N. YAGITA
KODAI MATH. J.
40 (2017), 537–552

NOTE ON RESTRICTION MAPS OF CHOW RINGS TO WEYL GROUP INVARIANTS

Nobuaki Yagita

Abstract

Let G be an algebraic group over $\mathbf{C}$ corresponding a compact simply connected Lie group. When $H^*(G)$ has p-torsion, we see $\rho_{CH}^*: CH^*(BG) \to CH^*(BT)^{W_G(T)}$ is always not surjective. We also study the algebraic cobordism version ρ_{Ω}^*. In particular when $G = Spin(7)$ and $p = 2$, we see each Griffiths element in $CH^*(BG)$ is detected by an element in $\Omega^*(BT)$.

1. Introduction

Let p be a prime number. Let G be a compact Lie group and T the maximal torus. Let us write $H^*(-) = H^*(-;\mathbf{Z}_{(p)})$, and BG, BT classifying spaces of G, T. Let $W = W_G(T) = N_G(T)/T$ be the Weyl group and $Tor \subset H^*(BG)$ be the ideal generated by torsion elements. Then we have the restriction map

$$\rho_H^*: H^*(BG) \to H^*(BG)/Tor \subset H^*(BT)^W$$

by using the Becker-Gottlieb transfer.

It is well known by Borel ([3]) that when $H^*(G)$ is p-torsion free (hence $H^*(BG)$ is p-torsion free), then ρ_H^* is surjective. However when $H^*(G)$ has p-torsion, there are cases that ρ_H^* are not surjective, which are founded by Feshbach [5].

Let us write by $G_{\mathbf{C}}$, $T_{\mathbf{C}}$ the reductive group over $\mathbf{C}$ and its maximal torus corresponding the Lie groups G, T. Let us write simply $CH^*(BG) = CH^*(BG_{\mathbf{C}})_{(p)}$, $CH^*(BT) = CH^*(BT_{\mathbf{C}})_{(p)}$ the Chow rings of $BG_{\mathbf{C}}$ and $BT_{\mathbf{C}}$ localized at p. We consider the Chow ring version of the restriction map

$$\rho_{CH}^*: CH^*(BG) \to CH^*(BG)/Tor \subset CH^*(BT)^W.$$

Our first observation is

2010 *Mathematics Subject Classification*. 55N20, 55R12, 55R40.
Key words and phrases. Chow ring, algebraic cobordism, $BSpin(n)$.
Received July 28, 2016; revised January 10, 2017.

Theorem 1.1. *Let G be simply connected. If $H^*(G)$ has p-torsion, then the map ρ_{CH}^* is always not surjective.*

In the proof, we use an element $x \in H^4(BG)$ with $\rho_H(x) \notin Im(\rho_{CH}^*)$. Hence $x \notin Im(cl)$ for the cycle map $cl : CH^*(BG) \to H^*(BG)$, from the commutative diagram

$$\begin{array}{ccc} CH^*(BG) & \xrightarrow{\rho_{CH}^*} & CH^*(BT)^W \\ {\scriptstyle cl}\downarrow & & \downarrow{\scriptstyle \cong} \\ H^*(BG) & \xrightarrow{\rho_H^*} & H^*(BT)^W \end{array}$$

The corresponding element $1 \otimes x \in CH^*(B\mathbf{G}_m \times BG)$ is the element founded as a counterexample for the integral Hodge and hence the integral Tate conjecture in [15].

Next, we consider elements in *Tor*. To study torsion elements, we consider the following restriction map

$$res_H : H^*(BG) \to \Pi_{A:abelian \subset G} H^*(BA)^{W_G(A)}.$$

There are cases such that res_H are not injective, while for many cases res_H are injective. We consider the Chow ring version ([21], [22]) of the above restriction map

$$res_{CH} : CH^*(BG) \to \Pi_{A:ab.} CH^*(BA)^{W_G(A)} \subset \Pi_{A:ab.} H^*(BA)^{W_G(A)}.$$

In general res_{CH} has nonzero kernel. In particular, elements in $Ker(cl)$ (i.e. Griffiths elements) for the cycle map cl are always in $Ker(res_{CH})$. Namely Griffiths elements are not detected by res_{CH}.

On the other hand, if the Totaro conjecture

$$CH^*(BG) \cong BP^*(BG) \otimes_{BP^*} \mathbf{Z}_{(p)}$$

(for the Brown-Peterson cohomology $BP^*(-)$) is correct, then of course all elements in $CH^*(BG)$ are detected by elements in $BP^*(BG)$. We show that there is a case that Griffiths elements are detected by ρ_Ω^* the restriction for algebraic cobordism theory $\Omega^*(-)$.

Let $\Omega^*(X) = MGL^{2*,*}(X) \otimes_{MU_{(p)}^*} BP^*$ be the BP-version of the algebraic cobordism defined by Voevodsky, Levine-Morel ([25], [13], [14]) such that $CH^*(X) \cong \Omega^*(X) \otimes_{BP^*} \mathbf{Z}_{(p)}$. In particular, we consider the case $G = Spin(7)$ and $p = 2$. We note that there are (nonzero) Griffiths elements in $CH^*(BG)$.

Theorem 1.2. *Let $G = Spin(7)$ and $p = 2$. Then each Griffiths element (in $CH^*(BG)$) is detected by an element in $\Omega^*(BT)^W \cong BP^*(BT)^W$.*

In §2 we study the map ρ_H^* for the ordinary cohomology theory, and recall Feshbach's result. In §3, we study the Chow ring version and show Theorem

1.1. In §4, we study the case $G = Spin(n)$. In §5, we study the BP^*-version and the algebraic cobordism version for the restriction ρ^*. In §6, we write down the case $G = Spin(7)$ quite explicitly, and show Theorem 1.2. In the last section, we note some partial results for the exceptional group $G = F_4$ and $p = 3$.

The author thanks Kirill Zainoulline to start considering this problem, and Masaki Kameko to let the author know the works by Benson-Wood and Feshbach.

2. Cohomology theory and Feshbach theorem

Let p be a prime number. Let G be a compact Lie group and T the maximal Torus. Then we have the restriction map

$$\rho_H^* : H^*(BG) \to H^*(BT)^W$$

where $H^*(-) = H^*(-;\mathbf{Z}_{(p)})$, BG, BT are classifying spaces and $W = W_G(T) = N_G(T)/T$ is the Weyl group.

It is well known by Borel ([3], [5], [2]) that when $H^*(G)$ is p-torsion free, then ρ_H^* is surjective (and hence is isomorphic). However when $H^*(G)$ has p-torsion, there are cases that ρ_H^* are not surjective by Feshbach.

For a connected compact Lie group G, we have the Becker-Gottlieb transfer $\tau : H^*(BT) \to H^*(BG)$ such that $\tau\rho_H^* = \chi(G/T)$ for the Euler number $\chi(-)$, and $\rho_H^*\tau(x) = \chi(G/T)x$ for $x \in H^*(BT)^W$. Let $\chi(G/T) = N$ and Tor be the ideal of $H^*(BG)$ generated by torsion elements. Then we have the injections

$$N \cdot H^*(BT)^W \subset H^*(BG)/Tor \subset H^*(BT)^W.$$

Feshbach found good criterion to see ρ_H^* is surjecive.

Theorem 2.1 (Feshbach [5]). *The restriction ρ_H^* is surjective if and only if $(H^*(BG)/Tor) \otimes \mathbf{Z}/p$ has no nonzero nilpotent elements.*

Proof. First note that $H^*(BT) \cong \mathbf{Z}_{(p)}[t_1, \dots, t_\ell]$ for $|t_i| = 2$. Hence if $x^m = px'$ in $H^*(BT)$, then $x = px''$ for $x'' \in H^*(BT)$. Moreover if $x = px' \in H^*(BT)^W$, then so is x' since $H^*(BT)$ is p-torsion free. Thus we see $H^*(BT)^W \otimes \mathbf{Z}/p$ has no nonzero nilpotent elements.

Assume that ρ_H^* is not surjective, and $x \in H^*(BT)^W$ but $x \notin Im(\rho_H^*)$. Let $s \geq 1$ be the smallest number such that $p^s x = \rho_H^*(y)$ for some $y \in H^*(BG)$. Hence $y \neq 0 \ mod(p)$. Then

$$\rho_H^*(y^N) = (p^s x)^N = p^{sN} x^N \in pN \cdot H^*(BT)^W \subset p\, Im(\rho_H^*).$$

This means that y is a nilpotent element in $(H^*(BG)/Tor) \otimes \mathbf{Z}/p$. $\square$

Using this theorem, Feshbach [5] showed ρ_H^* is surjective for $G = G_2$, $Spin(n)$ when $n \leq 10$, and is not surjective for $Spin(11)$, $Spin(12)$. Wood [27] showed that $Spin(13)$ is not surjective but $Spin(n)$ for $14 \leq n \leq 18$ are surjective. Benson and Wood [2] solved this problem completely, namely ρ_H^* is not surjective if and only if $n \geq 11$ and $n = 3, 4, 5 \; mod(8)$.

For odd prime, we consider $mod(p)$ version

$$\rho_{H/p} : H^*(BG; \mathbf{Z}/p) \to H^*(BT; \mathbf{Z}/p)^W \cong (H^*(BT)/p)^W.$$

It is known that $\rho_{H/p}^*$ is surjective when $G = F_4$ for $p = 3$ by Toda [20] using a completely different arguments. Also using different arguments (but without computations of $H^*(BT)^W$ for concrete cases), Kameko and Mimura [9] prove that $\rho_{H/p}^*$ are surjective when $G = E_6, E_7$ for $p = 3$ and $G = E_8$ for $p = 5$. (The only remain case is $G = E_8$, $p = 3$ for odd primes.)

Kameko-Mimura get more strong result. Recall the Milnor Q_i operation

$$Q_i : H^*(X; \mathbf{Z}/p) \to H^{*+2p^i-1}(X; \mathbf{Z}/p)$$

defined by $Q_0 = \beta$ and $Q_{i+1} = [P^{p^i} Q_i, Q_i P^{p^i}]$ for the Bockstein β and the reduced powers P^j.

Theorem 2.2 (Kameko-Mimura [9]). *Let $G = F_4, E_6.E_7$ for $p = 3$ or E_8 for $p = 5$. Let us write a generator by x_4 in $H^4(BG) \cong \mathbf{Z}_{(p)}$. Then we have*

$$H^*(BT; \mathbf{Z}/p)^W \cong H^{even}(BG; \mathbf{Z}/p)/(Q_1 Q_2 x_4).$$

Corollary 2.3. *For (G, p) in the above theorem, ρ_H^* is surjective.*

We can identify $Q_1 Q_2(x_4)$ is a p-torsion element in $H^*(BG)$, since its Q_0-image is zero. The above corollary is immediate from the following lemma.

Lemma 2.4. *If the composition*

$$\rho : (H^*(BG)/Tor) \otimes \mathbf{Z}/p \to H^*(BT)^W/p \to H^*(BT; \mathbf{Z}/p)^W$$

is injective, then ρ_H^* is surjective.

Proof. Let ρ_H^* be not surjecive and $y \in H^*(BT)^W$ with $y \notin Im(\rho_H^*)$. Then $p^s y = \rho_H^*(x)$ for some $s \geq 1$ and an additive generator $x \in H^*(BG)/Tor$. Of course $\rho(x) = 0 \in (H^*(BT)/p)^W$. $\square$

3. Chow rings

Let us write by $G_{\mathbf{C}}$, $T_{\mathbf{C}}$ the reductive group over $\mathbf{C}$ and its maximal torus corresponding the Lie group G and its maximal torus T. Let $CH^*(BG) = CH^*(BG_{\mathbf{C}})_{(p)}$ be the Chow ring of $BG_{\mathbf{C}}$ localized at p.

The arguments of Feshbach also work for Chow rings since the Becker-Gottlieb transfer is constructed by Totaro [22].

Lemma 3.1. *The restriction map ρ_{CH}^* of Chow rings is surjective if and only if $(CH^*(BG)/T) \otimes \mathbf{Z}/p$ has not nonzero nilpotent elements.*

However if $H^*(G)$ has p-torsion and G is simply connected, then $(CH^*(BG)/Tor) \otimes \mathbf{Z}/p$ always has nonzero nilpotent elements. In fact, $c_2 = px_4 \in CH^4(BG)$ in the proof of Theorem 3.3 below, is nilpotent in $(CH^*(BG)/(Tor)) \otimes \mathbf{Z}/p$. However from the proof of the above lemma, we note

Corollary 3.2. *If $x \in CH^*(BT)^W$ but $x \notin Im(\rho^*_{CH})$, then there is $y \in CH^*(BG)$ such that $\rho^*_{CH}(y) = p^s x$ for some $s \geq 1$ and y is nonzero nilpotent element in $(CH^*(BG)/(Tor)) \otimes \mathbf{Z}/p$.*

Voevodsky [25], [26] defined the Milnor operation Q_i on the mod p motivic cohomology (over a perfect field k of any $ch(k)$)

$$Q_i : H^{*,*'}(X;\mathbf{Z}/p) \to H^{*+2(p^i-1),*'+p^i-1}(X;\mathbf{Z}/p)$$

which is compatible with the usual topological Q_i by the realization map $t_{\mathbf{C}} : H^{*,*'}(X;\mathbf{Z}/p) \to H^*(X(\mathbf{C});\mathbf{Z}/p)$ when $ch(k) = 0$. In particular, note for smooth X,

$$Q_i|CH^*(X)/p = Q_i|H^{2*,*}(X;\mathbf{Z}/p) = 0.$$

(See §2 in [Pi-Ya] for details.) We will prove the following theorem without using Feshbach theorem (Lemma 3.1).

Theorem 3.3. *Let G be simply connected and $H^*(G)$ has p-torsion. Then the restriction map*

$$\rho^*_{CH} : CH^2(BG) \to CH^2(BT)^W$$

is not surjective.

Proof. (See §2, 3 in [15].) At first, we note that $H^*(BT)^W \cong CH^*(BT)^W$ since $H^*(BT) \cong CH^*(BT)$. Therefore we have the commutative diagram

$$\begin{array}{ccc} CH^*(BG) & \xrightarrow{\rho^*_{CH}} & CH^*(BT)^W \\ {\scriptstyle cl}\downarrow & & \downarrow{\scriptstyle \cong} \\ H^*(BG) & \xrightarrow{\rho^*_{H}} & H^*(BT)^W \end{array}$$

If $H^*(G)$ has p-torsion, then G has a subgroup isomorphic to G_2 (resp. F_4, E_8) for $p = 2$ (resp. $p = 3, 5$). (For details, see [29] or §3 in [15].) We prove the theorem for $p = 2$ but the other cases are proved similarly.

It is known that the inclusion $G_2 \subset G$ induces a surjection $H^4(BG) \to H^4(BG_2) \cong \mathbf{Z}_{(2)}$ and let us write by x_4 its generator. Then it is also known $Q_1 x_4 \neq 0$ in $H^*(BG_2;\mathbf{Z}/2)$ where Q_1 is the Milnor operation. Therefore $x_4 \in H^4(BG_2)$ is not in the image of the cycle map

$$cl : CH^2(BG_2) \to H^4(BG_2).$$

On the other hand, the element $2x_4$ is in $Im(cl)$ because it is represented by the second Chern class c_2. Since $\rho_H^* \otimes \mathbf{Q}$ is an isomorphism, $\rho_H^*(x_4) \neq 0$. But $\rho_H^*(x_4)$ is not in the image ρ_{CH}^* from the above diagram. □

Remark. The condition of simply connected is necessary. By Vistoli ([24], [9]), it is known that ρ_{CH}^* is surjective for $G = PGL(p)$.

Remark. The above theorem is also proved by seeing that x_4 is not generated by Chern classes, since $CH^2(X)$ is always generated by Chern classes [22].

Recall that for a smooth projective complex variety X, the integral Hodge conjecture is that the cycle map

$$cl_{/Tor} : CH^*(X) \to H^{2*}(X)/Tor \cap H^{*,*}(X)$$

is surjective where $H^{*,*}(X) \subset H^{2*}(X; \mathbf{C})$ is the submodule generated by $(*,*)$-forms. Since $px_4 = c_2$ in the proof of the above theorem and $c_2 \in H^{*,*}(X)$, we see $x_4 \in H^{*,*}(X)$.

We know [21], [15] that $B\mathbf{G}_m \times BG$ can be approximated by smooth projective varieties. Hence counterexamples for the integral Hodge conjecture with $X = B\mathbf{G}_m \times BG$ give the examples such that ρ_{CH}^* is not surjective.

LEMMA 3.4. *Let $1 \otimes y \notin Im(cl_{/Tor}) \subset H^*(B\mathbf{G}_m \times BG)/Tor$ be a counterexample of the integral Hodge conjecture. Then it gives an example such that ρ_{CH}^* is not surjective, namely, $\rho_H^*(y) \notin Im(\rho_{CH}^*)$.*

Proof. First note that $\rho_{H/Tor}^* : H^*(BG)/Tor \to H^*(BT)^W$ is injective. Since $CH^*(BT)^W \cong H^*(BT)^W$, we note $\rho_{CH}^* = \rho_{H/Tor}^* cl_{/Tor}$. Therefore $y \notin Im(cl_{/Tor})$ implies that $\rho_H^*(y) \notin Im(\rho_{H/Tor}^* cl_{/Tor}) = Im(\rho_{CH}^*)$. □

For each prime p, there are counterexamples $X = B\mathbf{G}_m \times BG$ for the integral Hodge conjecture, while they are not simply connected. Indeed, Kameko, Antieau and Tripaphy ([7], [8], [1], [23]) show this for $G = (SL_p \times SL_p)/\mathbf{Z}/p$ and $SU(p^2)/\mathbf{Z}/p$. Hence these mean that they give the examples such that ρ_{CH}^* are not surjective for non simply connected and all p cases. They proved these facts by using Chern classes.

We also note its converse. Recall [15] that the integral Tate conjecture over a finite field k is the $ch(k) > 0$ version of the integral Hodge conjecture.

LEMMA 3.5. *Let $x \in H^*(BT)^W$ such that $x \notin \rho_{CH}^*$ but $x = \rho_H^*(y)$. Moreover let $p^s y$ be represented by a Chern class for some $s \geq 1$. Then $1 \otimes y \in H^*(B\mathbf{G}_m \times BG)$ gives a counterexample of the integral Hodge conjecture. It also gives a counterexample of the integral Tate conjecture for a finite field k of $ch(k) \neq p$.*

Proof. Since $p^s y$ is represented by a Chern class, we see $p^s y \in Im(cl)$. Hence it is contained in the Hodge class $H^{*,*}(B\mathbf{G}_m \times BG)$. Hence so is y. Since $x \notin \rho^*_{CH}$, we see $y \notin cl_{/Tor}$. For details of the integral Tate conjecture see [15]. □

4. *Spin*(n) for $p = 2$

In this section, we study Chow rings for the cases $G = Spin(n)$, $p = 2$. Recall that the $mod(2)$ cohomology is given by Quillen [17]

$$H^*(BSpin(n); \mathbf{Z}/2) \cong \mathbf{Z}/2[w_2, \dots, w_n]/J \otimes \mathbf{Z}/2[e]$$

where $e = w_{2^h}(\Delta)$ and $J = (w_2, Q_0 w_2, \dots, Q_{h-2} w_2)$. Here w_i is the Stiefel-Whitney class for the natural covering $Spin(n) \to SO(n)$. The number 2^h is the Radon-Hurwitz number, dimension of the spin representation Δ (which is the representation $\Delta|C \neq 0$ for the center $C \cong \mathbf{Z}/2 \subset Spin(n)$). The element e is the Stiefel-Whitney class w_{2^h} of the spin representation Δ.

Hereafter this section we always assume $G = Spin(n)$ and $p = 2$.

By Kono [11], it is known that $H^*(BG; \mathbf{Z})$ has no higher 2-torsion, that is

$$H(H^*(BG; \mathbf{Z}/2); Q_0) \cong (H^*(BG)/Tor) \otimes \mathbf{Z}/2$$

where $H(A; Q_0)$ is the homology of A with the differential $d = Q_0$.

For ease of arguments, let n be odd i.e., $n = 2k + 1$. Let T' be a maximal Torus of $SO(n)$ and $W' = W_{SO(n)}(T')$ its Weyl group. Then $W' \cong S^{\pm}_k$ is generated by permutations and change of signs so that $|S^{\pm}_k| = 2^k k!$. Hence we have

$$H^*(BT')^{W'} \cong \mathbf{Z}_{(2)}[p_1, \dots, p_k] \subset H^*(BT') \cong \mathbf{Z}_{(2)}[t_1, \dots, t_k], \quad |t_i| = 2$$

where the Pontriyagin class p_i is defined by $\Pi_i(1 + t_i^2) = \sum_i p_i$.

For the projection $\pi : Spin(n) \to SO(n)$, the maximal torus of $Spin(n)$ is given $\pi^{-1}(T')$ and $W = W_{Spin(n)}(T) \cong W'$. To seek the invariant $H^*(BT)^W$ is not so easy, since the action $W \cong S^{\pm}_k$ is not given by permutations and change of signs. Benson and Wood decided the $H^*(BT')^{W'}$-algebra structure of $H^*(BT)^W$ (Theorem 7.1 in [2]) and proved

Theorem 4.1 (Benson-Wood). *Let $G = Spin(n)$ and $p = 2$. Then ρ^*_H is surjective if and only if $n \leq 10$ or $n \neq 3, 4, 5$ $mod(8)$ (i.e., it is not the quaternion case).*

Hereafter to study the Chow ring version, we assume $Spin(n)$ is in the real case [17], that is $n = 8\ell - 1, 8\ell, 8\ell + 1$ (hence ρ^*_H is surjective and $h = 4\ell - 1$, $4\ell - 1, 4\ell$ respectively).

In this case, it is known [17] that each maximal elementary abelian 2-group A has $rank_2\, A = h + 1$ and

$$e|A = \Pi_{x \in H^1(B\bar{A}; \mathbf{Z}/2)}(z + x)$$

where we identify $A \cong C \oplus \bar{A}$ and $H^1(B\bar{A}; \mathbf{Z}/2) \cong \mathbf{Z}/2\{x_1, \dots, x_h\}$ is the $\mathbf{Z}/2$-vector space generated by $x_1, \dots, x_h$, and

$$H^*(BC; \mathbf{Z}/2) \cong \mathbf{Z}/2[z], \quad H^*(B\bar{A}; \mathbf{Z}/2) \cong \mathbf{Z}/2[x_1, \dots, x_h].$$

The Dickson algebra is written as a polynomial algebra

$$\mathbf{Z}/2[x_1, \dots, x_h]^{GL_h(\mathbf{Z}/2)} \cong \mathbf{Z}/2[d_0, \dots, d_{h-1}].$$

where d_i is defined as

$$e|A = z^{2^h} + d_{h-1}z^{2^{h-1}} + \cdots + d_0 z.$$

We can also identify $d_i = w_{2^h - 2^i}(\Delta) \in H^*(BG; \mathbf{Z}/2)$ [17].

Lemma 4.2 (Lemma 2.1 in [19]). *Milnor operations act on d_i by*

$$Q_{h-1}d_i = d_0 d_i, \quad Q_{j-1}d_j = d_0, \quad \text{for } 1 \le j,$$
$$Q_i d_j = 0 \quad \text{for } i < h-1 \text{ and } i \ne j-1.$$

Lemma 4.3 (Corollary 2.1 in [19]). *We have*

$$Q_{h-1}e = d_0 e \quad \text{and} \quad Q_k e = 0 \quad \text{for } 0 \le k \le h-2.$$

Theorem 4.4. *Let $T \subset G = Spin(n)$ for $n = 8\ell, 8\ell \pm 1$. There is an $e' \in CH^*(BT)^W$ such that $e' \notin Im(\rho_{CH}^*)$ and $\rho_H^*(e) = e'$ $mod(2)$.*

Proof. First note that $e|C = z^{2^h}$ and $w_i|C = 0$. Hence $H^*(BG; \mathbf{Z}/2)|C \cong \mathbf{Z}/2[z^{2^h}]$, which implies that e is not in the Q_0-image. From the preceding Lemma 4.3 we see $Q_0 e = 0$. By Kono's result, we see

$$0 \ne e \in H(H^*(BG; \mathbf{Z}/2); Q_0) \cong (H^*(BG)/Tor) \otimes \mathbf{Z}/2).$$

Take $e'' \in H^*(BG)/Tor$ with that $e'' = e$ $mod(2)$. Then

$$e' = \rho_H^*(e'') \ne 0 \quad \text{in } H^*(BG)/Tor \subset H^*(BT)^W.$$

From the preceding Lemma 4.3, $Q_{h-1}(e) \ne 0$. Hence we see $e' \notin \rho_{CH}^*$ by the existence of Q_i in the motivic cohomology by Voevodsky. □

Let $\Delta_{\mathbf{C}}$ be the complex representation induced from the real representation Δ. Then we see (see Theorem 4.2 in [19])

$$c_{2^{h-1}}(\Delta_{\mathbf{C}})|C = 2w_{2^h}|C = 2z^{2^h}.$$

Of course this element $c_{2^{h-1}}(\Delta_{\mathbf{C}})$ is in the Chow ring $CH^*(BG)$. Hence we see that we can take $2e' \in Im(\rho_{CH}^*)$.

From the result by Benson-Wood, we know ρ_H^* is surjective in this (real) case. Hence from Lemma 3.5 (or $Q_{h-1}(e) \ne 0$), we have

Corollary 4.5. *Let* $X = B\mathbf{G}_m \times BSpin(n)$ *with* $n = 8\ell, 8\ell \pm 1$. *The element* $1 \otimes e \in H^{2^h}(X) \cap H^{2^{h-1},2^{h-1}}(X)$ *gives a counterexample for the integral Hodge and the integral Tate conjectures, namely* $1 \otimes e \notin Im(cl_{H/Tor})$.

5. Cobordism

Let $BP^*(X)$ be the Brown-Peterson cohomology theory with the coefficients ring $BP^* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ of degree $|v_i| = -2(p^i - 1)$ (see [16] for details). Let $\Omega^*(X) = MGL^{2*,*}(X) \otimes_{MU^*} BP^*$ be the BP^*-version of the algebraic cobordism ([25], [13], [14], [29]) such that $\Omega^*(X) \otimes_{BP^*} \mathbf{Z}_{(p)} \cong CH^*(X)$.

We consider the cobordism version of the map ρ_H^*

$$\rho_\Omega^* : \Omega^*(BG) \to \Omega^*(BT)^W \cong BP^*(BT)^W.$$

Although $\mathbf{A}^1$-homotopy category has the Becker-Gottlieb transfer τ (this fact is announced in [4]), we see

$$\tau \cdot \rho_\Omega^* = \chi(G/T) \quad mod(v_1, v_2 \ldots)$$

which is not $\chi(G/T)$ in general. So we can not have the Ω^*-version of Feshbach's theorem.

We are interesting in an element $x \in \Omega^*(BG)$ such that $\rho_\Omega^*(x) \neq 0$ in $\Omega^*(BT)$. Of course, x is torsion free in $\Omega^*(BG)$, but there is a case such that

$$0 \neq x \in CH^*(BG)/p \cong \Omega^*(BG) \otimes_{BP^*} \mathbf{Z}/p$$

and x is p-torsion in $CH^*(BG)$.

Lemma 5.1. *Let* $f \in H^*(BT)^W$, $f \neq 0$ $mod(p)$, *and identify* $f \in gr\,\Omega^*(BT) \cong \Omega^* \otimes H^*(BT)$. *Let* $f \notin Im(\rho_\Omega^*)$ *but* $v_m f \in Im(\rho_\Omega^*)$ *for* $m \geq 0$. *Then* $v_j f \in Im(\rho_\Omega^*)$ *for all* $0 \leq j \leq m$. *Namely, there is* $c_j \in \Omega^*(BG)$ *such that* $\rho_\Omega^*(c_j) = v_j f$,

$$c_j \neq 0 \in \Omega^*(BG) \otimes_{BP^*} \mathbf{Z}/p \cong CH^*(BG)/p.$$

Moreover $pc_j = 0$ *in* $CH^*(BG)$ *for* $j > 0$.

Proof. We consider the Landweber-Novikov cohomology operation r_a (see [16] for details) in $gr\,\Omega^*(BT) \cong \Omega^* \otimes H^*(BT)$. By Cartan formula,

$$r_a(v_m f) = \sum_{a=a'+a''} r_{a'}(v_m) r_{a''}(f).$$

Here $r_{a''}(f) = 0$ for $|a''| > 0$ in $gr\,\Omega^*(BT) \cong \Omega^* \otimes H^*(BT)$. It is known that there are operations $r_{\beta_j}(v_m) = v_j$ for $j \leq m$ ([16]). Thus we see the first statement.

From the assumption, f itself is not in the cycle map ρ_{Ω^*}. Hence $v_j f$ is a BP^*-module generator in $\Omega^*(BT)^W \cap Im(\Omega^*(BG)))$. Hence it is also nonzero in $CH^*(BG)/p$. Since $pv_j f = v_j pf \in v_j\, Im(\Omega^*(BG))$, we have $pc_j = 0 \in CH^*(BG)$. □

We consider the Atiyah-Hirzebruch spectral sequence (AHss)

$$E_2^{*,*'} \cong H^*(X;BP^{*'}) \Rightarrow BP^*(X)$$

It is known that

$$(*)\ \ d_{2p^i-1}(x) = v_i \otimes Q_i(x) \quad mod(p, v_1, \dots, v_{i-1}).$$

In general, there are many other types of nonzero differential. However we consider cases that differentials are only of this form.

LEMMA 5.2. *Let $X = BSpin(n)$ and $n = 8\ell, 8\ell \pm 1$. In AHss for $BP^*(X)$, assume all nonzero differentials are of form $(*)$. Then $2e, v_1e, \dots, v_{h-2}e$ are all permanent cycles.*

Proof. We use Lemma 4.2, 4.3 in the preceding section. First recall $Q_i(d_0) = 0$, $Q_i(e) = 0$ for $i < h-1$. Therefore d_0e exists in E_{2^h-1}.

Since $Q_{j-1}d_j = d_0$ and $Q_k(d_j) = 0$ for $k < j-1$, the differential in AHss is

$$d_{2^j-1}(d_je) = v_{j-1} \otimes Q_{j-1}(d_je) = v_{j-1}d_0e.$$

Hence we have $(2, v_1, v_2, \dots, v_{h-2})(d_0e) = 0$ in $E_{2^h-1}^{*,*'}$.

Now we study the differential

$$d_{2^h-1}(e) = v_{h-1}Q_{h-1}(e) = v_{h-1}d_0e.$$

Note that e is BP^*-free in $E_{2^h-1}^{*,*'}$, since $e|C = z^{2^h}$ and $e \notin Im(Q_i)$. Hence we have

$$Ker(d_{2^h-1}) \cap BP^*\{e\} \cong Ideal(2, v_1, \dots, v_{h-2})\{e\}.$$

(In this paper, $R\{a, b, \dots\}$ means the R-free module generated by $a, b, \dots$) By the assumption $(*)$ for differentials, $2e, v_1e, \dots, v_{h-2}e$ are all permanent cycles. $\square$

For $7 \le n \le 9$, AHss converging $BP^*(BSpin(n))$ is computed in [12], ([19] also), and it is known that $(*)$ is satisfied.

COROLLARY 5.3. *For $n = 7, 8$ (resp. $n = 9$), the elements $2e$, v_1e (resp. $2e$, v_1e, v_2e) are in $Im(\rho_{BP}^*) \subset BP^*(BT)^W$ (but e itself is not).*

Let $K(s)^*(X)$ be the Morava K-theory with the coefficients ring $K(s)^* \cong \mathbf{Z}/p[v_s, v_s^{-1}]$, and $AK(s)^*(X) = AK(s)^{2*,*}(X)$ its algebraic version [29]. Here we consider an assumption such that

$$(**)\ \ AK(s)^*(BG) \to K(s)^*(BG) \quad \textit{is surjecive.}$$

It is known by Merkurjev (see [21] for details) that $AK^*(BG) \cong K^*(BG)$ for the algebraic K-theory $AK^*(X)$ and the complex K-theory $K^*(X)$, which induces $AK(1)^*(BG) \cong K(1)^*(BG)$. Hence $(**)$ is correct when $s = 1$ for all G.

LEMMA 5.4. *Let* $X = BSpin(n)$, $n = 8\ell, 8\ell \pm 1$ *and suppose* $(*)$. *Moreover suppose* $(**)$ *for* $s = h-2$. *Then* $v_{h-2}e \in Im(\rho_\Omega^*)$, *and hence there is* $c_i \in CH^*(X)$ *for* $0 \le i \le h-2$ *in Lemma* 5.1.

Proof. First note $0 \neq v_{h-2}e \in K(h-2)^*(X)$ (hence so is e). On the other hand [29]

$$AK(h-2)^*(X) \cong K(h-2)^* \otimes CH^*(X)/I$$

for some ideal I of $CH^*(X)$. Therefore there is an element $c \in CH^*(X)$ which corresponds $v_{h-2}^s e$ that is $cl_\Omega(c) = v_{h-2}^s e$ for $cl_\Omega : \Omega^*(X) \to BP^*(X)$. Since $e \notin Im(cl_\Omega)$, we see s must be positive. The possibility of

$$|v_{h-2}^s e| = -2(2^{h-2}-1)s + 2^h > 0$$

is $s=1$ or $s=2$. When $s=2$, we note $|v_{h-2}^2 e| = 4$ and $cl_{CH}(c) = 0$. However it is known by Totaro (Theorem 15.1 in [22]),

$$cl : CH^2(X) \to H^4(X) \quad \textit{is injective}.$$

Hence $s=1$ and $cl_\Omega(c) = v_{h-2}e$. □

From Merkurjev's result for $K(1)^*(BG)$, we have $cl_\Omega(c) = v_1 e$.

COROLLARY 5.5. *For* $X = BSpin(n)$ $n = 7, 8$, *there is an element* $c \in CH^3(X)$ *such that* $c \neq 0 \in CH^*(X)/2$, $cl(c) = 0$ *but* $\rho_\Omega^*(c) \neq 0 \in \Omega^*(BT)^W$.

6. *Spin*(7) for $p = 2$

Let G be a compact Lie group. Consider the restriction map

$$res_{H/p} : H^*(BG; \mathbf{Z}/p) \to Lim_{V:el.ab.}\, H^*(BV; \mathbf{Z}/p)^{W_G(A)}$$

where $W_G(A) = N_G(A)/C_G(A)$ and V ranges in the conjugacy classes of elementary abelian p-groups. Quillen [18] showed this $res_{H/p}$ is an F-isomorphism (i.e. its kernel and cokernel are generated by nilpotent elements). We consider its integral version

$$res_H : H^*(BG) \to \Pi_{A:ab.} H^*(BA)^{W_G(A)},$$

where A ranges in the conjugacy classes of abelian subgroups of G.

Hereafter this section, we assume $G = Spin(7)$ and $p = 2$ and hence $h = 3$. The number of conjugacy classes of the maximal abelian subgroups of G is two, one is the torus T and the other is $A' \cong (\mathbf{Z}/2)^4$ which is not contained in T. The Weyl group is $W_G(A') \cong \langle U, GL_3(\mathbf{Z}/2)\rangle \subset GL_4(\mathbf{Z}/2)$ where U is the maximal unipotent group in $GL_4(\mathbf{Z}/2)$. It is well known

$$H^*(BG; \mathbf{Z}/2) \cong H^*(BA'; \mathbf{Z}/2)^{W_G(A')} \cong \mathbf{Z}/2[w_4, w_6, w_7, w_8]$$

where w_i for $i \le 7$ (resp. $i = 8$) are the Stiefel-Whitney class for the representation induced from $Spin(7) \to SO(7)$ (resp. the spin representation Δ and hence $w_8 = w_8(\Delta) = e$).

Since $H^*(BG)$ has just 2-torsion by Kono, the restriction map res_H injects Tor into $H^*(BA'; \mathbf{Z}/2)^{W_G(A')}$, and

$$(H^*(BG)/Tor) \otimes \mathbf{Z}/2 \cong H(H^*(BG; \mathbf{Z}/2); Q_0).$$

Since $Q_0 w_i = 0$ for $i \neq 6$ and $Q_0 w_6 = w_7$, we have

$$H(H^*(BG; \mathbf{Z}/2); Q_0) \cong \mathbf{Z}/2[w_4, c_6, w_8] \quad c_6 = w_6^2.$$

Of course the right hand side ring has no nonzero nilpotent elements. Hence we see that ρ_H^* is surjective and

$$H^*(BT)^W \otimes \mathbf{Z}/2 \cong \mathbf{Z}/2[w_4, c_6, w_8].$$

Thus the integral cohomogy is written as

$$H^*(BG) \cong \mathbf{Z}_{(2)}[w_4, c_6, w_8] \otimes (\mathbf{Z}_{(2)}\{1\} \oplus \mathbf{Z}/2[w_7]\{w_7\}).$$

In particular, we note res_H is injective.

Next we consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'} \cong H^*(BG) \otimes BP^* \Rightarrow BP^*(BG).$$

Its differentials have forms of $(*)$ in §5. Using $Q_1(w_4) = w_7$, $Q_2(w_7) = c_7$, $Q_2(w_8) = w_7 w_8$ and $Q_3(w_7 w_8) = c_7 c_8$, we can compute the spectral sequence

$$\begin{aligned} gr\, BP^*(BG) \cong\ & BP^*[c_4, c_6, c_8]\{1, 2w_4, 2w_8, 2w_4 w_8, v_1 w_8\} \\ & \oplus BP^*/(2, v_1, v_2)[c_4, c_6, c_7, c_8]\{c_7\}/(v_3 c_7 c_8). \end{aligned}$$

Hence $BP^*(BG) \otimes_{BP^*} \mathbf{Z}_{(2)}$ is isomorphic to

$$\begin{aligned} & \mathbf{Z}_{(2)}^*[c_4, c_6, c_8]\{1, 2w_4, 2w_8, 2w_4 w_8, v_1 w_8\}/(2v_1 w_8) \\ & \quad \oplus \mathbf{Z}/2[c_4, c_6, c_7, c_8]\{c_7\}. \end{aligned}$$

On the other hand, the Chow ring of BG is given by Guillot ([6], [29], [30])

$$\begin{aligned} CH^*(BG) & \cong BP^*(BG) \otimes_{BP^*} \mathbf{Z}_{(2)} \\ & \cong \mathbf{Z}_{(2)}[c_4, c_6, c_8] \otimes (\mathbf{Z}_{(2)}\{1, c_2', c_4'.c_6'\} \oplus \mathbf{Z}/2\{\xi_3\} \oplus \mathbf{Z}/2[c_7]\{c_7\}) \end{aligned}$$

where $cl(c_i) = w_i^2$, $cl(c_2') = 2w_4$, $cl(c_4') = 2w_8$, $cl(c_6') = 2w_4 w_8$, and $cl(\xi_3) = 0$, $|\xi_3| = 6$. Note $cl_\Omega(\xi_3) = v_1 w_8$ in $BP^*(BT)^W$, and $\xi_3 = c$ in Corollary 5.5. Hence we have

$$\begin{aligned} CH^*(BG)/Tor & \cong \mathbf{Z}_{(2)}[c_4, c_6, c_8]\{1, c_2', c_4'.c_6'\} \\ & \subset \mathbf{Z}_{(2)}[w_4, c_6, w_8] \cong CH^*(BT)^W. \end{aligned}$$

In fact the nilpotent ideal in $(CH^*(BG)/(Tor)) \otimes \mathbf{Z}/2$ is generated by c_2', c_4', c_6'.

Next we consider the Chow rings version for the restriction map

$$res_{CH} : CH^*(BG) \to \Pi_{A:ab.} CH^*(BA)^{W_G(A)}.$$

Recall $CH^*(BA') \cong \mathbf{Z}_{(2)}[y_1, \dots, y_4]$ with $cl(y_i) = x_i^2$. Hence we have

$$(CH^*(BA')/2)^{W_G(A')} \cong \mathbf{Z}/2[c_4, c_6, c_7, c_8].$$

Since Tor is just 2-torsion, we have

LEMMA 6.1. *For the torsion ideal $Tor \subset CH^*(BG)$, we have*

$$res_{CH}(Tor) \cong \mathbf{Z}/2[c_4, c_6, c_8, c_7]\{c_7\} \subset CH^*(BA').$$

Thus we see that $Ker(res_{CH}) \cong \mathbf{Z}/2[c_4, c_6, c_8]\{\xi_3\}$, which is the ideal of Griffiths elements. We write down the above results.

THEOREM 6.2. *Let $(G, p) = (Spin(7), 2)$. Let $Grif$ be the ideal generated by Griffiths elements and $D = \mathbf{Z}_{(2)}[c_4, c_6, c_8]$. Then we have*

$$CH^*(BG)/Tor \cong D\{1, 2w_4, 2w_8, 2w_4w_8\}$$
$$\subset D\{1, w_4, w_8, w_4w_8\} \cong CH^*(BT)^W, \quad \text{with } w_i^2 = c_i,$$
$$Tor/Grif \cong D/2[c_7]\{c_7\}, \quad Grif \cong D/2\{\xi_3\}.$$

Thus we see Theorem 1.2 in the introduction.

COROLLARY 6.3. *Take an element $\xi \in \Omega^*(BG)$ such that $\xi = \xi_3$ in $\Omega^*(BG) \otimes_{BP^*} \mathbf{Z}_{(2)} \cong CH^*(BG)$. Also identify c_i as an element in $\Omega^*(BG)$. Then we have $\mathbf{Z}/2[c_4.c_6, c_8]\{\xi\} \subset \Omega^*(BT)^W/2$.*

COROLLARY 6.4. *Let $J = (2^2, 2v_1, v_1^2, v_2, \dots) \subset BP^*$ so that $BP^*/J \cong \mathbf{Z}/4\{1\} \oplus \mathbf{Z}/2\{v_1\}$. For $D = \mathbf{Z}_{(2)}[c_4, c_6, c_8]$, we have*

$$\Omega^*(BG)/J \cong D \otimes (BP^*/J\{1, c_2', c_4', c_6', \xi_3\}/(2\xi_3 = v_1c_4')) \oplus \mathbf{Z}/2[c_7]\{c_7\}).$$

7. The exceptional group F_4, $p = 3$

In this section, we assume $(G, p) = (F_4, 3)$. (However similar arguments also work for $(G, p) = (E_6, 3), (E_7, 3)$ and $(E_8, 5)$ [10].) Toda computed the $mod(3)$ cohomology of BF_4. (For details see [20].)

$$H^*(BG; \mathbf{Z}/3) \cong C \otimes D, \quad \text{where}$$
$$C = F\{1, x_{20}, x_{20}^2\} \oplus \mathbf{Z}/3[x_{26}] \otimes \Lambda(x_9) \otimes \{1, x_{20}, x_{21}, x_{26}\}$$
$$D = \mathbf{Z}_{(3)}[x_{36}, x_{48}], \quad F = \mathbf{Z}_{(3)}[x_4, x_8].$$

Using that $H^*(BG)$ has no higher 3-torsion and $Q_0x_8 = x_9$, $Q_0x_{20} = x_{21}$, $Q_0x_{25} = x_{26}$, we can compute

$$H^*(BG) \cong D \otimes C' \quad where$$
$$C'/Tor \cong Z_{(3)}\{1, x_4\} \oplus E, \quad where\ \ E = F\{ab \mid a, b \in \{x_4, x_8, x_{20}\}\}$$
$$C' \supset Tor \cong \mathbf{Z}/3[x_{26}]\{x_{26}, x_{21}, x_9, x_9x_{21}\}.$$

Note $x_{26} = Q_2Q_1(x_4)$ in Theorem 2.2 and

$$H^*(BT; \mathbf{Z}/3)^W \cong H^{even}(BG; \mathbf{Z}/3)/(Q_2Q_1x_4) \cong D \otimes F\{1, x_{20}, x_{20}^2\}.$$

(For $x_{20}^3 \neq 0$, see [20]). Hence we have

$$(H^*(BG)/Tor) \otimes \mathbf{Z}/3 \cong D/3 \otimes (\mathbf{Z}/3\{1, x_4\} \oplus E) \subset D/3 \otimes F\{1, x_{20}, x_{20}^2\}.$$

From Lemma 2.3, we see ρ_H^* is surjective and

$$H^*(BT)^W \cong H^*(BG)/Tor \cong D \otimes (\mathbf{Z}_{(3)}\{1, x_4\} \oplus E).$$

Next we consider the Atiyah-Hirzebruch spectral sequence [12]

$$E_2^{*,*'} \cong H^*(BG) \otimes BP^* \Rightarrow BP^*(BG).$$

Its differentials have forms of $(*)$ in §5. Using $Q_1(x_4) = x_9$, $Q_1(x_{20}) = x_{25}$, $Q_1(x_{21}) = x_{26}$ and $Q_2x_9 = x_{26}$, we can compute

$$gr\ BP^*(BG) \cong D \otimes (BP^* \otimes (\mathbf{Z}_{(3)}\{1, 3x_4\} \oplus E) \oplus BP^*/(3, v_1, v_2)[x_{26}]\{x_{26}\}).$$

Hence we have

$$BP^*(BG) \otimes_{BP^*} \mathbf{Z}_{(3)} \cong D \otimes (\mathbf{Z}_{(3)}\{1, 3x_4\} \oplus E \oplus \mathbf{Z}/3[x_{26}]\{x_{26}\}).$$

Proposition 7.1. *Let $(G, p) = (F_4, 3)$ and $Tor \supset Grif$ be the ideal generated by Griffiths elements. Then we have*

$$CH^*(BG)/Tor \subset D \otimes (\mathbf{Z}_{(3)}\{1, 3x_4\} \oplus E) \subset H^*(BG)/Tor,$$
$$Tor/Grif \cong D \otimes \mathbf{Z}/3[x_{26}]\{x_{26}\}.$$

If Totaro's conjecture is correct, then $Grif = \{0\}$ and the first inclusion is an isomorphism. From [28], it is known that if $x_8^2 \in Im(cl)$ for the cycle map cl, then we can show that cl itself is surjective. However it seems still unknown whether $x_8^2 \in Im(cl)$ or not.

Corollary 7.2. *Let $(G, p) = (F_4, 3)$. If $(**)$ in §5 is correct for some $n \geq 2$, then the cycle map $CH^*(BG) \to BP^*(BG) \otimes_{BP^*} \mathbf{Z}_{(3)}$ is surjective and*

$$CH^*(BG)/Tor \cong D \otimes (\mathbf{Z}_{(3)}\{1, 3x_4\} \oplus E).$$

Proof. The corollary follows from $|v_nx_8^2| = 16 - 2(3^n - 1) \leq 0$. □

References

[1] B. Antieau, On the integral Tate conjecture for finite fields and representation theory, Algebraic Geometry **3** (2016), 138–149.

[2] D. Benson and J. Wood, Integral invariants and cohomology of $BSpin(n)$, Topology **34** (1994), 13–28.

[3] A. Borel, Topics in the homology theory of fiber bundles, Lect. notes in math. (second edition) **36**, Springer Verlag, 1967.

[4] G. Carlsson and R. Joshua, Motivic Spanier-Whitehead duality and motivic Becker-Gottlieb transfer, www.researchgate.net.

[5] M. Feshbach, The image of $H^*(BG;\mathbf{Z})$ in $H^*(BT;\mathbf{Z})$ for a compact Lie group with maximal torus T, Topology **20** (1985), 93–95.

[6] P. Guillot, The Chow rings of G_2 and $Spin(7)$, J. reine angew. Math. **604** (2007), 137–158.

[7] M. Kameko, On the integral Tate conjecture over finite fields, Math. Proc. Cambridge Philos. Soc. **158**, 2015, 531–546.

[8] M. Kameko, Representation theory and the cycle map of a classifying space, Algebraic Geometry **4** (2017), 221–228.

[9] M. Kameko and M. Mimura, Wely group invariants, arXiv: 1202.6459v1.

[10] M. Kameko and N. Yagita, The Brown-Peterson cohomology of the classifying spaces of the projective unitary group $PU(p)$ and exceptional Lie groups, Trans. Amer. Math. Soc. **360** (2008), 2265–2284.

[11] A. Kono, On the integral cohomology of $BSpin(n)$, J. Math. Kyoto Univ. **26** (1986), 333–337.

[12] A. Kono and N. Yagita, Brown-Peterson and ordinary cohomology theories of classifying spaces for compact Lie groups, Trans. Amer. Math. Soc. **339** (1993), 781–798.

[13] M. Levine and F. Morel, Cobordisme algébrique I, C. R. Acad. Sci. Paris **332** (2001), 723–728.

[14] M. Levine and F. Morel, Cobordisme algébrique II, C. R. Acad. Sci. Paris **332** (2001), 815–820.

[15] A. Pirutka and N. Yagita, Note on the counterexamples for the integral Tate conjecture over finite fields, Document Math. Extra Vol.: A. Merkurjev and A. Suslin's sixtieth birthday, 2015, 501–511.

[16] D. Ravenel, Complex cobordism and stable homotopy groups of spheres, Pure and applied mathematics **121**, Academic Press, London, 1986.

[17] D. Quillen, The mod 2 cohomology rings of extra-special 2-groups and the spinor groups, Math. Ann. **194** (1971), 197–212.

[18] D. Quillen, A cohomological criterion for p-nilpotence, J. of Pure and Applied Algebra **1** (1971), 361–372.

[19] B. Schuster and N. Yagita, Transfers of Chern classes in BP-cohomology and Chow rings, Trans. Amer. Math. Soc. **353** (2001), 1039–1054.

[20] H. Toda, Cohomology $mod(3)$ of the classifying space BF_4 of the exceptional group F_4, J. Math. Kyoto Univ. **13** (1973), 97–115.

[21] B. Totaro, The Chow ring of classifying spaces, Proc. of symposia in pure math. "Algebraic K-theory" (1997: University of Washington, Seattle) **67**, 1999, 248–281.

[22] B. Totaro, Group cohomology and algebraic cycles, Cambridge tracts in math. **204**, Cambridge Univ. Press, 2014.

[23] A. Tripathy, Further counterexamples to the integral Hodge conjecture, arXiv:1601.0617v1.

[24] A. Vistoli, On the cohomolgy and the Chow ring of the classifying space PGL_p, J. reine angew. Math. **610** (2007), 181–227.

[25] V. Voevodsky, The Milnor conjecture, www.math.uiuc.edu/K-theory/0170/.
[26] V. Voevodsky, Motivic cohomology with $\mathbf{Z}/2$ coefficient, Publ. Math. IHES **98** (2003), 59–104.
[27] J. Wood, Nilpotent elements in the Bockstein spectral sequence for $BSpin(n)$, Contemporary Math. **146** (1993), 453–469.
[28] N. Yagita, The image of the cycle map of classifying space of the exceptional Lie group F_4, J. Math. Kyoto Univ. **44** (2004), 181–191.
[29] N. Yagita, Applications of Atiyah-Hirzebruch spectral sequence for motivic cobordism, Proc. London Math. Soc. **90**, 2005, 783–816.
[30] N. Yagita, Coniveau filtration of cohomology of groups, Proc. London Math. Soc. **101**, 2010, 179–206.

Nobuaki Yagita
Faculty of Education
Ibaraki University
Mito, Ibaraki
Japan
E-mail: nobuaki.yagita.math@vc.ibaraki.ac.jp