^{-1}) - \hat{F}_{n,U}(z_{2j} - 2n^{-1}) - 4n^{-1}| + O(n^{-1/2}) \\ &= O_P(m_n n^{-1/2} \log n). \end{aligned}$$

With this, we have, for G_n defined in (A.18),

$$G_n \leq \sup_{T \in \mathcal{M}_n} \sup_{z_1, \dots, z_N} |\bar{\mathbb{S}}_n^{(+)}(z_{1j}, z_{2j}; T)| + O_P(m_n n^{-1/2} \log n).$$

Note that

$$\sup_{T \in \mathcal{M}_n} \sup_{z_1, \dots, z_N} |\bar{\mathbb{S}}_n^{(+)}(z_{1j}, z_{2j}; T)| = \sup_{\theta \in \Theta_n} \left| \sum_{t=1}^n \theta(t, U_t) \right|,$$

where

$$\Theta_n := \{ \theta(t, w) := n^{-1/2} I\{t \in T\} (I\{w \leq u\} - u - (I\{w \leq v\} - v)) \mid (u, v) = z_1, \dots, z_N, T \in \mathcal{M}_n \}.$$

We apply the independent blocks argument with Θ_n defined above and $\eta_n := n^{-1/4} m_n \log n$; note that $|\mathcal{M}_n| \leq (n + 1)^4$ by Lemma A.5 and $N < n^2$ by construction.

Simple computations yield (recall that $z_j \in \mathcal{Z}$)

$$\sup_{\theta \in \Theta_n} \sup_{t=1, \dots, n} |\theta(t, U_t)| \leq 2n^{-1/2}, \tag{A.20}$$

$$\sup_{\theta \in \Theta_n} \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in \mathcal{S}_j} \theta(t, U_t) \right) \leq C n^{-1/2} \log n =: V_n^2, \tag{A.21}$$

$$\sup_{\theta \in \Theta_n} \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in T_j} \theta(t, U_t) \right) \leq V_n^2.$$

Thus (i) from the independent blocks argument follows from (A.20) since

$$n^{-1/4} m_n^{1/2} \log n \gg m_n n^{-1/2}.$$

Moreover, (A.21) yields (ii), again from the independent blocks argument. Finally, verify $E_n = o(1)$ with E_n defined in (A.2) by direct calculation to conclude

$$\sup_{T \in \mathcal{M}_n} \sup_{z_1, \dots, z_N} |\bar{\mathbb{S}}_n^{(+)}(z_{1j}, z_{2j}; T)| = O_P(n^{-1/4} m_n \log n).$$

A similar result can be derived for $\mathbb{S}_{n, \omega, \delta}^{(-)}$. This completes the proof.

A.2.3. Two auxiliary lemmas

We now state and prove Lemmas A.4 and A.5 that have been used in Sections A.2.1 and A.2.2.

Lemma A.4. (i) Assume that, for any $\gamma > 0$ such that $[\alpha - \gamma, \beta - \gamma] \subset (0, 1)$,

$$\inf_{u \in [\alpha - \gamma, \beta + \gamma]} f(F^{-1}(u)) > 0.$$

Then, $\sup_{u \in [\alpha, \beta]} |F(\hat{F}_n^{-1}(u)) - u| = \text{Op}(n^{-1/2} \sqrt{\log n})$.

(ii) Define $\rho_n(a_n, \delta) := (\frac{a_n + n^{1/(1+\delta)} a_n^2 \log n}{n} \log n)^{1/2} \vee (n^{-\delta/(1+\delta)} \log n)$. If $\rho_n(a_n, \delta)$ is $o(a_n)$, then

$$\sup_{u, v \in [\alpha, \beta], |u-v| \leq a_n} |F(\hat{F}_n^{-1}(u)) - F(\hat{F}_n^{-1}(v)) - (u - v)| = \text{Op}(\rho_n(2a_n, \delta)).$$

Proof. Elementary analytic considerations show that, for any nondecreasing function g , $\sup_{w \in [u, v]} |g(w) - w| \leq a_n$ implies $\sup_{w \in [u+2a_n, v-2a_n]} |g^{-1}(w) - w| \leq a_n$. This, for $g(w) = \hat{F}_n(F^{-1}(w))$, $u = \alpha - \delta$, and $v = \beta + \delta$, along with Lemma A.7, yields part (i) of the lemma. Turning to part (ii), by Lemma A.7, for any bounded $\mathcal{Y} \subset \mathbb{R}$,

$$\sup_{y \in \mathcal{Y}} \sup_{|x| \leq a_n} |\hat{F}_n(y+x) - \hat{F}_n(y) - F(x+y) + F(y)| = \text{Op}(\rho_n(a_n, \delta)).$$

Since, for any $A \subset [0, 1]$, $\sup_{u, v \in A} |F^{-1}(u) - F^{-1}(v)| \leq C_A |u - v|$ for some positive constant C_A ,

$$\sup_{u, v \in [\alpha - \gamma, \beta + \gamma], |u-v| \leq a_n} |\hat{F}_n(F^{-1}(u)) - \hat{F}_n(F^{-1}(v)) - (u - v)| = \text{Op}(\rho_n(a_n, \delta)).$$

We now apply Lemma 3.5 from Wendler [41], with $F(w) = \hat{F}_n(F^{-1}(w))$, $l = a_n$, $c = D\rho_n(a_n, \delta)$, $C_1 = \hat{F}_n(F^{-1}(\alpha - \gamma))$, $C_2 = \hat{F}_n(F^{-1}(\beta + \gamma))$. By assumption, $l + 2c = a_n + 2D\rho_n(a_n, \delta) \leq 2a_n$ for sufficiently large n . By Lemma A.7, we have $C_1 = \alpha + \delta + \text{op}(1)$, $C_2 = \beta - \delta + \text{op}(1)$ and, for any strictly increasing continuous function G , $(F \circ G^{-1})^{-1} = G \circ F^{-1}$ (see Exercise 3 in Chapter 1 of Shorack and Wellner [38]); moreover, by part (i) of the present lemma, $F(\hat{F}_n^{-1}(u))$ is uniformly close to u for large n . Hence,

$$\sup_{u, v \in [\alpha, \beta], |u-v| \leq 2a_n} |F(\hat{F}_n^{-1}(u)) - F_n(\hat{F}_n^{-1}(v)) - (u - v)| > D\rho_n(2a_n, \delta)$$

implies

$$\sup_{u, v \in [\alpha - \delta, \beta + \delta], |u-v| \leq a_n} |\hat{F}_n(F^{-1}(u)) - \hat{F}_n(F^{-1}(v)) - (u - v)| > D\rho_n(a_n, \delta).$$

Part (ii) of the lemma follows on letting D tend to infinity. □

Lemma A.5. The cardinality of the set \mathcal{M}_n defined in (A.19) is at most $(n + 1)^4$.

Proof. Fix a Fourier frequency $\omega_{j,n} = 2\pi j/n \in \mathcal{F}_n$ and note that

$$\mathbf{c}_t(\omega_{j,n})' \boldsymbol{\delta} = \delta_1 + \delta_2 \cos(\omega_{j,n}t) + \delta_3 \sin(\omega_{j,n}t) = \delta_1 + \sqrt{\delta_2^2 + \delta_3^2} \cos(\omega_{j,n}t + \phi(\delta_2, \delta_3)),$$

where $\phi(\delta_2, \delta_3) \in [0, 2\pi]$ denotes a phase shift. Moreover, for any $v \in [0, 1]$, noting that the mapping $x \mapsto \cos(\omega_{j,n}x + \phi)$ is n/j -periodic,

$$\begin{aligned} & \left\{ t \in \{1, \dots, n\} \mid 0 \leq v \leq \delta_1 + \sqrt{\delta_2^2 + \delta_3^2} \cos(\omega_{j,n}t + \phi) \right\} \\ &= \left\{ \frac{nk}{j} + w \mid w \in [C_{1,\phi,v,\delta} - C_{0,\phi,v,\delta}, C_{1,\phi,v,\delta} + C_{0,\phi,v,\delta}], k = 0, \dots, n \right\} \cap \{1, \dots, n\}, \end{aligned}$$

where $C_{0,\phi,v,\delta} \in [0, n/2j]$ and $C_{1,\phi,v,\delta} \in [0, n/j]$ denote two real-valued constants (depending on ϕ, v, δ). Now, we have

$$\begin{aligned} & \left\{ \frac{nk}{j} + v \mid v \in [a_1, b_1], k = 0, 1, \dots, n \right\} \cap \{1, \dots, n\} \\ &= \left\{ \frac{nk}{j} + v \mid v \in [a_2, b_2], k = 0, 1, \dots, n \right\} \cap \{1, \dots, n\}, \end{aligned}$$

provided that $\lceil ja_1 \rceil = \lceil ja_2 \rceil, \lceil jb_1 \rceil = \lceil jb_2 \rceil$, where $\lceil a \rceil$ denotes the smallest integer larger or equal to a . The argument above holds for any Fourier frequency. In particular, it implies that

$$\begin{aligned} \mathcal{M}_n \subset & \left\{ T = \left\{ t \in \{1, \dots, n\} \cap \left\{ \frac{kn}{j} + v \mid v \in \left[\frac{a-b}{j}, \frac{a+b}{j} \right] \right\} \right\} \mid \right. \\ & \left. b = 0, \dots, \lceil n/2 \rceil, a, k = 0, \dots, n, j = 1, \dots, n \right\}, \end{aligned}$$

a collection of sets that contains at most $(n + 1)^4$ elements. This completes the proof. □

A.3. Two basic lemmas

Finally, we state and prove here Lemmas A.6 and A.7, which have been used at several places in this Appendix.

Lemma A.6. *Denote by $X_1, \dots, X_{\mu_n m_n}$ a sequence of μ_n independent blocks of m_n random variables such that $\sup_{i=1, \dots, \mu_n m_n} |X_i| \leq C_n$ a.s., and*

$$\sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{i=m_n(j-1)+1}^{m_n j} X_i \right) \leq V_n^2.$$

Then, for all $\lambda_n > 0$, $\mathbb{P}(|\sum_{j=1}^{\mu_n} X_j| > \lambda_n) \leq 2 \exp(-\frac{\log 2}{4} (\frac{\lambda_n^2}{2V_n^2} \wedge \frac{\lambda_n}{m_n C_n}))$. In particular, for $D > 0$, $\mathbb{P}(|\sum_{j=1}^{\mu_n} X_j| > 6 \max(DV_n \sqrt{\log n}, D^2 m_n C_n \log n)) \leq 4n^{-D^2}$.

Proof. Defining the random variables $U_k := \sum_{j=m_n(k-1)+1}^{m_n k} X_j, k = 1, \dots, \mu_n$, note that $U_1, U_2, \dots, U_{\mu_n}$ are independent, that $|U_j| \leq m_n C_n$ a.s. and that $\text{Var}(\sum_j U_j) \leq V_n^2$. Applying

Bennett’s inequality (see Pollard [36]) yields

$$\begin{aligned} & \mathbb{P}\left(\left|\sum_{j=1}^n X_j\right| > \lambda_n\right) \\ & \leq 2 \exp\left(-\frac{V_n^2}{m_n^2 C_n^2} h\left(\frac{m_n C_n \lambda_n}{2V_n^2}\right)\right) \leq 2 \exp\left(-\frac{1}{4} \frac{\lambda_n}{m_n C_n} \log\left(1 + \frac{m_n C_n \lambda_n}{2V_n^2}\right)\right) \\ & \leq 2 \exp\left(-\frac{\log 2}{4} \frac{\lambda_n}{m_n C_n} \left(\frac{m_n C_n \lambda_n}{2V_n^2} \wedge 1\right)\right) = 2 \exp\left(-\frac{\log 2}{2} \left(\frac{\lambda_n^2}{4V_n^2} \wedge \frac{\lambda_n}{2m_n C_n}\right)\right), \end{aligned}$$

where $h(x) := (1+x)\log(1+x) - x \geq \frac{1}{2}x \log(1+x) \geq \frac{\log(2)}{2}x(x \wedge 1)$. The second assertion follows by direct calculation. \square

Lemma A.7. *Let Assumptions (A1) and (A2) hold.*

(i) *Let $\mathcal{Y} \subset \mathbb{R}$ be a bounded set, $D > 1$, and $0 \leq a_n = o(1)$. Then,*

$$\sup_{y \in \mathcal{Y}} \sup_{|x| \leq a_n} |\hat{F}_n(y+x) - \hat{F}_n(y) - F(x+y) + F(y)| = \text{Op}(\rho_n(a_n, \delta)),$$

$$\text{where } \rho_n(a_n, \delta) := \left(\frac{a_n + n^{1/(1+\delta)} a_n^2 \log n}{n} \log n\right)^{1/2} \vee (n^{-\delta/(1+\delta)} \log n).$$

(ii) *For any bounded $\mathcal{Y} \subset \mathbb{R}$, $\sup_{x \in \mathcal{Y}} |\hat{F}_n(x) - F(x)| = \text{Op}(n^{-1/2} \sqrt{\log n})$.*

Proof. The bounded set $Z := \{(x, y) \in \mathbb{R}^2 | y \in \mathcal{Y}, |x| \leq a_n\}$ can be covered with $N = O(n^2)$ spheres of radius $\frac{1}{2}n^{-1}$ and centers $(z_{1j}, z_{2j}) \in Z, j = 1, \dots, N$. A Taylor expansion yields

$$\begin{aligned} & \sup_{\|(x,y)-(z_{1j},z_{2j})\| \leq 1/2n} |\hat{F}_n(y+x) - \hat{F}_n(y) - F(x+y) + F(y) \\ & \quad - (\hat{F}_n(z_{1j} + z_{2j}) - \hat{F}_n(z_{2j}) - F(z_{1j} + z_{2j}) + F(z_{2j}))| \\ & \leq n^{-1} \sum_{t=1}^n (I\{|Y_t - z_{2j}| \leq n^{-1}\} + I\{|Y_t - (z_{1j} + z_{2j})| \leq n^{-1}\}) + Cn^{-1}, \end{aligned}$$

where the constant C does not depend on t and j . Therefore,

$$\begin{aligned} & \sup_{y \in \mathcal{Y}} \sup_{|x| \leq a_n} |\hat{F}_n(y+x) - \hat{F}_n(y) - F(x+y) + F(y)| \\ & \leq \sup_{\theta \in \Theta_{1,n}} \left| \sum_{t=1}^n \theta(t, Y_t) \right| + \sup_{\theta \in \Theta_{2,n}} \left| \sum_{t=1}^n \theta(t, Y_t) \right|, \end{aligned}$$

where

$$\begin{aligned} \Theta_{1,n} := & \{\theta(t, y) := n^{-1}(I\{y \leq z_{1j} + z_{2j}\} - I\{y \leq z_{2j}\}) - F(z_{1j} + z_{2j}) + F(z_{2j}) | \\ & j = 1, \dots, N\}, \end{aligned}$$

and

$$\Theta_{2,n} := \left\{ \theta(t, y) := n^{-1} \left(I\{|y - z_{2j}| \leq n^{-1}\} + I\{|y - (z_{1j} + z_{2j})| \leq n^{-1}\} \right) + Cn^{-1} \mid j = 1, \dots, N \right\}.$$

We proceed to bound the suprema over $\Theta_{1,n}$ and $\Theta_{2,n}$ by applying the independent blocks argument with $\eta_n := D\rho_n(a_n, \delta)$ and a suitable constant D . Begin with $\Theta_{1,n}$. We have $E\theta(t, X_t) = 0$ for all $\theta \in \Theta_{1,n}$, $\sup_{\theta \in \Theta_{1,n}} \sup_t |\theta(t, X_t)| \leq 2n^{-1}$, and

$$\begin{aligned} & \sup_y \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in S_j} I\{X_t \leq y + x\} - I\{X_t \leq y\} - F(x + y) + F(y) \right) \\ & \leq C_2 \mu_n m_n (m_n |x|^2 + |x|) =: V_{1,n}^2 \end{aligned}$$

for some finite constant C_2 independent of x , and $m_n := \lceil n^{1/(1+\delta)} \log n \rceil$, defined as within the independent blocks argument (see Section A.1.1). The same bound holds with S_j replaced by T_j . This implies

$$\sup_{\theta \in \Theta_{1,n}} \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in S_j} \theta(t, X_t) \right) \leq \frac{C_2(m_n a_n^2 + a_n)}{n},$$

and

$$\sup_{\theta \in \Theta_{1,n}} \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in T_j} \theta(t, X_t) \right) \leq \frac{C_2(m_n a_n^2 + a_n)}{n}.$$

A simple calculation [observe that $n\rho_n(a_n, \delta) \geq n^{1/(1+\delta)} \log n \sim m_n$] shows that this implies (i) and (ii) from the independent blocks argument and that for E_n defined in (A.2) we have $E_n = o(1)$. Thus $\sup_{\theta \in \Theta_{1,n}} |\sum_{t=1}^n \theta(t, Y_t)| = o_P(\eta_n)$.

Next, apply the independent blocks argument with $\Theta_{n,2}$. Observe that

$$\sup_{\theta \in \Theta_{1,n}} \sup_t |\theta(t, X_t)| \leq (C + 2)n^{-1} \quad \text{a.s.}$$

This yields (i) from the independent blocks argument. Furthermore, we have

$$\sup_{\theta \in \Theta_{2,n}} \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in S_j} \theta(t, X_t) \right) \leq C'n^{-2}, \quad \sup_{\theta \in \Theta_{2,n}} \sum_{j=1}^{\mu_n} \text{Var} \left(\sum_{t \in T_j} \theta(t, X_t) \right) \leq C'n^{-2}$$

for a constant C' and the same bound holds with S_j replaced by T_j . Thus (ii) from the independent blocks argument is established. Based on this and the fact that

$$\sup_{\theta \in \Theta_{2,n}} \sup_t |E[\theta(t, X_t)]| = O(n^{-2}),$$

some simple calculations show that for E_n defined in (A.2) we have $E_n = o(1)$. This completes the independent blocks argument for $\Theta_{2,n}$. Combining the results obtained so far establishes the first part of this lemma. The second part follows from similar arguments. \square

Appendix B: Technical details for the proof of Theorem 4.1

The proof of Theorem 4.1 in Section 4 is relying on Equations (4.4) and (4.5), which we establish in Sections B.1 and B.2, respectively.

B.1. Proof of (4.4)

Putting

$$\begin{aligned} 4n^{-1}\tilde{\Delta}_n &:= (\hat{\mathbf{b}}_{n,\tau_1,\omega_{j,n}} - \mathbf{b}_{n,\tau_1,\omega_{j,n}})' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \mathbf{b}_{n,\tau_2,\omega_{j,n}} \\ &\quad + (\mathbf{b}_{n,\tau_1,\omega_{j,n}})' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} (\hat{\mathbf{b}}_{n,\tau_2,\omega_{j,n}} - \mathbf{b}_{n,\tau_2,\omega_{j,n}}) \\ &\quad + (\hat{\mathbf{b}}_{n,\tau_1,\omega_{j,n}} - \mathbf{b}_{n,\tau_1,\omega_{j,n}})' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} (\hat{\mathbf{b}}_{n,\tau_2,\omega_{j,n}} - \mathbf{b}_{n,\tau_2,\omega_{j,n}}), \end{aligned}$$

we obtain, from the definition of the Laplace periodogram,

$$\begin{aligned} L_{n,\tau_1,\tau_2}(\omega_{j,n}) &:= \frac{n}{4} (\hat{\mathbf{b}}_{n,\tau_1,\omega_{j,n}})' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \hat{\mathbf{b}}_{n,\tau_2,\omega_{j,n}} \\ &= \frac{n}{4} (\mathbf{b}_{n,\tau_1,\omega_{j,n}})' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \mathbf{b}_{n,\tau_2,\omega_{j,n}} + \tilde{\Delta}_n \\ &= \frac{1}{f(q_{\tau_1})f(q_{\tau_2})} (n^{-1}d_n(\tau_1, \omega_{j,n})d_n(\tau_2, -\omega_{j,n})) + \tilde{\Delta}_n. \end{aligned}$$

By (3.9), for $\tau \in \{\tau_1, \tau_2\}$,

$$n^{1/2} \sup_{\omega_{j,n} \in \mathcal{F}_n} \|\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}}\| = \text{Op}(n^{(1/8)(1-\delta)/(1+\delta)} (\log n)^{7/4}),$$

while Lemma A.2 implies that

$$n^{1/2} \sup_{\omega_{j,n} \in \mathcal{F}_n} \|\mathbf{b}_{n,\tau,\omega_{j,n}}\| = \text{Op}(\sqrt{\log n}),$$

so that $\|\tilde{\Delta}_n\| = \text{Op}(n \|\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}}\| \cdot \|\mathbf{b}_{n,\tau,\omega_{j,n}}\|) = \text{Op}(R_n)$.

B.2. Proof of (4.5)

Note that $L_{n,\tau_1,\tau_2}(\omega_{j,n})$ is the cross-periodogram of the bivariate time series

$$(\tau_1 - I\{Y_t \leq q_{\tau_1}\}, \tau_2 - I\{Y_t \leq q_{\tau_2}\}). \tag{B.1}$$

Let $\omega_{j,n}, \omega_{k,n} \in (0, \pi)$ be two sequences of Fourier frequencies. Corollary 7.2.2 in Brillinger [7] implies that

$$\text{Var}(L_{n,\tau_1,\tau_2}(\omega_{j,n})) = \mathfrak{f}_{1,1}(\omega_{j,n})\mathfrak{f}_{2,2}(\omega_{j,n}) + \frac{2\pi}{n}\mathfrak{f}_{1,2,1,2}(\omega_{j,n}, -\omega_{j,n}, -\omega_{k,n}) + O(1/n) \tag{B.2}$$

and, for $\omega_{j,n} \neq \omega_{k,n}$,

$$\text{Cov}(L_{n,\tau_1,\tau_2}(\omega_{j,n}), L_{n,\tau_1,\tau_2}(\omega_{k,n})) = \frac{2\pi}{n}\mathfrak{f}_{1,2,1,2}(\omega_{j,n}, -\omega_{j,n}, -\omega_{k,n}) + O(1/n^2), \tag{B.3}$$

where $\mathfrak{f}_{1,1}, \mathfrak{f}_{2,2}$ and $\mathfrak{f}_{1,2,1,2}$ are the spectra and cumulant spectra of the bivariate time series (B.1), which exist by Assumption (A4). Note that the orders $O(1/n)$ and $O(1/n^2)$ of the remainders in (B.2) and (B.3) hold uniformly with respect to j and k . The aforementioned spectra coincide with the Laplace spectra $\mathfrak{f}_{\tau_1,\tau_1}$, and $\mathfrak{f}_{\tau_2,\tau_2}$ and the cumulant spectra are also bounded (see Brillinger [7], page 26). Therefore,

$$\text{Cov}(L_{n,\tau_1,\tau_2}(\omega_{j,n}), L_{n,\tau_1,\tau_2}(\omega_{k,n})) = \begin{cases} \mathfrak{f}_{\tau_1,\tau_1}(\omega_{j,n})\mathfrak{f}_{\tau_2,\tau_2}(\omega_{j,n}) + \bar{R}_n, & \omega_{j,n} = \omega_{k,n}, \\ \bar{R}_n, & \omega_{j,n} \neq \omega_{k,n}, \end{cases}$$

where $\bar{R}_n = O(1/n)$ does not depend on j and k . The assertion follows by the fact that the variance and the bias of the random variable K_n in (4.5) both are of the order $O(1/n)$. For the variance, note that

$$\begin{aligned} \text{Var}(K_n) &= \frac{1}{f^2(q_{\tau_1})f^2(q_{\tau_2})} \\ &\times \left[\sum_{|k| \leq N_n} W_n^2(k) \text{Var}(L_{n,\tau_1,\tau_2}(\omega_{j+k,n})) \right. \\ &\quad \left. + \sum_{|k_1| \leq N_n} W_n(k_1) \sum_{\substack{|k_2| \leq N_n \\ k_2 \neq k_1}} W_n(k_2) \text{Cov}(L_{n,\tau_1,\tau_2}(\omega_{j+k_1,n}), \overline{L_{n,\tau_1,\tau_2}(\omega_{j+k_2,n})}) \right] \\ &= O(1/n), \end{aligned}$$

due to the second part of Assumption (A3) and (B.3). As for the bias, $E[K_n] = O(1/n)$ due to the fact that $EL_{n,\tau_1,\tau_2}(\omega_{j+k,n}) = \sum_{k=-\infty}^{\infty} \gamma_k(q_{\tau_1}, q_{\tau_2})e^{-i\omega_{j+k,n}k} + O(1/n)$ uniformly with respect to the frequencies (see Theorem 4.3.2 in Brillinger [7]).

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